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无穷维随机分析引论

黄志远 严加安 著

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内 容 简 介

本书系统地介绍了 Malliavin 分析和白噪声分析这两个无穷维随机分析重要领域. 全书分五章. 第一章介绍无穷维分析的基础知识, 包括 Hilbert 空间中的线性算子、Fock 空间、核空间及其对偶、拓扑线性空间上的 Borel 测度; 第二章介绍 Malliavin 随机变分的基本理论; 第三章介绍随机变分的若干重要应用, 包括 Hörmander 定理的概率证明、抽象 Wiener 空间上的位势理论和拟必然分析、非适应随机分析; 第四章介绍白噪声分析的一般理论, 包括一般框架、泛函空间的刻画、泛函的乘积和 Wick 积; 第五章介绍广义泛函的分析运算及广义泛函空间中的算子理论, 并简要介绍了它们在量子物理中的应用.

本书可供数学及有关专业研究生、教师及概率论研究工作者阅读和参考.

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《纯粹数学与应用数学专著》丛书

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前　　言

早在 19 世纪末和 20 世纪初，由于数学物理问题的需要，V. Volterra, R. Gâteaux, P. Lévy 和 Fréchet 等已开始了无穷维分析的研究（参看 Lévy[2]）. 然而其中最富有成果的研究方向，是由 N. Wiener 和 A. N. Kolmogorov 开始的，与随机过程理论紧密联系的无穷维积分理论. 1923 年，作为 Brown 运动的数学模型，Wiener[1] 首先在连续函数空间上构造了一个概率测度，即 Wiener 测度. 此后，R. Cameron 和 W. Martin[1, 2, 3] 的一系列工作揭示了 Wiener 积分的许多重要性质，特别是 Gauss 测度的拟不变性. 1931 年，Kolmogorov[1] 导出了扩散过程的转移概率所满足的二阶抛物型偏微分方程，从而建立了随机过程和微分方程之间的联系. 40 年代，K. Itô (同时还有 I. I. Gikhman[1]) 开创了随机过程轨道的无穷小分析，即随机分析学. 通过 Itô 随机微分方程，人们可以直接构造扩散过程的轨道，将扩散过程看作 Brown 运动的轨道泛函，即 Wiener 泛函. 于是提供了用概率方法来解微分方程等一系列纯分析问题的可能性. 与此同时，R. Feynman 和 M. Kac 用泛函积分方法解数学物理方程的著名工作以及量子场论的发展给了无穷维分析以新的推动力.

在有限维经典分析中，经典的函数和微分概念已远远不能满足数学物理发展的要求. 在 1936 年，S. L. Sobolev[1] 从数学物理方程的求解出发拓广了函数和微分的概念，引进了广义函数与广义导数，建立了 Sobolev 空间理论. 40 年代，L. Schwartz 系统地发展了广义函数理论，使之成为解决数学物理问题的强有力的工具. 物理中常用的 Dirac δ 函数等获得了满意的数学解释. 然而长期以来，物理学中仍然使用着大量的直观概念和形式演算，对它们建立严格的数学基础并由此推动理论物理学的发展，是对数

学物理学家的巨大挑战.

在无穷维分析中，也面临着同样的情况. 由于常见的泛函 (例如扩散过程作为 Wiener 泛函) 并不都是 (在 Fréchet 意义下) 可微的，因此也有必要拓广泛函和微分的概念. 1976 年 P. Malliavin[1] 创立了随机变分学，拓广了微分的概念，使常见的 Wiener 泛函可以无限次微分，并将梯度、散度、Ornstein-Uhlenbeck 算子等成功地推广到了无穷维空间. 在此基础上，S. Watanabe[1], I. Shigekawa[1], D. W. Stroock[1], P. A. Meyer[1] 等建立了无穷维的 Sobolev 理论. Malliavin 分析在偏微分算子、热核的正则性和渐近估计、随机振荡积分以及随机系统的滤波与控制等方面成功的应用已使它成为当今随机分析领域中最瞩目的成果之一.

几乎与 Malliavin 同时，T. Hida 开创了白噪声分析. 白噪声是 Brown 运动的广义导数，其样本空间是 Schwartz 的广义函数空间. Hida 将 Wiener 泛函看作白噪声泛函，在此基础上建立了无穷维的 Schwartz 理论. 白噪声分析有着深刻的物理背景，它在 Feynman 积分以及量子场论中的成功应用已越来越引起物理学界的重视.

上述两种无穷维分析的框架，都是建立在 Gauss 测度的拟不变性质基础之上. 它们可以统一在所谓 Gauss 概率空间的框架中. 其理论基础可以回溯到 50 年代 I. E. Segal[1, 2] 关于 Hilbert 空间上的抽象积分理论，I. M. Gel'fand 的装备 (Rigged) Hilbert 空间理论 (参看 Gel'fand & Vilenkin[1]) 和 60 年代 L. Gross[1] 的抽象 Wiener 空间理论. 自然，框架的选择依赖于所要解决的实际问题. 例如 Malliavin 分析要求试验泛函空间足够丰富，以使常见的泛函成为光滑，而 Hida 分析则要求广义泛函足够广阔以包罗物理中许多直观概念和形式演算，使之具有严格的数学意义. 它们的关系和有限维空间的 Sobolev 理论与 Schwartz 理论颇为相似.

我们写这本书的目的，就是要对无穷维随机分析这一迅速发展着的研究领域提供一本入门的读物，力求做到简明而自封，以期使具有随机分析基础知识的读者，能较快掌握无穷维随机分析的

基本理论和方法，直接阅读现代文献，进入这一领域研究工作的前沿．本书分五章．第一章是无穷维分析的基础知识，包括 Hilbert 空间中的线性算子， Fock 空间，赋可列范空间，核空间及其对偶空间，拓扑线性空间上的 Borel 测度等内容．为使本书基本自封和方便读者查考，我们将一般局部凸拓扑线性空间的基本概念和结果作为附录放在书的末尾．第一章的内容既是以后各章的基础，也有独立的价值．第二章是 Malliavin 随机分析的基本理论，包括 Gauss 概率空间上泛函的混沌分解和微分运算， Ornstein-Uhlenbeck 半群， Meyer 不等式和 Sobolev 空间理论以及泛函分布密度的存在性和光滑性等．所有结果的证明都力求做到简明而富有启发性．第三章是 Malliavin 随机分析的若干重要应用．着重讨论了 Itô 随机微分方程的解的分布密度，即相应的二阶抛物型方程基本解的正则性，用概率方法证明了 Hörmander 关于微分算子亚椭圆性的著名定理，讨论了抽象 Wiener 空间上的位势理论和拟必然分析以及非适应随机分析等．第四章是白噪声分析的一般理论，内容包括建立白噪声分析的一般框架，泛函空间的刻画，泛函的乘积和 Wick 积以及广义泛函的矩刻画，并简要介绍其在 Feynman 积分、 $P(\phi)_2$- 场及 Brown 运动自交局部时等方面的应用．第五章是广义泛函空间中的算子理论 (包括广义泛函的分析运算) 及其在量子物理中的应用．由于篇幅所限，未涉及算子理论在无穷维调和分析中的应用，有兴趣的读者可参看 N. Obata[2].

本书第一章 §1, §4 和第四、第五章由严加安执笔，第一章 §2, §3, 第二、第三章和附录由黄志远执笔．任佳刚教授、骆顺龙博士分别为第三章 §2 和第五章 §4 提供了部分材料并提出了许多宝贵意见，我们向他们表示谢意．本书的写作和出版分别得到了国家自然科学基金会的资助 (项目编号 19131040) 及中国科学院科学出版基金的资助，特此表示感谢．

黄志远、严加安　　1996 年 8 月

目　　录

第一章 无穷维分析的基础知识

§1. Hilbert空间中的线性算子

令 $\mathbb{K}$ 表示实数域 $\mathbb{R}$ 或复数域 $\mathbb{C}$, H,K 及 E 表示 $\mathbb{K}$ 上的 Hilbert 空间. 不同 Hilbert 空间中的内积和范数统一用 $(\cdot,\cdot)$ 及 $\|\cdot\|$ 表示. 我们约定内积 (x,y) 关于 x 线性、关于 y 共轭线性. 如不特别指明数域 $\mathbb{R}$ 或 $\mathbb{C}$, 所有结果同时适用于两种情形.

1.1 基本概念、记号及若干引理

我们用 $L(H,K)$ 及 $\mathcal{L}(H,K)$ 分别表示 H 到 K 中的线性算子及有界线性算子全体, 并用 $L(H)$ 及 $\mathcal{L}(H)$ 分别简记 $L(H,H)$ 及 $\mathcal{L}(H,H)$. 设 $A\in L(H,K)$, 我们用 $\mathcal{D}(A)$ 表示其定义域, 它是 H 的线性子空间. 今后对 $A\in\mathcal{L}(H,K)$ 恒假定 $\mathcal{D}(A)$ 在 H 中稠, 从而可进一步假定 $\mathcal{D}(A)=H$. 对无界线性算子 A, 它的定义域 $\mathcal{D}(A)$ 必须连同算子一同给定. 设 $A\in L(H,K)$, 令

$$\mathcal{N}(A)=\{x\in\mathcal{D}(A):\ Ax=0\},\qquad \mathcal{R}(A)=\{Ax:\ x\in\mathcal{D}(A)\},$$

分别称 $\mathcal{N}(A)$ 及 $\mathcal{R}(A)$ 为 A 的 **核**(或 **零空间**) 及 **值域**. 如果 $\mathcal{D}(A)$ 在 H 中稠, 则称 A 是 **稠定的**. 若 $\mathcal{N}(A)=\{0\}$, 则称 A 是 **可逆的**. 对可逆算子 A, 定义 A 的 **逆**A^{-1} 如下: $\mathcal{D}(A^{-1})=\mathcal{R}(A)$; 若 $Ax=y$, 则令 $A^{-1}y=x$.

乘积空间 $H\times K$ 按如下内积 $(\cdot,\cdot)$ 成为一 Hilbert 空间:

$$(\{x,y\},\{z,w\})=(x,z)+(y,w),\quad x,z\in H,\ y,w\in K.$$

(即 Hilbert 空间直和 $H\oplus K$.) 设 $A\in L(H,K)$, 令

$$\mathcal{G}(A)=\{\,\{x,Ax\}\ :\quad x\in\mathcal{D}(A)\},\tag{1.1}$$

$$\mathcal{W}(A)=\{\,\{Ax,x\}\ :\quad x\in\mathcal{D}(A)\},\tag{1.2}$$

则 $\mathcal{G}(A)$ 及 $\mathcal{W}(A)$ 分别为 $H\oplus K$ 及 $K\oplus H$ 的线性子空间. 我们分别称它们为 A 的 **图象** 和 **逆图象**. 若 A 可逆, 则 $\mathcal{W}(A)=\mathcal{G}(A^{-1})$.

设 $A_1,A_2\in L(H,K)$, 若 $\mathcal{G}(A_1)\subset\mathcal{G}(A_2)$, 即 $\mathcal{D}(A_1)\subset\mathcal{D}(A_2)$ 且限制在 $\mathcal{D}(A_1)$ 上 A_2 与 A_1 一致, 则称 A_2 是 A_1 的 **延拓**, 称 A_1 是 A_2 在 $\mathcal{D}(A_1)$ 上的 **限制**, 记为 $A_1\subset A_2$ 或 $A_2\supset A_1$.

设 $A\in L(H,K)$. 如果 $\mathcal{G}(A)$ 是 $H\oplus K$ 的闭子空间 (即 $\mathcal{W}(A)$ 是 $K\oplus H$ 的闭子空间), 则称 A 为 **闭算子**. 若 $\mathcal{G}(A)$ 在 $H\oplus K$ 中的闭包 $\overline{\mathcal{G}(A)}$ 是某个线性算子 $\widetilde{A}$ 的图象, 则称 A 是 **可闭的**, 并称 $\widetilde{A}$ 是 A 的 **闭包**. 显然, A 是可闭的当且仅当 $\{0,y\}\in\overline{\mathcal{G}(A)}$ 蕴含 $y=0$. 若 A 是闭算子, 且 $\mathcal{D}(A)=H$, 则由闭图象定理知 A 是有界算子. 闭算子的零空间为闭子空间.

设 $A\in L(H,K)$ 为稠定的, 令

$$\mathcal{D}(A^*)=\{y\in K:\exists c_y>0,\text{使得}\forall x\in\mathcal{D}(A),|(Ax,y)|\le c_y\|x\|\},$$

则由 Riesz 表现定理, $\forall y\in\mathcal{D}(A^*)$, 存在 H 中唯一元素, 记为 A^*y, 使得

$$(x,A^*y)=(Ax,y)\quad\forall x\in\mathcal{D}(A). \tag{1.3}$$

显然有 $A^*\in L(K,H)$, 我们称 A^* 为 A 的 **共轭算子**. 若 $A,B\in L(H,K)$ 为稠定的, 且 $A\subset B$, 则 $B^*\subset A^*$.

设 $A\in L(H)$. 如果 A 稠定且 $A\subset A^*$, 即

$$(Ax,y)=(x,Ay)\quad\forall x,y\in\mathcal{D}(A),$$

则称 A 是 **对称的**; 若进一步有 $A=A^*$, 则称 A 是 **自共轭的**.

引理 1.1 设 $A\in L(H)$ 为稠定算子, 且 $(Ax,x)=0,\forall x\in\mathcal{D}(A)$.

(1) 若 H 为复空间, 则 A 为零算子 (即 $Ax=0,\forall x\in\mathcal{D}(A)$);

(2) 若 H 为实空间, 且 A 为对称算子, 则 A 为零算子.

证明 (1) 设 $x,y\in\mathcal{D}(A)$, 则

$$(Ax,y)+(Ay,x)=(A(x+y),x+y)-(Ax,x)-(Ay,y)=0. \tag{1.4}$$

在上式两边同乘 $i(=\sqrt{-1})$ 并用 iy 代替 y 得

$$(Ax,y)-(Ay,x)=0. \tag{1.5}$$

于是由 (1.4) 及 (1.5) 得 $(Ax,y)=0$, $\forall y\in\mathcal{D}(A)$. 由于 $\mathcal{D}(A)$ 在 H 中稠, 这表明 $Ax=0$.

(2) 由 (1.4) 及 A 的对称性推得. ∎

设 $A\in\mathcal{L}(H,K)$, 我们用 $\|A\|$ 表示算子 A 的**范数**, 即

$$\|A\|=\sup\{\|Ax\| \ : \ \|x\|=1\}.$$

下一引理给出了对称有界算子范数的另一表达式.

引理 1.2 设 $A\in\mathcal{L}(H)$ 为对称算子, 则

$$\|A\|=\sup_{\|x\|=1}|(Ax,x)|. \tag{1.6}$$

证明 我们有

$$\begin{aligned}(Ax,y)+(y,Ax)&=(Ax,y)+(Ay,x)\\&=\frac{1}{2}[(A(x+y),x+y)-(A(x-y),x-y)].\end{aligned}$$

故有

$$\begin{aligned}|(Ax,y)+(y,Ax)|&\le\frac{1}{2}(\|x+y\|^2+\|x-y\|^2)\sup_{\|z\|=1}|(Az,z)|\\&=(\|x\|^2+\|y\|^2)\sup_{\|z\|=1}|(Az,z)|.\end{aligned} \tag{1.7}$$

最后一等式是由于平行四边形定律. 为证 (1.6), 不妨设 A 是非零算子. 记 $a=\sup_{\|x\|=1}|(Ax,x)|$, 则由引理 1.1 知 $a>0$. 在 (1.7) 中令 $y=a^{-1}Ax$, 则得

$$2a^{-1}\|Ax\|^2\le(\|x\|^2+a^{-2}\|Ax\|^2)a,$$

即有 $\|Ax\|^2 \le a^2\|x\|^2$, 从而 $\|A\| \le a$. 但相反的不等式恒成立，故 (1.6) 得证. ∎

设 $A \in L(H,K)$, $B \in L(K,E)$. B 与 A 的 **乘积** 定义如下:

$$\mathcal{D}(BA) = \{x \in \mathcal{D}(A):\ Ax \in \mathcal{D}(B)\}, \tag{1.8}$$

$$(BA)x = B(Ax),\ x \in \mathcal{D}(BA). \tag{1.9}$$

于是 $BA \in L(H,E)$.

引理 1.3 设 $A \in L(H,K), B \in L(K,E)$. 如果 A, B 及 BA 都是稠定的，则

$$A^*B^* \subset (BA)^*. \tag{1.10}$$

若进一步 B 是有界算子，则

$$A^*B^* = (BA)^*. \tag{1.11}$$

证明 (1.10) 可以从共轭算子定义出发直接验证. 为证 (1.11), 只需证 $(BA)^* \subset A^*B^*$. 设 $B \in \mathcal{L}(K,E)$, 由于 $\mathcal{D}(A) = \mathcal{D}(BA)$, $\mathcal{D}(B^*) = E$, 故对任一 $y \in \mathcal{D}((BA)^*)$, 有

$$(Ax, B^*y) = ((BA)x, y) = (x, (BA)^*y), \quad \forall x \in \mathcal{D}(A).$$

这表明 $B^*y \in \mathcal{D}(A^*)$(从而 $y \in \mathcal{D}(A^*B^*)$) 且有 $A^*B^*y = (BA)^*y$, 于是 $(BA)^* \subset A^*B^*$. (1.11) 得证. ∎

设 M 为 H 的一个闭子空间， $M^\perp$ 为 M 在 H 中的正交补，则对任给 $x \in H$, x 有如下唯一分解:

$$x = y + z,$$

其中 $y \in M, z \in M^\perp$. 我们用 Px 表示 y, 称 Px 为 x 到 M 上的 **投影**. 显然 P 为 H 上的有界对称线性算子, 且是 **幂等的**, 即 $P^2 = P$. 我们称幂等的有界对称线性算子为 **投影算子**.

下一引理给出了投影算子的一个刻画.

引理 1.4 设 $P \in \mathcal{L}(H)$. 则当且仅当 $\mathcal{R}(P) = \mathcal{N}(P)^\perp$ 且 $P^2 = P$ 时 P 为投影算子.

证明 设 $\mathcal{R}(P)=\mathcal{N}(P)^{\perp}$, 且 $P^2=P$. 则 $\forall x,y\in H$, $x-Px\in\mathcal{N}(P)$, $y-Py\in\mathcal{N}(P)$, 故有

$$\begin{aligned}(Px,y)&=(Px,Py+(y-Py))=(Px,Py)\\&=(Px+(x-Px),Py)=(x,Py).\end{aligned}$$

这表明 P 是对称的, 从而依定义 P 是投影算子. 反之, 设 P 是投影算子, 则

$$\begin{aligned}x\in\mathcal{N}(P)&\iff\forall y\in H,\ (x,Py)=(Px,y)=0\\&\iff x\perp\mathcal{R}(P),\end{aligned}$$

即有 $\mathcal{N}(P)=\mathcal{R}(P)^{\perp}$. 又由 $P^2=P$ 推知 $\mathcal{R}(P)=\mathcal{N}(I-P)$, 从而 $\mathcal{R}(P)$ 为 H 的闭子空间. 因此有 $\mathcal{R}(P)=\mathcal{R}(P)^{\perp\perp}=\mathcal{N}(P)^{\perp}$. ∎

1.2 可闭算子、对称算子与自共轭算子

定理 1.5 设 $A\in L(H,K)$ 为稠定的, 则

(1) A^* 为闭的, 且 $\mathcal{G}(A^*)=\mathcal{W}(-A)^{\perp}$;

(2) 若 A 为闭的, 则 A^* 稠定, 且 $A^{**}=A$;

(3) 当且仅当 A^* 稠定时 A 可闭. 这时 A^{**} 为 A 的闭包.

证明 (1) 设 $y\in K$, $z\in H$, 则有

$$\begin{aligned}\{y,z\}\in\mathcal{G}(A^*)&\iff y\in\mathcal{D}(A^*),z=A^*y\\&\iff(z,x)=(y,Ax),\ \forall x\in\mathcal{D}(A)\\&\iff(\{y,z\},\{-Ax,x\})=0,\ \forall x\in\mathcal{D}(A).\end{aligned}$$

这表明 $\mathcal{G}(A^*)=\mathcal{W}(-A)^{\perp}$. 特别 $\mathcal{G}(A^*)$ 为 $K\oplus H$ 的闭子空间, 即 A^* 为闭算子.

(2) 由于 $-A$ 为闭算子, 故 $\mathcal{G}(-A)$ 是 $H\oplus K$ 的闭子空间, 从而 $\mathcal{W}(-A)$ 是 $K\oplus H$ 的闭子空间. 由 (1) 知 $K\oplus H$ 有如下正交分解:

$$K\oplus H=\mathcal{W}(-A)\oplus\mathcal{G}(A^*).\tag{1.12}$$

现设 $z \in K$, 且 $z \perp \mathcal{D}(A^*)$, 则 $\{z,0\} \perp \mathcal{G}(A^*)$. 故由 (1.12) 知 $\{z,0\} \in \mathcal{W}(-A)$, 从而 $z = -A0 = 0$. 这表明 $\mathcal{D}(A^*)$ 在 K 中稠. 对 A^* 及 $-A$ 应用 (1.12) 得

$$H \oplus K = \mathcal{W}(-A^*) \oplus \mathcal{G}(A^{**}), \tag{1.13}$$

$$K \oplus H = \mathcal{W}(A) \oplus \mathcal{G}(-A^*). \tag{1.14}$$

但 (1.14) 等价于 $H \oplus K$ 的如下正交分解:

$$H \oplus K = \mathcal{G}(A) \oplus \mathcal{W}(-A^*). \tag{1.15}$$

比较 (1.13) 及 (1.15) 得 $\mathcal{G}(A) = \mathcal{G}(A^{**})$, 即有 $A = A^{**}$.

(3) 设 A 可闭, $\widetilde{A}$ 是 A 的闭包, 则 $\widetilde{A} \supset A$. 由共轭算子定义知 $A^* \supset \widetilde{A}^*$. 特别 $\mathcal{D}(A^*) \supset \mathcal{D}(\widetilde{A}^*)$, 故由 (2) 知 A^* 是稠定的. 对 $-A$ 应用 (1) 得

$$K \oplus H = \overline{\mathcal{W}(A)} \oplus \mathcal{G}(-A^*). \tag{1.16}$$

上式等价于

$$H \oplus K = \overline{\mathcal{G}(A)} \oplus \mathcal{W}(-A^*). \tag{1.17}$$

比较 (1.13) 及 (1.17) 知 $\overline{\mathcal{G}(A)} = \mathcal{G}(A^{**})$, 即 A^{**} 为 A 的闭包.

反之, 设 A^* 稠定, 往证 A 可闭. 由于 A^* 为闭算子, (1.13) 仍成立. 另一方面恒有 (1.17), 故得 $\overline{\mathcal{G}(A)} = \mathcal{G}(A^{**})$, 这表明 A 是可闭的. ∎

定理 1.6 设 $A \in L(H)$ 且对称, 则有下述结论:

(1) A 可闭, A^{**} 为 A 的闭包, A^{**} 对称;

(2) 若 $\mathcal{D}(A) = H$, 则 A 为有界自共轭算子;

(3) 若 A 自共轭且可逆, 则 $\mathcal{R}(A)$ 在 H 中稠且 A^{-1} 自共轭;

(4) 若 $\mathcal{R}(A)$ 在 H 中稠, 则 A 可逆;

(5) 若 $\mathcal{R}(A) = H$, 则 A 自共轭且 A^{-1} 为有界自共轭算子.

证明 (1) 由于 $A^* \supset A$, 故 A^* 稠定, 从而由定理 1.5(3) 知 A 可闭, 且 A^{**} 为 A 的闭包. 此外, 由于 $A \subset A^*$, 故 $A^{**} \subset A^{***}$, 从而 A^{**} 对称.

(2) 由于 $A \subset A^*$, 且 $\mathcal{D}(A) = H$, 故 $A = A^*$, 即 A 自共轭. 特别 A 是闭的. 故由闭图象定理知 A 是有界算子.

(3) 设 $y \in H$, $y \perp \mathcal{R}(A)$, 则 $\forall x \in \mathcal{D}(A)$, $(Ax, y) = 0$. 故 $y \in \mathcal{D}(A^*) = \mathcal{D}(A)$. 于是 $\forall x \in \mathcal{D}(A)$, $(x, Ay) = (Ax, y) = 0$. 这表明 $Ay = 0$. 由于假定 A 可逆, 必有 $y = 0$. 这样我们证明了 $\mathcal{R}(A)$ 在 H 中稠. 往证 A^{-1} 是自共轭的. 由于 $A = A^*$, 故由 (1.12) 知

$$\mathcal{G}((A^{-1})^*) = \mathcal{W}(-A^{-1})^{\perp} = \mathcal{G}(-A)^{\perp} = \mathcal{W}(A),$$

但恒有 $\mathcal{W}(A) = \mathcal{G}(A^{-1})$, 故 $\mathcal{G}((A^{-1})^*) = \mathcal{G}(A^{-1})$, 即 $(A^{-1})^* = A^{-1}$.

(4) 设 $y \in \mathcal{D}(A)$ 且 $Ay = 0$, 则 $\forall x \in \mathcal{D}(A)$, $(Ax, y) = (x, Ay) = 0$, 即 $y \perp \mathcal{R}(A)$. 由假定 $\mathcal{R}(A)$ 在 H 中稠, 故必有 $y = 0$. 这表明 $\mathcal{N}(A) = \{0\}$, 即 A 可逆.

(5) 首先, 由 (4) 知 A 可逆, 且由假定 $\mathcal{D}(A^{-1}) = \mathcal{R}(A) = H$. 设 $x, y \in H$, 则

$$(A^{-1}x, y) = (A^{-1}x, A(A^{-1}y)) = (x, A^{-1}y).$$

这表明 A^{-1} 对称. 于是由 (2) 及 (3) 推知 A^{-1} 有界自共轭和 A 自共轭. ∎

下一重要定理属于 von Neumann.

定理 1.7 设 $A \in L(H, K)$ 为稠定的闭算子, 则 A^*A 为 H 中的自共轭算子, 且 $\mathcal{G}_0 \equiv \big\{\{y, Ay\} : y \in \mathcal{D}(A^*A)\big\}$ 在 $\mathcal{G}(A)$ 中稠. 此外, AA^* 也是 K 中的自共轭算子.

证明 设 $x \in H$. 由 (1.12) 知, 存在 $u \in \mathcal{D}(A), v \in \mathcal{D}(A^*)$ 使得

$$\{0, x\} = \{-Au, u\} + \{v, A^*v\}.$$

故有 $v = Au$, 从而

$$x = u + A^*v = (I + A^*A)u.$$

令 $S = I + A^*A$, 显然 S^{-1} 对称, 且 $\|S^{-1}\| \le 1$, 故 S^{-1} 自共轭, 于是由定理 1.6(3) 知 S 自共轭. 这蕴含 A^*A 自共轭.

为证 $\mathcal{G}_0$ 在 $\mathcal{G}(A)$ 中稠, 只需证: $\forall x \in \mathcal{D}(A)$, 若 $\{x, Ax\}$ 与 $\mathcal{G}_0$ 正交, 则 $x=0$. 此正交性蕴含 $(x,y)+(Ax,Ay)=(x,y+A^*Ay)=0$, $\forall y \in \mathcal{D}(A^*A)$. 但是 $\mathcal{R}(I+A^*A)=H$, 故必须有 $x=0$.

最后, 由定理 1.5 知 A^* 为从 K 到 H 的稠定闭算子, 且 $A^{**}=A$. 对 A^* 应用已证结果即知 AA^* 为 K 中的自共轭算子. ∎

设 $A \in L(H)$ 为对称的. 如果 A 的闭包 (即 A^{**}) 是自共轭算子, 则称 A 是 **本性自共轭的**.

下一定理给出了本性自共轭算子的一个等价描述.

定理 1.8 设 $A \in L(H)$ 为对称. 为要 A 是本性自共轭的, 必须且只需 A^* 自共轭 (即 A^* 对称).

证明 若 A^* 自共轭, 则 $A^*=A^{**}$, 从而 $A^{***}=A^{**}$, 即 A^{**} 自共轭. 反之, 设 A^{**} 自共轭, 则 $A^{**}=A^{***}$. 但由于 A^* 是闭稠定算子, 根据定理 1.5(2), 我们有 $A^{***}=A^*$, 从而最终有 $A^*=A^{**}$, 即 A^* 自共轭. ∎

设 $A \in L(H)$ 对称, 如果存在实数 c 使得

$$(Ax,x) \geq c\|x\|^2, \quad \forall x \in \mathcal{D}(A),$$

则记为 $A \geq c$ 并称 A 是 **下半有界的**. 如果 c 可取为 0(正数), 则称 A 是 **正的** (**有正下界的**).

下一定理给出了下半有界对称算子为自共轭算子或本性自共轭算子的一个有用的刻画.

定理 1.9 设 $A \in L(H)$ 为一下半有界的对称算子, $A \geq c$, $\epsilon > 0$. 令 $B=(\epsilon-c)I+A$, $\mathcal{D}(B)=\mathcal{D}(A)$, 则

(1) A 为自共轭, 当且仅当 $\mathcal{R}(B)=H$;

(2) A 为本性自共轭, 当且仅当 $\mathcal{R}(B)$ 在 H 中稠 (或等价地, $\mathcal{N}(B^*)=\{0\}$).

证明 (1) 设 A 自共轭, 则 B 自共轭且可逆, 故由定理 1.6(3) 知 $\mathcal{R}(B)$ 在 H 中稠. $\forall x \in H, \exists y_n \in \mathcal{D}(A)$, 使得 $\|x-By_n\| \to 0$, $n \to \infty$. 令 $x_n=By_n$. 由于 $A-cI \geq 0$ 以及

$$x_n-x_m=\epsilon(y_n-y_m)+(A-c)(y_n-y_m),$$

我们有

$$\epsilon\|y_n - y_m\|^2 \le (x_n - x_m, y_n - y_m) \le \|x_n - x_m\|\|y_n - y_m\|.$$

从而存在 $y \in H$, 使 $y_n \to y$. 但是 $x_n = (\epsilon - c)y_n + Ay_n$, 故 Ay_n 在 H 收敛. 由于 A 是闭算子, 必有 $y \in \mathcal{D}(A)$ 且 $Ay_n \to Ay$. 这样一来, 我们有 $x = (\epsilon - c)y + Ay \in \mathcal{R}(B)$. 这表明 $\mathcal{R}(B) = H$.

反之, 设 $\mathcal{R}(B) = H$. 则由定理 1.6(5) 知 B 为自共轭, 从而 A 为自共轭.

(2) 设 A 本性自共轭, 则 A 的闭包 $\widetilde{A}$ 为自共轭. 令 $\widetilde{B} = (\epsilon - c)I + \widetilde{A}$, 则由 (1) 知 $\mathcal{R}(\widetilde{B}) = H$. 由 $A - cI \ge 0$, 与上述证明类似可证 $\mathcal{R}(\widetilde{B}) = \overline{\mathcal{R}(B)}$, 从而 $\mathcal{R}(B)$ 在 H 中稠. 反之, 设 $\mathcal{R}(B)$ 在 H 中稠. 令 $\widetilde{A}$ 为 A 的闭包, $\widetilde{B} = (\epsilon - c)I + \widetilde{A}$, 则可以证明 $\mathcal{R}(\widetilde{B}) = \overline{\mathcal{R}(B)}$. 从而由 (1) 知 $\widetilde{B}$ 自共轭, 即 $\widetilde{A}$ 自共轭. 依定义, A 本性自共轭. ∎

系 1.10 设 A 为 H 中有正下界的对称算子, 则 A 为本性自共轭当且仅当 $\mathcal{R}(A)$ 在 H 中稠 (或者等价地 $\mathcal{N}(A^*) = \{0\}$).

证明 设 $A \ge \epsilon$, $\epsilon > 0$. 令 $A_1 = A - \epsilon I$, 则 $A_1 \ge 0$, $A = A_1 + \epsilon I$. 由定理 1.9(2) 立得本系结论. ∎

例 令 $H = L^2(\mathbb{R}^d)$, $A = -\Delta + I$, $\mathcal{D}(A) = C_0^\infty(\mathbb{R}^d)$ (这里 Δ 是 Laplace 算子, $C_0^\infty(\mathbb{R}^d)$ 表示 $\mathbb{R}^d$ 上具有紧支撑无穷次可微的函数全体), 则 A 为本性自共轭的. 事实上, 显然 A 为有正下界的对称算子. 为证 A 本性自共轭, 只需证 $\mathcal{N}(A^*) = \{0\}$. 设 $g \in H$, $A^*g = 0$, 则在 Schwartz 分布意义下 $Ag = 0$, 因为 $\forall f \in C_0^\infty(\mathbb{R}^d)$ 有 $\langle Af, g\rangle = (Af, g) = (f, A^*g) = 0$, 这里 $\langle\cdot,\cdot\rangle$ 表示 $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}^*(\mathbb{R}^d)$ 上的典则双线性型. 我们用 $\mathcal{F}f$ 表示 f 的 Fourier 变换, 则

$$\mathcal{F}(-\Delta + I)g(\xi) = (|\xi|^2 + 1)\mathcal{F}g(\xi).$$

从而 $\mathcal{F}g(\xi) = 0$, 故 $g = 0$. 这表明 $\mathcal{N}(A^*) = \{0\}$.

令 $\widetilde{A}$ 为 A 的闭包, 则 $\widetilde{A}$ 为自共轭算子. 易证

$$\mathcal{D}(\widetilde{A}) = \mathcal{H}^2(\mathbb{R}^d) \equiv \left\{ f \in L^2(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |\xi|^2 |\mathcal{F}f(\xi)|^2 d\xi < \infty \right\}.$$

1.3 下半有界对称算子的自共轭延拓

对称算子不一定有自共轭延拓. 下面我们介绍的 Friedrichs 定理表明: 下半有界对称算子恒有自共轭延拓. 为了证明这一定理, 我们需要一个有关闭的正对称共轭双线性型的表示定理, 它也属于 Friedrichs[1].

定义 1.11 设 H 为数域 $\mathbb{K}$ 上的 Hilbert 空间, V 为 H 的一稠线性子空间. 令 $a(\cdot,\cdot): V \times V \to \mathbb{K}$ 为 V 上的二元函数, 称 a 为 H 中的 **对称共轭双线性型**(或 **Hermite 型**), 如果

(1) $a(x,y)$ 关于 x 线性, 关于 y 共轭线性;

(2) $a(\cdot,\cdot)$ 对称, 即 $a(x,y)=\overline{a(y,x)}$.

我们称 V 为 a 的定义域, 记为 $\mathcal{D}(a)$. 如果 $a(x,x) \geq 0$, $\forall x \in \mathcal{D}(a)$, 则称 a 为 **正的**. 若进一步有

$$x \neq 0 \Longrightarrow a(x,x) > 0, \tag{1.18}$$

则称 a 为 **严格正的**.

设 a 为一正 Hermite 型. 在 $\mathcal{D}(a)$ 上定义内积:

$$(x,y)_a \equiv a(x,y)+(x,y)\ , \quad x,y \in \mathcal{D}(a), \tag{1.19}$$

则 $\mathcal{D}(a)$ 关于内积 $(\cdot,\cdot)_a$ 为一内积空间. 如果 $\mathcal{D}(a)$ 关于范数 $\|\cdot\|_a$ 完备, 则称 a 是 **闭的**.

定理 1.12 设 a 为 H 中闭的正 Hermite 型, 则存在唯一的正自共轭算子 A, 使得 $\mathcal{D}(A) \subset \mathcal{D}(a)$, 且

$$(Ax,y)=a(x,y)\ , \quad \forall x \in \mathcal{D}(A),\ y \in \mathcal{D}(a). \tag{1.20}$$

证明 令

$$\mathcal{D}(A)=\{x \in \mathcal{D}(a):\ \exists c_x>0, \text{使得} |a(x,y)| \leq c_x\|y\|,\ \forall y \in \mathcal{D}(a)\}. \tag{1.21}$$

由 Riesz 表现定理, $\forall x \in \mathcal{D}(A)$, 存在 H 中唯一的元素, 记为 Ax, 使得

$$a(x,y)=(Ax,y)\ , \quad \forall y \in \mathcal{D}(a). \tag{1.22}$$

显然 $A \in L(H)$, A 为正的.

设 $z \in H$, 由 (1.19)

$$|(z,y)| \le \|z\|\|y\| \le \|z\|\|y\|_a\ , \quad \forall y \in \mathcal{D}(a). \tag{1.23}$$

故由 Riesz 表现定理存在 $\mathcal{D}(a)$ 中唯一的元素, 记为 Bz, 使得

$$(z,y) = (Bz,y)_a = a(Bz,y) + (Bz,y), \quad \forall y \in \mathcal{D}(a). \tag{1.24}$$

往证 $\mathcal{D}(A) = \mathcal{R}(B)$ 且在 H 中稠. 由 (1.24) 及 (1.21) 知 $Bz \in \mathcal{D}(A)$. 从而 $\mathcal{R}(B) \subset \mathcal{D}(A)$. 另一方面, 由于 $\mathcal{D}(B) = H$ 及 $x = B(x + Ax), \forall x \in \mathcal{D}(A)$, 我们有 $\mathcal{D}(A) \subset \mathcal{R}(B)$. 于是 $\mathcal{D}(A) = \mathcal{R}(B)$. 但是由 (1.24) 知, 若 $y \in \mathcal{D}(a)$ 与 $\mathcal{R}(B)$ 正交, 则 $y = 0$. 故 $\mathcal{R}(B)$(即 $\mathcal{D}(A)$) 在 $\mathcal{D}(a)$ 中按范数 $\|\cdot\|_a$ 稠, 从而也按比它弱的 H 范数 $\|\cdot\|$ 稠. 因此, 最终它在 H 中稠.

最后证明 A 为自共轭算子. 由 (1.24), $Bz = 0$ 蕴含 $(z,y) = 0$, $\forall y \in \mathcal{D}(a)$, 从而 $z = 0$. 这表明 B 可逆. 由于 $\mathcal{D}(B^{-1}) = \mathcal{R}(B)$, 故 B^{-1} 是稠定的, 且有 $\mathcal{D}(A) = \mathcal{D}(B^{-1})$. 由 (1.24) 及 (1.22) 得

$$(Ax,y) = a(x,y) = (B^{-1}x,y) - (x,y)\ , \quad \forall x \in \mathcal{D}(A), y \in \mathcal{D}(a). \tag{1.25}$$

因此有 $A = B^{-1} - I$. 由于 $\mathcal{R}(B^{-1}) = H$ 且 B^{-1} 为对称算子, 由定理 1.6(5) 知 B^{-1} 自共轭, 从而 A 亦为自共轭算子. 满足 (1.22) 的算子 A 的唯一性由引理 1.1 推得. ∎

有了上述准备以后, 我们可以证明如下的

定理 1.13(Friedrichs 定理) 设 $A \in L(H), A \ge c$ 为对称下半有界, 则 A 有自共轭延拓 $\widehat{A}$, $\widehat{A}$ 下半有界, 且 $\widehat{A} \ge c$.

证明 首先设 $A \ge 1$, 令

$$a(x,y) = (Ax,y)\ , \quad \forall x,y \in \mathcal{D}(A), \tag{1.26}$$

则 a 为 H 中严格正 Hermite 型, 它确定了 $\mathcal{D}(A)$ 上的一个内积 $(\cdot,\cdot)^*$. 我们用 $\mathcal{D}(\widetilde{a})$ 表示 $\mathcal{D}(A)$ 关于范数 $\|\cdot\|^*$ 的完备化, 并用

$\widetilde{a}(\cdot,\cdot)$ 表示 $a(\cdot,\cdot)$ 到 $\mathcal{D}(\widetilde{a})\times\mathcal{D}(\widetilde{a})$ 上的连续延拓. 由于范数 $\|\cdot\|^*$ 比 H 上的范数 $\|\cdot\|$ 强, 故 $\mathcal{D}(\widetilde{a})$ 可取为 H 的子空间, 从而 $\widetilde{a}$ 为 H 中的闭的严格正 Hermite 型. 故由定理 1.12 存在唯一的自共轭算子 $\widetilde{A}$ 使得

$$(\widetilde{A}x,y)=\widetilde{a}(x,y)\ ,\quad x\in\mathcal{D}(\widetilde{A}),\ y\in\mathcal{D}(\widetilde{a}).$$

设 $x\in\mathcal{D}(A)$, 则由 (1.26)

$$|\widetilde{a}(x,y)|=|(Ax,y)|\le\|Ax\|\|y\|\le\|Ax\|\|y\|_a,\quad\forall y\in\mathcal{D}(A).$$

由于 $\mathcal{D}(A)$ 在 $\mathcal{D}(\widetilde{a})$ 中按范数 $\|\cdot\|^*$ 稠, 故上一不等式对一切 $y\in\mathcal{D}(\widetilde{a})$ 也成立, 从而依 $\widetilde{A}$ 的定义知: $x\in\mathcal{D}(\widetilde{A})$, 且 $(\widetilde{A}x,y)=\widetilde{a}(x,y)$, $\forall y\in\mathcal{D}(\widetilde{a})$. 特别由 (1.26) 得

$$(\widetilde{A}x,y)=\widetilde{a}(x,y)=(Ax,y),\quad\forall y\in\mathcal{D}(A).$$

故有 $\widetilde{A}x=Ax,\forall x\in\mathcal{D}(A)$. 这表明 $\widetilde{A}$ 是 A 的延拓, 此外显然有 $\widetilde{A}\ge 1$.

对一般情形 $A\ge c$, 令 $A_1=A+(1-c)I$, 则 $A_1\ge 1$. 由前所证, 存在 A_1 的自共轭延拓 $\widetilde{A}_1$, 且 $\widetilde{A}_1\ge 1$. 令 $\widetilde{A}=\widetilde{A}_1-(1-c)I$, 则 $\widetilde{A}\ge c$, $\widetilde{A}$ 为 A 的自共轭延拓. ∎

1.4 自共轭算子的谱分解

定义 1.14 设 H 为数域 $\mathbb{K}$ 上的 Hilbert 空间, A 为 H 中的闭算子. 令

$$\rho(A)=\{\lambda\in\mathbb{K}:\mathcal{N}(\lambda I-A)=\{0\},\overline{\mathcal{R}(\lambda I-A)}=H,(\lambda I-A)^{-1}\in\mathcal{L}(H)\},$$

称 $\rho(A)$ 为 A 的 **预解集**. $\rho(A)$ 在 $\mathbb{K}$ 中的补集称为 A 的 **谱集**, 记为 $\sigma(A)$. 又令

$$\sigma_p(A)=\{\lambda\in\mathbb{K}\ :\ \mathcal{N}(\lambda I-A)\ne\{0\}\},$$

$\sigma_p(A)$ 称为 A 的 **特征值集**. 设 $\lambda \in \sigma_p(A)$, 称 $\mathcal{N}(\lambda I - A)$ 为 A 的对应于 λ 的 **特征子空间**, 每个非零 $x \in \mathcal{N}(\lambda I - A)$ 称为 A 的对应于 λ 的 **特征向量**.

自共轭算子谱分解通常是对复 Hilbert 空间情形给出的, 这时可以用对称算子的 Cayley 变换将问题转化为有界自共轭算子的谱分解. 对实 Hilbert 空间情形中的自共轭算子, 我们可以通过 "复化" 方法将它变成复 Hilbert 空中的自共轭算子. 所以关于自共轭算子的谱分解, 对这两种 Hilbert 空间有统一的描述. 下面我们将介绍有关结果, 但只对紧自共轭算子及其逆的谱分解给出证明.

以下我们用 $\mathcal{P}(H)$ 表示 H 上的投影算子全体. 设 $P_1, P_2 \in \mathcal{P}(H)$. 如果 $P_1(H) \subset P_2(H)$, 我们记为 $P_1 \leq P_2$, 这时 $P_2 - P_1 \in \mathcal{P}(H)$.

定义 1.15 设 $\{E_\lambda, \lambda \in \mathbb{R}\} \subset \mathcal{P}(H)$. 称它为 H 的一个 **单位分解**, 如果它满足下述条件:

(1) 单调增: $\lambda_1 \leq \lambda \Longrightarrow E_{\lambda_1} \leq E_{\lambda_2}$;

(2) 右连续: $E_{\lambda+} \equiv s\text{-}\lim_{s\downarrow\downarrow\lambda} E_s = E_\lambda$;

(3) $E_{-\infty} \equiv s\text{-}\lim_{\lambda\to-\infty} E_\lambda = 0, E_\infty \equiv s\text{-}\lim_{\lambda\to\infty} E_\lambda = I$,

这里 s-lim 表示算子的强极限.

下一定理是自共轭算子的谱分解定理.

定理 1.16(von Neumann) 设 A 为 H 中的自共轭算子, 则存在 H 的唯一单位分解 $\{E_\lambda, \lambda \in \mathbb{R}\}$, 使得 $\forall x, y \in \mathcal{D}(A)$

$$(Ax, y) = \int_{\mathbb{R}} \lambda d(E_\lambda x, y) , \tag{1.27}$$

这里右端为 Lebesgue-Stieltjes 积分. 我们称 $\{E_\lambda, \lambda \in \mathbb{R}\}$ 为 A 的 **谱族**.

通常我们用如下 "谱积分" 表达式来表示算子 A:

$$A = \int_{\mathbb{R}} \lambda\, dE_\lambda . \tag{1.28}$$

我们有

$$\mathcal{D}(A) = \left\{ x \in H \ : \ \int_{\mathbb{R}} \lambda^2 d(E_\lambda x, x) < \infty \right\}. \tag{1.29}$$

注 对 H 的任一单位分解 $\{E_\lambda, \lambda\in I\!R\}$，我们如 (1.29) 定义 $\mathcal{D}(A)$，则 $\mathcal{D}(A)$ 在 H 中稠. 对给定 $x\in\mathcal{D}(A)$，由 Riesz 表现定理可唯一确定 H 的一元素，记为 Ax，使得 (1.27) 式对一切 $y\in\mathcal{D}(A)$ 成立. 容易证明：如上定义的算子 A 是 H 中的自共轭算子，其谱族为 $\{E_\lambda, \lambda\in I\!R\}$.

下一定理给出了 $\varphi(A)$ 的定义，其中 φ 为 $I\!R$ 上的实值 Borel 可测函数， A 为 H 中的自共轭算子.

定理 1.17 设 A 为 H 中的自共轭算子， $\{E_\lambda, \lambda\in I\!R\}$ 为其谱族， φ 为 $I\!R$ 上的实值 Borel 可测函数. 令

$$\mathcal{D}(\varphi(A))\equiv\left\{x\in H\ :\ \int_{I\!R}\varphi(\lambda)^2 d(E_\lambda x, x)<\infty\right\}, \tag{1.30}$$

则 $\mathcal{D}(\varphi(A))$ 在 H 中稠，且 $\forall x,y\in\mathcal{D}(\varphi(A))$

$$\int_{I\!R}|\varphi(\lambda)||\,d(E_\lambda x,y)|\le\|y\|\left(\int_{I\!R}\varphi(\lambda)^2 d(E_\lambda x,x)\right)^{1/2}.$$

对 $x\in\mathcal{D}(\varphi(A))$，令 $\varphi(A)x$ 为 H 中唯一的元素，使得

$$(\varphi(A)x,y)=\int_{I\!R}\varphi(\lambda)d(E_\lambda x,y)\ ,\ \forall y\in\mathcal{D}(A). \tag{1.31}$$

我们用如下“谱积分”表达 $\varphi(A)$:

$$\varphi(A)=\int_{I\!R}\varphi(\lambda)dE_\lambda\ . \tag{1.32}$$

则 $\varphi(A)$ 为 H 中的自共轭算子.

下一定理用谱族给出了自共轭算子特征值集的刻画.

定理 1.18 设 A 为 H 中的自共轭算子， $\{E_\lambda, \lambda\in I\!R\}$ 为其谱族. 则

$$\sigma_p(A)=\{\lambda\in I\!R\ :\ E_\lambda\ne E_{\lambda-}\}. \tag{1.33}$$

定理 1.19 设 A 为 H 中的下半有界自共轭算子，$\{E_\lambda, \lambda \in \mathbb{R}\}$ 为其谱族，令

$$c = \sup\{\lambda : E_\lambda = 0\},$$

则 $c \in \mathbb{R}$. 这时 (1.27) 及 (1.31) 中的积分区域 $\mathbb{R}$ 可以用区间 $[c, \infty)$ 代替. 特别，如果 A 为正的 (即 $c \geq 0$)，则对任意 $p \in \mathbb{R}$ 可定义 A 的 p- 次幂：

$$A^p = \int_{[0,\infty)} \lambda^p dE_\lambda, \tag{1.34}$$

则 A^p 为自共轭算子. 我们称 $A^{1/2}$ 为 A 的 **平方根**.

下一定理是对定理 1.12 的重要补充.

定理 1.20 设 a 为 H 中的一闭的正 Hermite 型，A 为按 (1.20) 与 a 联系的正自共轭算子，则 $\mathcal{D}(A^{1/2}) = \mathcal{D}(a)$，且有

$$a(x,y) = (A^{1/2}x, A^{1/2}y), \qquad x, y \in \mathcal{D}(a).$$

证明 令 $a'(x,y) = (A^{1/2}x, A^{1/2}y), x, y \in \mathcal{D}(A^{1/2})$，则易见 a' 为 H 中的闭的正 Hermite 型. 由于 $\{\{x, Ax\} : x \in \mathcal{D}(A)\}$ 显然在 $\mathcal{G}(A^{1/2})$ 中稠，且 a' 与 a 在 $\mathcal{D}(A)$ 上一致，故有 $a = a'$. ∎

下一定理给出了稠定闭算子的极分解.

定理 1.21 设 A 为 H 到 K 的稠定闭算子. 令 $T = (A^*A)^{1/2}$，则 T 为 H 中的正自共轭算子，且 $\mathcal{D}(T) = \mathcal{D}(A)$. 此外，存在从 $\mathcal{R}(T)$ 到 K 中的唯一线性等距算子 U(即 $\|Ux\| = \|x\|$)，使得 $A = UT$. 我们称这一分解为 A 的 **极分解**，称 T 为 A 的 **绝对值**，记为 $|A|$.

证明 由定理 1.7 知，A^*A 为 H 中的正自共轭算子，令

$$a(x,y) \equiv (Ax, Ay), \quad x, y \in \mathcal{D}(A).$$

则 a 为 H 中闭的正 Hermite 型. 由定理 1.12 及 1.20 易知 $\mathcal{D}(T) = \mathcal{D}(a) = \mathcal{D}(A)$，且有

$$\|Ax\|^2 = a(x,x) = \|Tx\|^2, \quad x \in \mathcal{D}(A) = \mathcal{D}(T).$$

从而 $Ax=0 \Longleftrightarrow Tx=0$. 对 $y=Tx\in\mathcal{R}(T)$, 令 $Uy=Ax$, 则 U 在 $\mathcal{R}(T)$ 上的定义是不含混的, $A=UT$, 且 $\|Uy\|=\|Ax\|=\|Tx\|=\|y\|$. ∎

定义 1.22 设 $A\in\mathcal{L}(H,K)$, $\mathcal{D}(A)=H$. 如果 A 将 K 中的单位球 (或任一有界子集) 映成 K 中相对紧集, 则称 A 为**紧算子**(或**全连续算子**).

我们今后用 $\mathcal{K}(H,K)$ 表示 H 到 K 中的紧算子全体. 显然紧算子为有界算子, $\mathcal{K}(H,K)$ 为 $\mathcal{L}(H,K)$ 的闭子空间.

下一定理是自共轭紧算子的谱分解定理. 为方便读者, 我们给出它的证明.

定理 1.23 设 A 为 H 中的非零自共轭紧算子. 则存在 H 中的一标准正交系 $\{e_n\}$ 及一列非零实数 $\{\lambda_n\}$, 使得 $Ae_n=\lambda_n e_n$, 且有

$$Ax=\sum_n \lambda_n(x,e_n)e_n\ ,\ \forall x\in H\ . \tag{1.35}$$

如果 A 退化 (即 $\mathcal{R}(A)$ 为 H 的有穷维子空间), 则上述级数只含有限项; 如果 A 非退化, 则 $\lim_{n\to\infty}\lambda_n=0$.

证明 由于 A 自共轭, (Ax,x) 为实数. 令

$$m=\inf_{\|x\|=1}(Ax,x)\ ,\quad M=\sup_{\|x\|=1}(Ax,x)\ .$$

由引理 1.2 知, m 与 M 中至少一个不等于零. 令 λ_1 为 m 与 M 中绝对值较大者, 则 $|\lambda_1|=\sup_{\|x\|=1}|(Ax,x)|$. 选取 $x_n\in H$, $\|x_n\|=1$, 使得 $\lambda_1=\lim_n(Ax_n,x_n)$. 由于 A 为紧算子, $\{Ax_n,n\geq 1\}$ 在 H 中相对紧, 必要时取子列, 不妨设 $\{Ax_n\}$ 在 H 中收敛于某极限 y. 由引理 1.2 知, $\|A\|=|\lambda_1|$, 故 $\|y\|\leq\|A\|=|\lambda_1|$. 另一方面,

$$\begin{aligned}\lim_{n\to\infty}\|Ax_n-\lambda_1x_n\|^2&=\lim_{n\to\infty}(\|Ax_n\|^2-2\lambda_1(Ax_n,x_n)+\lambda_1^2)\\&=\|y\|^2-\lambda_1^2.\end{aligned}$$

于是必有 $\|y\|=\lambda_1$, 且有 $\lim_{n\to\infty}\|Ax_n-\lambda_1x_n\|=0$. 令 $e_1=\lambda_1^{-1}y$, 则 $\|e_1\|=1$, $Ae_1=\lambda_1e_1$.

现令 $V(e_1)$ 为 e_1 张成的线性子空间，即 $V(e_1)=\{\alpha e_1,\alpha\in \mathbb{K}\}$. 令 $H_1=V(e_1)^{\perp}$, 则

$$(Ay,e_1)=(y,Ae_1)=\lambda_1(y,e_1).$$

于是 $y\in H_1\Longleftrightarrow Ay\in H_1$. 易见 A 限于 H_1 为 H_1 上的自共轭紧算子. 若 A 不是 H_1 上的零算子, 则可重复上述步骤, 得到 $e_2\in H_1$ 及非零实数 λ_2, 使 $|\lambda_2|=\sup_{x\in H_1,\|x\|=1}|(Ax,x)|$, 且 $Ae_2=\lambda_2 e_2$. 依此类推, 若 A 退化, 且 $\dim\mathcal{R}(A)=N$, 我们可得 H 中一标准正交系 $\{e_1,e_2,\cdots,e_N\}$, 及一列非零实数 $\{\lambda_1,\cdots,\lambda_N\}$, 使得 $Ae_j=\lambda_j e_j, 1\le j\le N$, $|\lambda_1|\ge|\lambda_2|\ge\cdots|\lambda_N|$, 且对 $1\le j\le N$ 有

$$|\lambda_j|=\sup_{\|x\|=1,x\in H_{j-1}}|(Ax,x)|\,, \tag{1.36}$$

其中 $H_0=H$,H_j 为由 $e_1,\cdots,e_j$ 张成的线性子空间 $V(e_1,\cdots,e_j)$ 的正交补, $1\le j\le N-1$. 这时公式 (1.35) 显然成立. 若 A 非退化, 则我们得到一标准正交系 $\{e_n,n\ge1\}$ 及一列非零实数 $\{\lambda_n,n\ge1\}$, 使得 $\{|\lambda_n|,n\ge1\}$ 为单调非增, $Ae_n=\lambda_n e_n$, 且 (1.36) 对一切 j 成立. 令 $x_n=\lambda_n^{-1}e_n$, 则 $e_n=Ax_n$. 由于 $\{e_n,n\ge1\}$ 不是 H 中的相对紧集, 故 $\{x_n,n\ge1\}$ 不可能是 H 中的有界集. 因此, 必须有 $\lim_{n\to\infty}\lambda_n=0$.

剩下要证 (1.35) 式. 设 $x\in H$, 令 $y_m=x-\sum_{n=1}^m(x,e_n)e_n$, 则 $y_m\in H_m$. 由 (1.36) 及引理 1.2 知

$$\|Ay_m\|\le|\lambda_{m+1}|\,\|y_m\|\le|\lambda_{m+1}|\,\|x\|,$$

于是 $\lim_{m\to\infty}|Ay_m|=0$. 这表明 (1.35) 成立. ∎

注 设 $\{e_n,n\ge1\}$ 为 H 中的一标准正交系, $\{\lambda_n,n\ge1\}$ 为一列非零实数, 满足 $\lim_{n\to\infty}\lambda_n=0$. 令 A 为由 (1.35) 定义的 H 中的线性算子, 则 A 为自共轭紧算子. 为要 A 是正的 (即 $(Ax,x)\ge0,\forall x\in H$), 必须且只需 $\lambda_n>0,\forall n\ge1$.

1.5 Hilbert-Schmidt 算子与迹算子

本节只讨论可分 Hilbert 空间中的有界线性算子. 这时我们把 H 中的完备标准正交系称为 H 的**标准正交基**, 或简称为 H 的**基**.

引理 1.24 设 $A \in \mathcal{L}(H, K)$, $\{e_n\}$ 及 $\{f_n\}$ 分别为 H 及 K 的基, 则有

$$\sum_n \|Ae_n\|^2 = \sum_n \|A^* f_n\|^2. \tag{1.37}$$

特别, $\sum_n \|Ae_n\|^2$ 不依赖基 $\{e_n\}$ 的选取.

证明 由于

$$Ae_n = \sum_m (Ae_n, f_m) f_m\ , \quad A^* f_m = \sum_n (A^* f_m, e_n) e_n\ ,$$

故得

$$\begin{aligned}\sum_n \|Ae_n\|^2 &= \sum_n \sum_m |(Ae_n, f_m)|^2 \\ &= \sum_n \sum_m |(e_n, A^* f_m)|^2 = \sum_m \|A^* f_m\|^2 .\end{aligned}$$ ∎

上一引理导致如下的

定义 1.25 设 $A \in \mathcal{L}(H, K)$. 若对 H 的某个基 $\{e_n\}$ 有

$$\sum_n \|Ae_n\|^2 < \infty,$$

则称 A 为 **Hilbert-Schmidt 算子**(简称为 **H-S 算子**). 这时令

$$\|A\|_2 \equiv \left(\sum_n \|Ae_n\|^2\right)^{1/2}, \tag{1.38}$$

称 $\|A\|_2$ 为 A 的 **Hilbert-Schmidt 范数**(也记为 $\|A\|_{\mathrm{HS}}$).

今后, 我们用 $\mathcal{L}_{(2)}(H, K)$ 表示 H 到 K 中的 H-S 算子全体. 对 $A, B \in \mathcal{L}_{(2)}(H, K)$, 令

$$(A, B)_2 \equiv \sum_n (Ae_n, Be_n)\ , \tag{1.39}$$

其中 $\{e_n\}$ 为 H 的基. 由于 $(A,B)=\frac{1}{4}(\|A+B\|_2^2-\|A-B\|_2^2)$, 故按照 (1.39) 式定义的 $(A,B)_2$ 不依赖于基 $\{e_n\}$ 的选取, 且 $(\cdot,\cdot)_2$ 为 $\mathcal{L}_{(2)}(H,K)$ 上的内积.

定理 1.26 $\mathcal{L}_{(2)}(H,K)$ 按内积 $(\cdot,\cdot)_2$ 为一可分 Hilbert 空间.

证明 首先我们任意取定 H 及 K 的基 $\{e_n\}$ 及 $\{f_n\}$. 对于 $A\in\mathcal{L}_{(2)}(H,K)$, 令 $a_{n,k}(A)=(Ae_n,f_k)$, 则

$$\|A\|_2^2=\sum_n\|Ae_n\|^2=\sum_n\sum_k|\,a_{n,k}(A)\,|^2\,.$$

令 $M=\{(a_{n,k})_{n,k\ge1}\,:\,a_{n,k}\in I\!K,\sum_{n,k=1}^{\infty}|a_{n,k}|^2<\infty\}$. 熟知, M 按如下内积成为一可分 Hilbert 空间:

$$\big((a_{n,k}),(b_{n,k})\big)=\sum_{n,k}a_{n,k}\bar{b}_{n,k}\,,$$

但 $A\mapsto(a_{n,k}(A))_{n,k\ge1}$ 为 $\mathcal{L}_{(2)}(H,K)$ 到 M 之上的一个线性保范同构映射, 故 $\mathcal{L}_{(2)}(H,K)$ 为一可分 Hilbert 空间. ∎

定理 1.27 H-S 算子为紧算子, 且对 $A\in\mathcal{L}_{(2)}(H,K)$ 有

$$\|A\|\le\|A\|_2 \tag{1.40}$$

证明 首先证明 (1.40). 为此, 令 (f_n) 为 K 的基. 由 (1.37) 及 (1.38) 得

$$\begin{aligned}\|Ax\|^2&=\sum_n|\,(Ax,f_n)\,|^2=\sum_n|\,(x,A^*f_n)\,|^2\\&\le\|x\|^2\sum_n\|A^*f_n\|^2=\|x\|^2\|A\|_2^2\,,\end{aligned}$$

(1.40) 得证. 现设 $\{e_n\}$ 为 H 的基. 对每个 $k\ge1$, 令

$$A_kx=\sum_{n=1}^{k}(x,e_n)Ae_n\,,\quad x\in H\,. \tag{1.41}$$

则每个 A_k 为退化算子，从而为紧算子．但我们有

$$\begin{aligned}\|A-A_k\|^2 \le \|A-A_k\|_2^2 &= \sum_n \|(A-A_k)e_n\|^2 \\ &= \sum_{n\ge k+1} \|Ae_n\|^2 ,\end{aligned}$$

故由级数 $\sum_n \|Ae_n\|^2$ 的收敛性推知 $\lim_{k\to\infty}\|A-A_k\|=0$．因此 A 为紧算子 (因为紧算子全体为 $\mathcal{L}(H,K)$ 的闭子空间)． ∎

注 由上述证明看出，退化算子全体在 $\mathcal{L}_{(2)}(H,K)$ 中稠．

定义 1.28 设 B 为 H 中的一非负自共轭紧算子．令 $\mathrm{Tr}B \equiv \|B^{1/2}\|_2^2$．称 $\mathrm{Tr}B$ 为 B 的**迹**．

由引理 1.24 知，对 H 的任一基 $\{e_n\}$，我们有

$$\mathrm{Tr}\,B = \sum_n \|B^{1/2}e_n\|^2 = \sum_n (Be_n, e_n) . \tag{1.42}$$

下一定理给出了 H-S 算子的一个刻画．

定理 1.29 设 A 为 H 到 K 的紧算子，则 $A \in \mathcal{L}_{(2)}(H,K) \Longleftrightarrow \mathrm{Tr}(A^*A) < \infty$．这时有

$$\|A\|_2^2 = \mathrm{Tr}(A^*A) . \tag{1.43}$$

证明 设 $A = UT$ 为 A 的极分解 (定理 1.21)，其中 $T = (A^*A)^{1/2}$．设 $\{e_n\}$ 为 H 的一个基，则有

$$\sum_n \|Ae_n\|^2 = \sum_n \|Te_n\|^2 = \sum_n (A^*Ae_n, e_n) , \tag{1.44}$$

由此推得定理的结论． ∎

注 设 $A \in \mathcal{K}(H,K), A = UT$ 为 A 的极分解，

$$Tx = \sum_n \lambda_n (x, e_n) e_n , \quad x \in H \tag{1.45}$$

为 T 的一个谱分解 (见定理 1.23). 则定理 1.29 的一个等价说法是

$$A \in \mathcal{L}_{(2)}(H,K) \Longleftrightarrow \sum_n \lambda_n^2 < \infty ,$$

且有

$$\|A\|_2^2 = \sum_n \lambda_n^2. \tag{1.46}$$

定义 1.30 设 $A \in \mathcal{K}(H,K)$, $A = UT$ 为 A 的极分解, (1.45) 为 T 的一个谱分解. 如果 $\sum_n \lambda_n < \infty$, 则称 A 为 **迹算子**(或 **核算子**). 令

$$\|A\|_1 = \sum_n \lambda_n, \tag{1.47}$$

称 $\|A\|_1$ 为 A 的 **迹范数**. 我们用 $\mathcal{L}_{(1)}(H,K)$ 表示 H 到 K 中迹算子全体.

由 (1.46) 知, 迹算子必为 H-S 算子, 从而亦为紧算子. 此外, 设 $A \in \mathcal{K}(H,K)$, 则

$$\begin{aligned} A \in \mathcal{L}_{(1)}(H,K) &\Longleftrightarrow \mathrm{Tr}[(A^*A)^{1/2}] < \infty \\ &\Longleftrightarrow (A^*A)^{1/4} \in \mathcal{L}_{(2)}(H,K), \end{aligned}$$

且有

$$\|A\|_1 = \mathrm{Tr}[(A^*A)^{1/2}] = \|(A^*A)^{1/4}\|_2^2. \tag{1.48}$$

下一定理给出了迹范数的一个表达式.

定理 1.31 设 $A \in \mathcal{L}_{(1)}(H,K)$, 则

$$\|A\|_1 = \sup \sum_n |\,(Af_n, g_n)\,| , \tag{1.49}$$

其中上确界是对 H 及 K 的一切基 $\{f_n\}$ 及 $\{g_n\}$ 取的. 此外, $\mathcal{L}_{(1)}(H,K)$ 按迹范数 $\|\cdot\|_1$ 为一可分 Banach 空间.

证明 设 $A=UT$ 为 A 的极分解, (1.45) 为 T 的一个谱分解. 则 $\forall x\in H$, 有

$$Ax=UTx=\sum_n \lambda_n(x,e_n)Ue_n\ . \tag{1.50}$$

将 $\{e_n\}$ 扩充成为 H 的基 $\{f_n'\}$, 将 $\{Ue_n\}$ 扩充成为 K 的基 $\{g_n'\}$, 并保持 e_n 与 Ue_n 的对应关系. 则由 (1.50) 得

$$\sum_n |(Af_n',g_n')|=\sum_n \lambda_n=\|A\|_1. \tag{1.51}$$

另一方面, 对 H 及 K 的任意基 $\{f_n\}$ 及 $\{g_n\}$, 由 (1.50) 有

$$\begin{aligned}\sum_n |(Af_n,g_n)| &= \sum_n \Big|\sum_m \lambda_m(f_n,e_m)(Ue_m,g_n)\Big| \\ &\le \sum_m \lambda_m \sum_n |(f_n,e_m)(Ue_m,g_n)| \\ &\le \frac{1}{2}\sum_m \lambda_m \sum_n(|(f_n,e_m)|^2+|(Ue_m,g_n)|^2) \\ &= \sum_m \lambda_m=\|A\|_1\ .\end{aligned} \tag{1.52}$$

故由 (1.51) 及 (1.52) 推得 (1.49). ∎

定理 1.32 设 $A\in\mathcal{L}_{(1)}(H)$. 则对 H 的任一基 $\{f_n\}$ 有

$$\sum_n |(Af_n,f_n)|\le \|A\|_1. \tag{1.53}$$

此外, $\sum_n(Af_n,f_n)$ 不依赖于基 $\{f_n\}$ 的选取. 我们令

$$\mathrm{Tr}A=\sum_n(Af_n,f_n)\ , \tag{1.54}$$

称 $\mathrm{Tr}A$ 为 A 的 **迹**(参见定义 1.28).

证明 (1.53) 是 (1.49) 的直接推论. 设 $A = UT$ 为 A 的极分解, (1.45) 为 T 的谱分解. 则 $\forall x \in H$ 有

$$Ax = UTx = \sum_n \lambda_n (x, e_n) U e_n = \sum_n (x, e_n) A e_n \ . \tag{1.55}$$

由 (1.52) 知二重级数 $\sum_n \sum_m (f_n, e_m)(Ae_m, f_n)$ 绝对收敛, 从而求和顺序可以交换, 故由 (1.55) 得

$$\begin{aligned}\sum_n (Af_n, f_n) &= \sum_n \sum_m (f_n, e_m)(Ae_m, f_n) \\ &= \sum_m \sum_n (f_n, e_m)(Ae_m, f_n) \\ &= \sum_m \Big(Ae_m, \sum_n (e_m, f_n) f_n\Big) = \sum_m (Ae_m, e_m) \ .\end{aligned}$$

这表明 $\sum_n (Af_n, f_n)$ 不依赖于基 $\{f_n\}$ 的选取. ∎

我们将下一定理的证明留给读者作为习题.

定理 1.33 设 $B \in \mathcal{L}(H, K)$, $A \in \mathcal{L}(K, E)$, 则有

$$\begin{aligned}&\|AB\|_2 \le \|A\| \|B\|_2 \ , \qquad \|AB\|_2 \le \|A\|_2 \|B\| \ , \\ &\|AB\|_1 \le \|A\| \|B\|_1 \ , \qquad \|AB\|_1 \le \|A\|_1 \|B\| \ , \\ &\|AB\|_1 \le \|A\|_2 \|B\|_2 \ .\end{aligned}$$

作为本节的结束, 我们介绍如下重要结果 (证明见 Meyer[3]).

定理 1.34 $\mathcal{L}_{(1)}(H, K)$ 为 $\mathcal{K}(H, K)$ 的拓扑对偶, $\mathcal{L}(H, K)$ 为 $\mathcal{L}_{(1)}(H, K)$ 的拓扑对偶, 其典则双线性型分别为

$$\langle B, A \rangle \equiv \sum_n (Bf_n, \overline{Af_n}) \ , \ A \in \mathcal{L}_{(1)}(H, K) \ , \ B \in \mathcal{K}(H, K) \ ;$$

$$\langle A, B \rangle \equiv \sum_n (Af_n, \overline{Bf_n}) \ , \ A \in \mathcal{L}_{(1)}(H, K) \ , \ B \in \mathcal{L}(H, K) \ ,$$

其中 $\{f_n\}$ 为 H 的任一基.

§ 2. Fock 空间与二次量子化

本节假定所有 Hilbert 空间都是域 $\mathbb{K}$ (实数域 $\mathbb{R}$ 或复数域 $\mathbb{C}$) 上的 **可分** Hilbert 空间；范数一律用 (带下标或不带下标的) $\|\cdot\|$ 表示；标准正交基均简称为基 (或 ONB).

2.1 Hilbert 空间的张量积

设 H_1 与 H_2 为 Hilbert 空间，其内积分别为 $(\cdot,\ \cdot)_1$ 与 $(\cdot,\ \cdot)_2$，对 $\varphi_1 \in H_1$ 及 $\varphi_2 \in H_2$，我们定义其张量积为 $H_1 \times H_2$ 上的一个共轭双线性型：

$$\varphi_1 \otimes \varphi_2(\xi_1, \xi_2) \equiv (\varphi_1, \xi_1)_1(\varphi_2, \xi_2)_2, \quad \xi_1 \in H_1, \xi_2 \in H_2. \tag{2.1}$$

以 $\mathcal{E}$ 表示由 $\{\varphi_1 \otimes \varphi_2 :\ \varphi_1 \in H_1, \varphi_2 \in H_2\}$ 生成的线性空间，对 $\varphi_1 \otimes \varphi_2,\ \psi_1 \otimes \psi_2 \in \mathcal{E}$，定义：

$$b(\varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2) \equiv (\varphi_1, \psi_1)_1(\varphi_2, \psi_2)_2, \tag{2.2}$$

并将其线性开拓到 $\mathcal{E}$ 上.

命题 2.1 由 (2.2) 式确定了 $\mathcal{E}\times\mathcal{E}$ 上一个严格正 Hermite 型，从而 $(\mathcal{E},\ b)$ 为内积空间.

证明 首先要证明 b 在 $\mathcal{E}$ 上的开拓是确定的. 若 $F \in \mathcal{E}$ 有两种不同的表示：

$$F = \sum_{j=1}^{n}(\varphi_{1j} \otimes \varphi_{2j}) = \sum_{k=1}^{m}(\varphi'_{1k} \otimes \varphi'_{2k}),$$

则由 (2.1) 式， $\forall \xi_1 \in H_1, \xi_2 \in H_2$

$$\begin{aligned} F(\xi_1, \xi_2) &= \sum_{j=1}^{n}(\varphi_{1j}, \xi_1)_1(\varphi_{2j}, \xi_2)_2 \\ &= \sum_{k=1}^{m}(\varphi'_{1k}, \xi_1)_1(\varphi'_{2k}, \xi_2)_2, \end{aligned}$$

从而由 (2.2) 式, $\forall\psi_1\in H_1,\psi_2\in H_2$ 有

$$\begin{aligned}b\left(\textstyle\sum_{j=1}^{n}(\varphi_{1j}\otimes\varphi_{2j}),\psi_1\otimes\psi_2\right)&=\sum_{j=1}^{n}(\varphi_{1j},\psi_1)_1(\varphi_{2j},\psi_2)_2\\&=\sum_{k=1}^{m}(\varphi'_{1k},\psi_1)_1(\psi'_{2k},\psi_2)_2\\&=b\left(\textstyle\sum_{k=1}^{m}(\varphi'_{1k}\otimes\varphi'_{2k}),\psi_1\otimes\psi_2\right),\end{aligned}$$

即 b 的定义不依赖于 $\mathcal{E}$ 中元素的表示.

由 (2.2) 式易知 b 为 Hermite 型, 现证其严格正性.

设 $F=\sum_{j=1}^{n}(\varphi_{1j}\otimes\varphi_{2j})\neq 0$, 取 $(e_1,\cdots,e_n)$ 为 $(\varphi_{11},\cdots,\varphi_{1n})$ 生成的子空间的基, 则存在 $f_1,\cdots,f_n\in H_2$, 不全为 0, 使 $F=\sum_{j=1}^{n}(e_j\otimes f_j)$. 于是

$$\begin{aligned}b(F,F)&=\sum_{j,k=1}^{n}b(e_j\otimes f_j,e_k\otimes f_k)\\&=\sum_{j,k=1}^{n}(e_j,e_k)_1(f_j,f_k)_2\\&=\sum_{j=1}^{n}\|f_j\|_2^2>0.\end{aligned}$$ ∎

定义 2.2 由上述内积空间 $(\mathcal{E},b)$ 完备化而得的 Hilbert 空间称为 H_1 和 H_2 的 **Hilbert张量积** (或简称张量积), 记为 $H_1\otimes H_2$.

命题 2.3 若 $\{e_j\}$ 和 $\{f_k\}$ 分别为 Hilbert 空间 H_1 和 H_2 之基, 则 $\{e_j\otimes f_k\}_{j,k\in\mathbb{N}}$ 为 Hilbert 空间 $H_1\otimes H_2$ 之基.

证明 正交性由 (2.2) 式即可看出, 为证其完备性, 只要证 $\mathcal{E}$ 含于由 $\{e_j\otimes f_k\}_{j,k\in\mathbb{N}}$ 生成之闭子空间 S 中.

任给 $\varphi_1\otimes\varphi_2\in\mathcal{E}$, 设 $\varphi_1=\sum_j c_je_j$, $\varphi_2=\sum_k d_kf_k$ 且系数满足 $\sum_j|c_j|^2<\infty$, $\sum_k|d_k|^2<\infty$. 则

$$\sum_{j,k}c_jd_k(e_j\otimes f_k)\in S.$$

由直接计算可知

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\left\|\varphi_1\otimes\varphi_2-\sum_{j=1}^{n}\sum_{k=1}^{m}c_jd_k(e_j\otimes f_k)\right\|=0. \quad ∎$$

平方可积函数空间是常见的 Hilbert 空间. 这类 Hilbert 空间的张量积有着非常直观而自然的含义. 设 (X,μ) 为一测度空间, $L^2(X,\mu)$ 表示 X 上关于测度 μ 平方可积的函数 (等价类) 所构成的 Hilbert 空间, 其内积由下式给出:

$$(f,g)=\int_X f(x)\overline{g(x)}\mu(dx). \tag{2.3}$$

若 H 为任一可分 Hilbert 空间, 我们以 $L^2(X,\mu;H)$ 表示 X 上取值 H 的平方可积函数 (μ- 等价类) 所构成的 Hilbert 空间, 具有内积:

$$(f,g)=\int_X (f(x),g(x))\mu(dx). \tag{2.4}$$

定理2.4 设 (X,μ) 和 (Y,ν) 为测度空间, 且 $L^2(X,\mu)$ 和 $L^2(Y,\nu)$ 可分, 则

1° 存在唯一同构关系:

$$L^2(X,\mu)\otimes L^2(Y,\nu)\cong L^2(X\times Y,\ \mu\times\nu),$$

使 $f\otimes g$ 对应于 $f(x)g(y)$;

2° 对任一可分 Hilbert 空间 H, 存在唯一同构关系:

$$L^2(X,\mu)\otimes H\cong L^2(X,\mu;H),$$

使 $f\otimes h$ 对应于 $f(x)\cdot h$.

证明 1° 设 $\{e_j(x)\}$ 和 $\{f_k(y)\}$ 分别为 $L^2(X,\mu)$ 和 $L^2(Y,\nu)$ 之基, 易见函数系 $\{e_j(x)f_k(y)\}_{j,k\in\mathbb{N}}$ 构成 $L^2(X\times Y,\mu\times\nu)$ 之基, 于是映射

$$U: f\otimes g\longmapsto f(x)g(y)$$

可唯一开拓为 $L^2(X,\mu)\otimes L^2(Y,\nu)$ 至 $L^2(X\times Y,\mu\times\nu)$ 上的酉算子.

2° 设 $\{e_j\}$ 为 H 之基, 则 $\forall g\in L^2(X,\mu;H)$, 由直接计算可知

$$\lim_{n\to\infty}\left\|g(x)-\sum_{j=1}^{n}(g(x),e_j)e_j\right\|=0.$$

这说明 $\{g_j(x)\cdot e_j:g_j\in L^2(X,\mu),j\in\mathbb{N}\}$ 之有限线性组合全体在 $L^2(X,\mu;H)$ 中稠密, 而映射

$$U:\quad \sum_{j=1}^{n}(g_j\otimes e_j)\longmapsto\sum_{j=1}^{n}g_j(x)\cdot e_j$$

定义在 $L^2(X,\mu)\otimes H$ 之稠密子集上, 保持内积不变, 从而可唯一开拓为 $L^2(X,\mu)\otimes H$ 到 $L^2(X,\mu;H)$ 上的酉算子. ∎

由 (2.2) 式可知

$$\|\varphi_1\otimes\varphi_2\|=\|\varphi_1\|_1\|\varphi_2\|_2,\tag{2.5}$$

从而双线性映射 $(\varphi_1,\varphi_2)\longmapsto\varphi_1\otimes\varphi_2$ 为 $H_1\times H_2$ 到 $H_1\otimes H_2$ 中的连续映射. 设 $\{e_j\}$ 和 $\{f_k\}$ 分别为 H_1 及 H_2 之基, 在 (2.1) 式中令 $\xi_1=e_j,\xi_2=f_k$, 两边取模的平方, 对 $j,k=1,2,\cdots$ 求和得

$$\begin{aligned}\sum_{j,k=1}^{\infty}|\varphi_1\otimes\varphi_2(e_j,f_k)|^2&=\|\varphi_1\|_1^2\|\varphi_2\|_2^2\\&=\|\varphi_1\otimes\varphi_2\|^2.\end{aligned}\tag{2.6}$$

由此可见, (2.6) 式左边可以作为 $H_1\otimes H_2$ 中范数的定义, 此和不依赖于基的选择, 称为 **Hilbert-Schmidt范数** .

对任意有限多个 Hilbert 空间的张量积, 可以归纳地定义, 但根据前面所述事实, 可以直接给出如下定义.

定义 2.5 设 $\{e_k^j\}_{k\in\mathbb{N}}$ 为 Hilbert 空间 H_j 之基 $(1\leq j\leq n)$, 对乘积空间 $\prod_{j=1}^n H_j$ 上 (共轭) n- 线性型 F, 定义其 Hilbert-Schmidt 范数

$$\|F\|_{\mathrm{HS}}^2 \equiv \sum_{(k_1,\cdots,k_n)\in\mathbb{N}^n} |F(e_{k_1}^1,\cdots,e_{k_n}^n)|^2, \tag{2.7}$$

此范数不依赖于基的选取. 具有有限 HS 范数的 (共轭) n- 线性型总体, 构成 **Hilbert空间张量积** $H_1\otimes H_2\otimes\cdots\otimes H_n$, 简记为 $\otimes_{j=1}^n H_j$.

将 (2.1) 推广为 n 元素张量积, 则 $\{\otimes_{j=1}^n e_{k_j}^j : (k_1,\cdots,k_n)\in\mathbb{N}^n\}$ 构成 $\otimes_{j=1}^n H_j$ 之基. 特别, 当 $H_1=H_2=\cdots=H_n=H$ 时, 其 n 重张量积记为 $H^{\otimes n}$. 若 $\{e_k\}_{k\in\mathbb{N}}$ 为 H 之基, 则 $\{\otimes_{j=1}^n e_{k_j} : (k_1,\cdots,k_n)\in\mathbb{N}^n\}$ 为 $H^{\otimes n}$ 之基. 在数学物理中, 常常考虑它的两个子空间: 对称和反称张量积子空间. 这里我们只介绍对称张量积空间.

设 $\mathfrak{S}$ 为 $\{1,2,\cdots,n\}$ 上的 n 阶置换群, 对 $\sigma\in\mathfrak{S}$, 定义:

$$\pi_\sigma(\varphi_1\otimes\cdots\otimes\varphi_n)\equiv\varphi_{\sigma(1)}\otimes\cdots\otimes\varphi_{\sigma(n)},$$

则 π_σ 可开拓为 $H^{\otimes n}$ 的自同构映射, 且对 $\sigma,\tau\in\mathfrak{S}$ 有 $\pi_{\sigma\tau}=\pi_\sigma\pi_\tau$. 于是自同态映射

$$\pi_n \equiv \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}}\pi_\sigma \tag{2.8}$$

为 $H^{\otimes n}$ 上的正交投影.

定义 2.6 由 (2.8) 式给出的正交投影 π_n 的值域 (它是 $H^{\otimes n}$ 的闭子空间) 称为 H 之 **n 重对称张量积空间**, 记为 $H^{\widehat{\otimes} n}$. 对 $\varphi_1,\cdots,\varphi_n\in H$, $\otimes_{j=1}^n\varphi_j$ 之投影

$$\begin{aligned}\widehat{\otimes}_{j=1}^n\varphi_j &\equiv \pi_n(\otimes_{j=1}^n\varphi_j)\\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}}\otimes_{j=1}^n\varphi_{\sigma(j)}\end{aligned} \tag{2.9}$$

称为 $\varphi_1,\cdots,\varphi_n$ 之 **对称张量积**. 我们约定对 $F,G\in H^{\widehat{\otimes} n}$

$$(F,G)_{H^{\widehat{\otimes} n}} \equiv n!(F,G)_{H^{\otimes n}}, \tag{2.10}$$

但为了记号方便，我们将一概折算成 $H^{\otimes n}$ 中的内积以省去内积符号中的下标 $H^{\widehat{\otimes} n}$ 和 $H^{\otimes n}$.

若 $H=L^2(X,\mu),(X,\mu)$ 为一测度空间，则由定理 2.4 可知，$H^{\otimes n}\cong L^2(X^n,\mu^n), AFdVP(X^n,\mu^n)$ 表示 (X,μ) 之 n 重乘积测度空间，而

$$H^{\widehat{\otimes} n}\cong \widehat{L^2}(X^n,n!\mu^n),$$

即 $H^{\widehat{\otimes} n}$ 等距同构于 X 上 n 元对称、关于测度 $n!\mu^n$ 平方可积的函数 (等价类) 所构成的 Hilbert 空间.

注 设 H 为任一线性空间，F 为 $\overbrace{H\times\cdots\times H}^{n}$ 上的对称 n 线性型. 若令

$$A(\varphi)\equiv F(\varphi,\cdots,\varphi),\qquad \varphi\in H,$$

则由简单计算可得

$$F(\varphi_1,\cdots,\varphi_n)=\frac{1}{2^n n!}\sum_{\substack{\epsilon_j^2=1\\ 1\le j\le n}}\epsilon_1\cdots\epsilon_n A\left(\textstyle\sum_{j=1}^n\epsilon_j\varphi_j\right),\qquad(2.11)$$

其中 $\sum$ 是对 $\epsilon_j=\pm1\ (j=1,\cdots,n)$ 所有 2^n 种情形求和. 此即所谓 **极化公式**. 作为直接推论，我们有

$$\widehat{\otimes}_{j=1}^n\varphi_j=\frac{1}{2^n n!}\sum_{\substack{\epsilon_j^2=1\\ 1\le j\le n}}\epsilon_1\cdots\epsilon_n\left(\textstyle\sum_{j=1}^n\epsilon_j\varphi_j\right)^{\otimes n},$$

$$\varphi_1,\cdots,\varphi_n\in H.\qquad(2.12)$$

从而 $H^{\widehat{\otimes} n}$ 是由 $\{h^{\otimes n}:h\in H\}$ 所生成的闭子空间.

设 Λ 为有限多项不为零的非负整数序列 $\alpha=\{\alpha_j\}_{j\in\mathbb{N}}$ 的全体，对 $\alpha\in\Lambda$，令 $|\alpha|\equiv\sum_j\alpha_j,\alpha!\equiv\prod_j(\alpha_j!)$(注意上述和、积实际上只含有限项)．$\forall n\in\mathbb{N}$，记 $\Lambda_n\equiv\{\alpha\in\Lambda:|\alpha|=n\}$，则 $\Lambda=\sum_n\Lambda_n$.

命题 2.7 设 $\{e_j\}_{j\in\mathbb{N}}$ 为 Hilbert 空间 H 之基，对 $\alpha\in\Lambda_n$，令

$$\widehat{e}_\alpha\equiv\pi_n\Big(\bigotimes_j e_j^{\otimes\alpha_j}\Big),\qquad(2.13)$$

(括号中张量积只含 n 个因子，其中 e_j 出现 α_j 次， $j=1,2,\cdots$)，则 $\{(\alpha!)^{-1/2}\widehat{e}_\alpha : \alpha \in \Lambda_n\}$ 构成 $H^{\widehat{\otimes} n}$ 之基.

证明 对 $(k_1,\cdots,k_n)\in \mathbb{N}^n$，令 β_j 为此 n 个自然数中等于 j 的个数，即

$$\beta_j \equiv \#\{i : 1\le i\le n, k_i=j\}, \quad j=1,2,\cdots,$$

得到的序列 $\beta=\{\beta_j\}\in\Lambda_n$. 记此映射为 $\pi:\mathbb{N}^n\to\Lambda_n$，显然 π 为满射，且当 $(k_1,\cdots,k_n)\in\pi^{-1}(\alpha)$ 时

$$\pi_n(\otimes_{j=1}^n e_{k_j})=\widehat{e}_\alpha.$$

若 $\beta\in\Lambda_n$, $(k_1',\cdots,k_n')\in\pi^{-1}(\beta)$，则

$$\begin{aligned}(\widehat{e}_\alpha,\widehat{e}_\beta) &= \big(\pi_n(\otimes_{j=1}^n e_{k_j}),\pi_n(\otimes_{j=1}^n e_{k_j'})\big)\\ &= \big(\otimes_{j=1}^n e_{k_j},\pi_n(\otimes_{j=1}^n e_{k_j'})\big)\\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}}\prod_{j=1}^n(e_{k_j},e_{k'_{\sigma(j)}}).\end{aligned}$$

当 $\alpha=\beta$ 时上式为 $\alpha!/(n!)$，当 $\alpha\ne\beta$ 时上式为 0，故 $\{(\alpha!)^{-1/2}\widehat{e}_\alpha : \alpha\in\Lambda_n\}$ 为 $H^{\widehat{\otimes} n}$ 之基. ∎

2.2 Fock 空间

在量子物理中，常以 $H^{\otimes n}$ 表示 n 个相同粒子的系统，由于相同粒子无法区分，且根据自旋的不同服从玻色 (Bose) 或费米 (Fermi) 统计，故我们考虑 H 的 n 重对称或反称张量积子空间. 但粒子可能增生或湮灭，粒子总数是不固定的，因此用 Fock 空间来作模型. 在无穷维随机分析中，最重要的事实之一是无穷维 Gauss 概率空间上平方可积泛函空间和对称 Fock 空间的同构关系，即所谓 Wiener-Itô-Segal 的混沌 (chaos) 分解 (参看第二章 §1 及第四章 §1).

我们先介绍 Hilbert 空间无穷直和的概念.

设 $(H_n, (\cdot, \cdot)_n), n=1,2,\cdots$ 为一列 Hilbert 空间, H 为乘积空间 $\prod_n H_n$ 之线性子空间, 由满足 $\sum_n \|\varphi_n\|_n^2 < \infty$ 之 $\varphi = \{\varphi_n\}$ 构成. 对 $\varphi = \{\varphi_n\}, \psi = \{\psi_n\} \in H$, 令

$$b(\varphi, \psi) \equiv \sum_{n=1}^{\infty} (\varphi_n, \psi_n)_n. \tag{2.14}$$

由 Schwarz 不等式

$$|(\varphi_n, \psi_n)_n| \leq \|\varphi_n\|_n \|\psi_n\|_n \leq \frac{1}{2}(\|\varphi_n\|_n^2 + \|\psi_n\|_n^2)$$

可知 (2.14) 式对 $\varphi, \psi \in H$ 有定义.

命题2.8 由 (2.14) 确定的 b 为 $H \times H$ 上的严格正 Hermite 型, (H, b) 为 Hilbert 空间, 称为 $\{H_n\}_{n\in\mathbb{N}}$ 之 **Hilbert直和空间**, 其代数直和空间在其中稠密.

证明 显然 b 为 Hermite 型. 若 $\varphi \neq 0$, 则

$$b(\varphi, \varphi) = \sum_{n=1}^{\infty} \|\varphi_n\|_n^2 > 0,$$

故 b 为严格正, 因代数直和是由仅仅有限项不为 0 之 $\varphi = \{\varphi_n\}$ 构成, 显然它在 H 中稠密.

设 $\varphi^{(m)} = \{\varphi_n^{(m)}\}_{n\in\mathbb{N}}, m = 1,2,\cdots$ 为 H 中关于范数 $\|\varphi\| = \sqrt{b(\varphi,\varphi)}$ 的基本列, 则 $\forall n, \{\varphi_n^{(m)}\}_{m\in\mathbb{N}}$ 为 H_n 中基本列, 从而在 H_n 中收敛于某 $\varphi_n^{(0)}$. 易见, $\varphi^{(0)} \equiv \{\varphi_n^{(0)}\} \in H$, 且 $\|\varphi^{(n)} - \varphi^{(0)}\| \longrightarrow 0$, 故 H 完备. ∎

今后, 我们将此无穷直和空间记为

$$H = \bigoplus_{n=1}^{\infty} H_n. \tag{2.15}$$

注意, 不要和 $\{H_n\}$ 之代数直和混淆.

定义 2.9 设 H 为 Hilbert 空间，则

$$\mathcal{F}(H) \equiv \bigoplus_{n=0}^{\infty} H^{\otimes n} \tag{2.16}$$

(约定 $H^{\otimes 0} = \mathbb{K}$) 称为 H 上的 **Fock空间** (或完全 Fock 空间，自由 Fock 空间). 而

$$\Gamma(H) \equiv \bigoplus_{n=0}^{\infty} H^{\widehat{\otimes} n} \tag{2.17}$$

则称为 H 上的 **对称 Fock 空间** 或玻色 (Bose) Fock 空间.

若 $\{e_k\}_{k\in\mathbb{N}}$ 为 H 之基，则形如

$$(0,0,\cdots,\otimes_{j=1}^{n} e_{k_j},0,\cdots) \tag{2.18}$$

(其中 $(k_1,\cdots,k_n)\in\mathbb{N}^n, \otimes_{j=1}^{n} e_{k_j}$ 位于第 $n+1$ 项，$\forall n\in\mathbb{N}$) 的序列以及 $(1,0,0,\cdots)$ 构成 $\mathcal{F}(H)$ 之基，而形如

$$(0,0,\cdots,(\alpha!)^{-1/2}\widehat{e}_{\alpha^{(n)}},0,\cdots) \tag{2.19}$$

(其中 $\alpha^{(n)}\in\Lambda_n, \widehat{e}_{\alpha^{(n)}}$ 位于第 $n+1$ 项，$\forall n\in\mathbb{N}$) 的序列以及 $(1,0,0,\cdots)$ 构成 $\Gamma(H)$ 之基.

2.3 二次量子化算子

所谓二次量子化就是由 Hilbert 空间 H 上的算子出发来构造 Fock 空间 $\mathcal{F}(H)$ 或 $\Gamma(H)$ 上的算子. 为此，我们要定义算子的张量积.

设 $H_i, K_i (i=1,2)$ 为 Hilbert 空间，A_i 为 H_i 到 K_i 中的稠定线性算子，$\mathcal{D}(A_i)\subset H_i$ 为其定义域. 令 $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ 为 $\{\varphi_1\otimes\varphi_2 : \varphi_i\in\mathcal{D}(A_i), i=1,2\}$ 张成的线性空间，则 $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ 在 $H_1\otimes H_2$ 中稠密. 定义

$$A_1\otimes A_2(\varphi_1\otimes\varphi_2)\equiv A_1\varphi_1\otimes A_2\varphi_2,\quad \varphi_i\in\mathcal{D}(A_i),\ i=1,2, \tag{2.20}$$

并将其线性开拓到 $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ 上. 容易证明, 此开拓是确定的, 即不依赖于 $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ 中元素的具体表示 (参照命题 2.1 之证明). 定义

$$A_1+A_2\equiv A_1\otimes I+I\otimes A_2. \tag{2.21}$$

命题2.10 由 (2.20) 及 (2.21) 式定义的算子 $A_1\otimes A_2$ 及 A_1+A_2 均为 $H_1\otimes H_2$ 到 $K_1\otimes K_2$ 中的稠定线性算子. 若 A_1 及 A_2 可闭, 则 $A_1\otimes A_2$ 及 A_1+A_2 均可闭.

证明 只需证 $A_1\otimes A_2$ 的可闭性. 设 A_i 可闭, (若将 K_i^* 与 K_i 等同, 将 H_i^* 与 H_i 等同) 则其共轭算子 A_i^* 为 K_i 到 H_i 中的稠定线性算子 (见定理 1.5), 且 $\forall F\in\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$, $G\in\mathcal{D}(A_1^*)\otimes\mathcal{D}(A_2^*)$ 有

$$((A_1\otimes A_2)F,\ G)=(F,\ (A_1^*\otimes A_2^*)G). \tag{2.22}$$

故 $\mathcal{D}(A_1^*)\otimes\mathcal{D}(A_2^*)\subset\mathcal{D}((A_1\otimes A_2)^*)$, 从而 $(A_1\otimes A_2)^*$ 稠定, 亦即 $A_1\otimes A_2$ 可闭. ∎

由定理 1.5 知 $(A_1^*\otimes A_2^*)^*$ 为 $A_1\otimes A_2$ 之闭包, 称为 A_1 与 A_2 之**张量积**, 仍记为 $A_1\otimes A_2$. 同样, A_1+A_2 之闭包仍记为 A_1+A_2. 于是有

$$(A_1\otimes A_2)^*=A_1^*\otimes A_2^*, \tag{2.23}$$

$$(A_1+A_2)^*=A_1^*+A_2^*. \tag{2.24}$$

命题2.11 设 A_1 和 A_2 分别为 Hilbert 空间 H_1 与 H_2 上的有界算子, 则

$$\|A_1\otimes A_2\|=\|A_1\|\|A_2\|. \tag{2.25}$$

证明 设 $\{e_j\}$ 和 $\{f_k\}$ 分别为 H_1 和 H_2 之基. 对任一有限和 $\sum_{j,k}c_{jk}(e_j\otimes f_k)$, 我们有

$$\begin{aligned}\|(A_1\otimes I)\textstyle\sum_{j,k}c_{jk}(e_j\otimes f_k)\|^2&=\sum_k\|\textstyle\sum_j c_{jk}A_1e_j\|^2\\&\le\sum_k\|A_1\|^2\sum_j|c_{jk}|^2\\&=\|A_1\|^2\|\textstyle\sum_{j,k}c_{jk}(e_j\otimes f_k)\|^2.\end{aligned}$$

由于上述形式的有限和在 $H_1\otimes H_2$ 中稠密，故有 $\|A_1\otimes I\|\le\|A_1\|$，于是由 $A_1\otimes A_2=(A_1\otimes I)(I\otimes A_2)$ 得

$$\|A_1\otimes A_2\|\le\|A_1\otimes I\|\|I\otimes A_2\|\le\|A_1\|\|A_2\|.$$

反之，任给 $\epsilon>0$，必有单位向量 $\varphi\in H_1,\psi\in H_2$ 使 $\|A_1\varphi\|\ge\|A_1\|-\epsilon$, $\|A_2\psi\|\ge\|A_2\|-\epsilon$，于是

$$\begin{aligned}\|(A_1\otimes A_2)(\varphi\otimes\psi)\|&=\|A_1\varphi\|\|A_2\psi\|\\&\ge\|A_1\|\|A_2\|-\epsilon\|A_1\|-\epsilon\|A_2\|+\epsilon^2.\end{aligned}$$

由于 ϵ 可任意小，故 $\|A_1\otimes A_2\|\ge\|A_1\|\|A_2\|$, (2.25) 得证. ∎

任意有限个算子的张量积可以归纳地定义. 特别，若 A 为 Hilbert 空间上闭线性算子，可以定义其 n 重张量积 $A^{\otimes n}$，它是 $H^{\otimes n}$ 上的闭线性算子，限制于 $H^{\widehat{\otimes} n}$ 上仍为一闭线性算子. 类似地，可以定义

$$\begin{aligned}A^{(n)}\equiv&\overbrace{A\otimes I\otimes\cdots\otimes I}^{n}+\overbrace{I\otimes A\otimes\cdots\otimes I}^{n}+\cdots\\&+\overbrace{I\otimes I\otimes\cdots\otimes A}^{n}.\end{aligned}\tag{2.26}$$

由 (2.23) 及 (2.24) 推知，若将 H 与 H^* 视为等同，当 A 为自共轭时， $A^{\otimes n}$ 及 $A^{(n)}$ 亦为自共轭算子.

现在来构造 Fock 空间上的算子.

定义 2.12 设 A 为 Hilbert 空间 H 上稠定闭线性算子，令 $\mathcal{D}\equiv\{\varphi\in\mathcal{F}(H):\varphi=\{\varphi_n\},\varphi_n\in\mathcal{D}(A)^{\otimes n},\forall n$, 除有限项外均为 $0\}$. 若 $\varphi=\{\varphi_n\}\in\mathcal{D}$, 在 $\mathcal{D}$ 上定义

$$\Gamma(A)\varphi\equiv\{A^{\otimes n}\varphi_n\};\qquad d\Gamma(A)\varphi\equiv\{A^{(n)}\varphi_n\}.\tag{2.27}$$

(约定 $A^{\otimes 0}=I,A^{(0)}=0$), 则 $\Gamma(A)$ 及 $d\Gamma(A)$ 均为 $\mathcal{F}(H)$ 上稠定、可闭线性算子，其闭包仍记为 $\Gamma(A)$ 及 $d\Gamma(A)$, 分别称为 A 之 **二次量子化** 及 **微分二次量子化** 算子[1].

[1] 在物理文献中，一般称 $d\Gamma(A)$ 为二次量子化算子，而 $\Gamma(A)$ 没有特别的名称.

由定义及 (2.23),(2.24) 式容易看出,

$$\Gamma(A^*) = \Gamma(A)^*, \qquad d\Gamma(A^*) = d\Gamma(A)^*. \tag{2.28}$$

特别,若 A 为 H 上的自共轭算子,将 H 与 H^* 视为等同,则 $\Gamma(A)$ 及 $d\Gamma(A)$ 均为 $\mathcal{F}(H)$ 上的自共轭算子,它们与对称 Fock 空间投影可交换,从而限制于 $\Gamma(H)$ 上仍为自共轭算子. 特别 $\Gamma(I) = I, d\Gamma(I) = N$ 为 **计数算子(number operator)**, 即

$$d\Gamma(I)\Big|_{H^{\otimes n}} = n\cdot\ . \tag{2.29}$$

其中 $n\cdot$ 表示乘以数目 n.

容易证明如下事实 (详细讨论参看 Cook[1], Reed-Simon[1] 和 Simon[1]).

命题2.13 1° 若 A 为 H 上的压缩算子, 则 $\Gamma(A)$ 为 $\Gamma(H)$ 上的压缩算子;

2° 若 A 生成 H 上的强连续半群, 则 $d\Gamma(A)$ 生成 $\Gamma(H)$ 上的强连续压缩半群, 且

$$\exp\{-td\Gamma(A)\} = \Gamma(\exp\{-tA\}), \quad t \geq 0; \tag{2.30}$$

3° 若 A 为 H 上的自共轭算子, 生成酉算子群 $\exp\{itA\}$, $t \in \mathbb{R}$, 则 $d\Gamma(A)$ 为 $\Gamma(H)$ 上的自共轭算子, 且生成酉算子群:

$$\exp\{itd\Gamma(A)\} = \Gamma(\exp\{itA\}), \quad t \in \mathbb{R}. \tag{2.31}$$

在 Fock 空间的分析中, **指数向量**

$$\mathcal{E}(h) \equiv 1 \oplus h \oplus \frac{h^{\otimes 2}}{2!} \oplus \cdots \oplus \frac{h^{\otimes n}}{n!} \oplus \cdots \tag{2.32}$$

起着重要的作用. 显然, 对 $h, g \in H$ 有

$$(\mathcal{E}(h), \mathcal{E}(g))_{\Gamma(H)} = \exp\{(h, g)_H\}. \tag{2.33}$$

若 $h \in \mathcal{D}(A)$, 则

$$\Gamma(A)\mathcal{E}(h) = \mathcal{E}(Ah). \tag{2.34}$$

此外，我们有

命题 2.14 指数向量族 $\{\mathcal{E}(h), h \in H\}$ 为线性独立集，所生成的线性子空间在 $\Gamma(H)$ 中稠密.

证明 设 $h_1, \cdots, h_n \in H$ 且互不相同，易见 $\exists g \in H$, 使 $\{(h_j, g)_H, 1 \leq j \leq n\}$ 互不相同. 若 $\exists a_1, \cdots, a_n \in \mathbb{K}$, 使 $a_1\mathcal{E}(h_1) + \cdots + a_n\mathcal{E}(h_n) = 0$, 则 $\forall \lambda \in \mathbb{K}$

$$0 = \left(\sum_{j=1}^{n} a_j \mathcal{E}(h_j), \mathcal{E}(\overline{\lambda} g)\right)_{\Gamma(H)} = \sum_{j=1}^{n} a_j e^{\lambda (h_j, g)_H},$$

从而 $a_1 = a_2 = \cdots = a_n = 0$,$\{\mathcal{E}(h_j), 1 \leq j \leq n\}$ 的线性独立性得证.

设 $\{\mathcal{E}(h), h \in H\}$ 之线性闭包为 S, 则 $\mathcal{E}(0) \in S$. 我们以 $h^{\otimes k}$ 表示向量 $0 \oplus \cdots \oplus h^{\otimes k} \oplus 0 \oplus \cdots$, 假定 $\{h^{\otimes k}, k = 0, 1, \cdots, n-1, h \in H\} \subset S$, 则由

$$h^{\otimes n} = n! \lim_{t \to 0} t^{-n} \left\{ \mathcal{E}(th) - \bigoplus_{j=0}^{n-1} (j!)^{-1} t^j h^{\otimes j} \right\}$$

可知 $h^{\otimes n} \in S$. 由极化公式 (2.12) 可知任一 $H^{\widehat{\otimes} n}$ 均含于 S 中，于是 $S = \Gamma(H)$. ∎

§3. 赋可列范空间与核空间

在局部凸空间中，应用中十分重要的是赋范空间，有限维空间中的许多结果可以直接推广到无穷维赋范空间中去. 然而它在应用中也有局限性，特别是在广义函数论中，需要研究由一族半范生成的拓扑线性空间，其中最重要的一类空间就是所谓核空间.

核空间的理论是 1955 年由 A. Grothendieck[1] 建立的，其名称来源于 L. Schwartz 的核定理. 设 $H = L^2(\mathbb{R})$, 则 $H \otimes H \cong L^2(\mathbb{R}^2)$.

熟知由平方可积核 $K \in L^2(\mathbb{R}^2)$ 可以确定 $L^2(\mathbb{R})$ 中的一个有界线性算子

$$\widetilde{K}f(x) \equiv \int_{\mathbb{R}} K(x,y)f(y)dy, \tag{3.1}$$

即 $\widetilde{K} \in \mathcal{L}(H)$. 然而，并不是每个有界线性算子都具有平方可积核，例如恒等算子 I 就没有如 (3.1) 式的表示，除非其中 $K(x,y)$ 代以 $\delta(x-y)$, 但我们知道，$\delta(x-y)$ 是广义函数，不属于 $L^2(\mathbb{R}^2)$.

引进广义函数后，Schwartz 证明了：$\forall \widetilde{K} \in \mathcal{L}(\mathcal{S}(\mathbb{R}), \mathcal{S}^*(\mathbb{R}))$, 必有核 $K \in \mathcal{S}^*(\mathbb{R}^2)$ 使 (3.1) 式成立，即

$$\mathcal{L}\big(\mathcal{S}(\mathbb{R}), \mathcal{S}^*(\mathbb{R})\big) \cong \mathcal{S}^*(\mathbb{R}^2) \cong \mathcal{S}^*(\mathbb{R}) \otimes \mathcal{S}^*(\mathbb{R}),$$

其中 $\mathcal{S}(\mathbb{R})$ 为 $\mathbb{R}$ 上的速降 C^∞ 函数空间，$\mathcal{S}^*(\mathbb{R})$ 为其对偶空间，即缓增广义函数空间，它们都是核空间. 关于一般局部凸空间的拓扑张量积，我们将在本节中介绍.

核空间还具有许多优良性质. 例如在完备核空间中，有界闭集都是紧集 (这在无穷维赋范空间中是不可能的), 序列的强收敛和弱收敛等价. 可以说，无论从理论或应用角度来说，核空间的重要性都不亚于赋范空间.

有关局部凸空间及其对偶空间的概念及基本性质可参考本书的附录 B.

3.1 赋可列范空间及其对偶空间

设 X 为一局部凸拓扑线性空间，其拓扑由一族半范 Γ 生成. 对 $p \in \Gamma$, $p^{-1}(0)$ 为 X 的线性子空间，若 $Q_p : X \longrightarrow X/p^{-1}(0)$ 为商映射，则由 $\widehat{p}(Q_p x) \equiv p(x)$ 在商空间 $X/p^{-1}(0)$ 上定义了一个范数. 关于此范数完备化所得的 Banach 空间记为 X_p. 若 $p, q \in \Gamma$, 且 $p \prec q$, 则 $q^{-1}(0) \subset p^{-1}(0)$, 从而映射 $I_{pq} \equiv Q_p Q_q^{-1}$ 可延拓为 X_q 到 X_p 的连续线性映射. 易见，X 中的拓扑是使所有 $\{Q_p, p \in \Gamma\}$ 为连续的最弱局部凸拓扑，从而是关于 $\{X_p, Q_p; p \in \Gamma\}$ 的投影拓扑 (参看附录 B).

注意，上述 I_{pq} 一般来说未必是单射，故我们不能将 X_q 看作 X_p 的子集. Gel'fand - Silov[1] 引进了一类重要的局部凸空间，其拓扑由可列个范数生成，但需要满足如下相容性条件：

定义 3.1 设 p, q 是线性空间 X 上的两个范数，如果依两个范数都是基本的序列依其中一个范数收敛于 0 时依另一个范数也收敛于 0, 则称 p 与 q**相容**.

注意对范数 p, q 来说，$p^{-1}(0) = q^{-1}(0) = \{0\}$. 此时若 $p \prec q$, 则关于 q 的基本序列必关于 p 也是基本序列，既然它们关于 p 和 q 收敛于同一极限，因此 $I_{pq}: X_q \longrightarrow X_p$ 为单射. 此时可以将 X_q 看成 X_p 的线性子空间，且 I_{pq} 将 X_q 连续、稠密地嵌入 X_p.

若 X 的拓扑由可列个范数 $\{p_n\}_{n\in\mathbb{N}}$ 生成，不失一般性，总可假定它们是单调不减的，即

$$p_1 \prec p_2 \prec \cdots \prec p_n \prec \cdots .$$

假若不然，可以将每一 p_n 换成 $p'_n \equiv \max\{p_1, \cdots, p_n\}$, 则 $\{p'_n\}_{n\in\mathbb{N}}$ 是单调不减的，与原来可列个范数 $\{p_n\}_{n\in\mathbb{N}}$ 等价.

定义 3.2 设 X 为一局部凸空间，其拓扑由一列相容的范数 $\{\|\cdot\|_n, n \in \mathbb{N}\}$ 生成，则称它为 **赋可列范空间**.

不妨设它们单调不减：

$$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \leq \|\cdot\|_n \leq \cdots, \tag{3.2}$$

并以 X_n 表示 X 关于范数 $\|\cdot\|_n$ 完备化所得的 Banach 空间，则

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots, \tag{3.3}$$

其中 X_{n+1} 连续、稠密地嵌入 X_n. 显然，完备的赋可列范空间是 Fréchet 空间. X 作为完备局部凸空间的投影极限是完备的，且

$$X = \bigcap_n X_n. \tag{3.4}$$

引进距离

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}, \quad x, y \in X, \tag{3.5}$$

则 X 为一完备距离空间．此距离所确定的拓扑与 X 中的拓扑等价，序列在 X 中收敛之充要条件为在每个 X_n 中收敛．但要注意，X 中的有界集并不是按距离定义的有界集．

$B \subset X$ 为有界集的充要条件是

$$\sup_{x \in B} \|x\|_n < \infty, \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

也就是说，B 是每一 Banach 空间 X_n 中的有界集．然而，除非 X 可赋范(即其拓扑可以由一个范数生成)，每一 Banach 空间 X_n 中的单位球都不可能是 X 中的有界集(否则 0 点有有界邻域从而可赋范)．因此，不可赋范的赋可列范空间中，一切有界集都是无处稠密的，且整个空间不能表示成有界子集的可列并集(因为完备距离空间是第二纲集)．由此可见，它和赋范空间中的有界集有多么大的不同之处！

注 作为 Fréchet 空间，X 是桶式的，即原有拓扑与强拓扑一致，又由 Mackey 定理，一切与对偶 $\langle X, X^* \rangle$ 相容的拓扑有相同的有界集族，从而弱有界集也是强有界集．所以我们在这里不加区分，一律称之为 **有界集**．

现在考虑赋可列范空间 X 的对偶空间 X^*，即 X 上连续线性泛函全体所构成的线性空间，并考察其中拓扑．

我们知道，每一个 Banach 空间 X_n 之对偶空间 X_n^* 仍为 Banach 空间，其范数由下式定义：

$$\|f\|_{-n} \equiv \sup_{\|x\|_n \le 1} |\langle f, x \rangle|, \quad n \in \mathbb{N}. \tag{3.7}$$

若 f 是 X 上的连续线性泛函，则它必在某一邻域 $\{x : \|x\|_n < \epsilon\}$ 上有界，因而是 X_n 中的连续线性泛函，故

$$X^* = \bigcup_n X_n^*. \tag{3.8}$$

由 (3.7) 式可知

$$\|\cdot\|_{-1} \ge \|\cdot\|_{-2} \ge \cdots \ge \|\cdot\|_{-n} \cdots. \tag{3.9}$$

于是

$$X_1^* \subset X_2^* \subset \cdots \subset X_n^* \subset \cdots, \tag{3.10}$$

且 X^* 是 $\{X_n^*\}$ 的归纳极限.

在 X^* 中可以有各种不同的拓扑，其中最重要的是强拓扑和弱 * 拓扑，即在有界集上一致收敛拓扑和点点收敛拓扑，它们分别由半范族

$$\|f\|_B \equiv \sup_{x\in B} |\langle f,x\rangle| \tag{3.11}$$

及

$$\|f\|_x \equiv |\langle f,x\rangle|, \quad x \in X \tag{3.12}$$

生成，其中 B 跑遍 X 中有界集. 一般说来，X^* 已不再是赋可列范空间，甚至不再有可数邻域基，从而其中收敛性不能只考虑序列的收敛. 当然，序列收敛性在 X^* 中仍然是相当重要的.

我们先给出 X 中有界集的另一种刻画.

命题 3.3 赋可列范空间 X 中子集 B 为有界的充要条件是：X 上每一连续线性泛函都在其上有界.

证明 必要性很明显，往证充分性. 先设 X 为赋范空间，考虑 B 的极：

$$B^\circ \equiv \{f \in X^* : \sup_{x\in B} |\langle f,x\rangle| \le 1\},$$

它是 X^* 中闭绝对凸集. 由假定，每个 $f \in X^*$ 在 B 上有界，例如，$\sup_{x\in B}|\langle f,x\rangle| \le c$, 则 $c^{-1}f \in B^\circ$, 从而 B° 为吸收集，即 B° 为桶. 由于 Banach 空间为桶式空间，B° 必含 X^* 零点某一邻域，例如 $\{f : \|f\| \le \epsilon\}$, 也就是说

$$\|f\| \le \epsilon \Longrightarrow \sup_{x\in B} |\langle f,x\rangle| \le 1.$$

从而

$$\sup_{x\in B}\|x\| = \sup_{x\in B}\sup_{\|f\|\le\epsilon} |\langle f,x\rangle| /\epsilon \le 1/\epsilon,$$

即 B 为 X 中有界集.

若 X 为赋可列范空间，X 中的子集 B 也是每一赋范空间 X_n 的子集，由假定每一 $f \in X_n^*$ 在 B 上有界，从而 B 为 X_n 中有界集，由 n 的任意性得知 B 为 X 中有界集. ∎

在赋可列范空间 X 的对偶空间 X^* 中，弱*有界集和强有界集也是相同的.

命题 3.4 设 X 为赋可列范空间，X^* 为其对偶空间，则 X^* 中弱*有界集也是强有界集.

证明 设 F 为 X^* 中弱*有界集，它的极

$$F^\circ \equiv \{x \in X : \sup_{f \in F} | \langle f, x \rangle | \leq 1\}$$

是 X 中的桶，从而包含 X 的零点某邻域 U, 即

$$\sup_{f \in F} \sup_{x \in U} | \langle f, x \rangle | \leq 1.$$

既然对 X 中任一有界集 B, 必有 $\lambda > 0$ 使 $B \subset \lambda U$, 故

$$\sup_{f \in F} \|f\|_B = \sup_{f \in F} \sup_{x \in B} | \langle f, x \rangle | < \infty,$$

亦即 F 为 X^* 中强有界集. ∎

因此，在 X^* 中我们将不区分强有界集和弱*有界集，而一概称之为 **有界集**. 下面给出 X^* 中有界集的另一种刻画.

命题 3.5 在赋可列范空间 X 的对偶空间 X^* 中，子集 F 为有界集的充要条件是：$\exists n \in \mathbb{N}$ 使 $F \subset X_n^*$ 且 F 在 X_n^* 中有界.

证明 设 F 在 X^* 中有界，由命题 3.4 之证明可知，存在 X 零点邻域 U, 例如 $\{x : \|x\|_n < \epsilon\}$, 使半范 $p_F(x) \equiv \sup_{f \in F} |\langle f, x \rangle|$ 在其上有界，即 $\exists n \in \mathbb{N}$ 及 $C > 0$ 使

$$\|x\|_n < \epsilon \Longrightarrow \sup_{f \in F} | \langle f, x \rangle | < C,$$

亦即 F 为 X_n^* 中有界集.

反之，设 $\exists n\in\mathbb{N}$ 及 $C>0$ 使 $F\subset X_n^*$, 且

$$\sup_{f\in F}\|f\|_{-n}=\sup_{f\in F}\sup_{\|x\|_n\le 1}|\langle f,x\rangle|\le C,$$

则 F 在 X 零点某邻域 $U=\{x:\|x\|_n\le 1\}$ 上有界，于是 F 为 X^* 中有界集. ∎

由于收敛**序列**必有界，由命题 3.5 可得

定理 3.6 在赋可列范空间 X 的对偶空间 X^* 中，序列 $\{f_n\}$ 弱 * 收敛于 f 的充要条件是：$\exists m\in\mathbb{N}$, 使 $\{f_n\}\subset X_m^*$ 且 $\forall x\in X_m$

$$\lim_{n\to\infty}\langle f_n,x\rangle=\langle f,x\rangle\,. \tag{3.13}$$

证明 充分性显然，往证必要性. 若 $\{f_n\}$ 弱 * 收敛于 f, 则必有界，从而 $\exists m\in\mathbb{N}$, 使 $\{f_n\}\subset X_m^*$ 且在 X_m^* 中有界. 由弱 * 收敛定义，(3.13) 式对一切 $x\in X$ 成立. 由于 X 在 X_m 中稠密且 X_m 为赋范空间，因此可开拓到 X_m 上使 (3.13) 式成立. ∎

由此容易看出，**$\mathbf{X}^*$ 关于弱 * 拓扑是序列完备的**, 即若序列 $\{f_n\}\subset X^*$ 且 $\forall x\in X,\{\langle f_n,x\rangle:n\in\mathbb{N}\}$ 为基本数列，则存在 $f\in X^*$ 使 f_n 弱 * 收敛于 f. 此外，还可以看出，每个子空间 **$\mathbf{X}_n^*$ 在 $\mathbf{X}^*$ 中弱 * 稠密**.

3.2 核空间及其对偶空间

我们先给出一般核空间的定义，然后着重讨论应用中十分重要的可列 Hilbert 核空间及其对偶空间.

定义 3.7 设局部凸空间 X 之拓扑由一族 H 半范 Γ 生成，若 $\forall p\in\Gamma,\exists q\in\Gamma$ 使 $p\prec q$, 且映射

$$I_{pq}:\quad X_q\longrightarrow X_p \tag{3.14}$$

为核算子 (或称迹算子，即 $I_{pq}\in\mathcal{L}_{(1)}(X_q,X_p)$), 则称 X 为**核空间**.

由于两个 Hilbert-Schmidt 算子之积为核算子，上述定义也可叙述为："$\forall p\in\Gamma,\exists q\in\Gamma$ 使 $p\prec_{\rm HS} q$", 其中 $p\prec_{\rm HS} q$ 表示 pHS 囿于 q, 即 $I_{pq}\in\mathcal{L}_{(2)}(X_q,X_p)$(参看附录 B).

若 Γ 可数，则 X 为一列 Hilbert 空间的投影极限，从而可距离化，其拓扑由可列 H 半范族生成的完备核空间称为 **Fréchet 核空间**. 特别，我们有

定义 3.8 完备的赋可列 H 范空间称为 **可列 Hilbert 空间**.(不妨设其 H 范序列满足 (3.2) 式，从而 (3.3) 为一列 Hilbert 空间.) 若 $\forall n \in \mathbb{N}$, $\exists m > n$ 使嵌入映射 $I_{nm} : X_m \hookrightarrow X_n$ 为核算子 (或者 Hilbert- Schmidt 算子), 则 X 称为 **可列 Hilbert 核空间**.

显然，可列 Hilbert 核空间是 Fréchet 核空间.

命题 3.9 可列 Hilbert 空间是自反空间.

证明 局部凸空间为自反空间的充要条件是: 它是桶式的, 且有界集相对弱紧 (参看附录 B). 可列 Hilbert 空间是 Fréchet 空间, 从而是桶式空间. 设 B 为可列 Hilbert 空间 X 中任一有界集，则它在每一 Hilbert 空间 X_n 中为有界集，从而相对 $\sigma(X_n, X_n^*)$- 紧. 但 X 可视为乘积空间 $\prod_n X_n$ 之子集，且 X 中弱拓扑为诸 X_n 中弱拓扑之乘积拓扑，由 Tychonoff 定理可知 B 在 X 中相对 $\sigma(X, X^*)$-紧. ∎

定理 3.10 完备核空间中有界集是相对紧集，从而有界闭集为紧集.

证明 设 B 为完备核空间 X 中的有界集. 则对每一连续 H 半范 p, 有连续 H 半范 q, 使 $p \prec q$, 且 $I_{pq} : X_q \longrightarrow X_p$ 为核算子 (从而更是紧算子). 既然 B 在每一 X_q 中有界，故 I_{pq} 将 B 映射为 X_p 中相对紧集. 由于 p 是任意的，将 B 看成乘积空间 $\prod_{p\in\Gamma} X_p$ 中的子集，由 Tychonoff 定理可知 B 在 X 中相对紧. ∎

系 3.11 Banach 空间如同时为核空间，则必为有限维空间.

证明 Banach 空间每点具有有界邻域，由空间核性，每点具有紧邻域，从而为局部紧空间. 但 Banach 空间当且仅当有限维时为局部紧. ∎

定理 3.12 在 Fréchet 核空间 X 及其对偶空间 X^* 中，**序列的强收敛和弱收敛是等价的**.

证明 只要证明弱收敛序列必强收敛. 设 $\{x_n\}$ 在 X 中弱收敛

于 0, 则 $\{x_n\}$ 有界, 因 X 为完备核空间, 故 $\{x_n\}$ 相对紧, 但由于其弱极限为 0, 其任一强收敛子序列只能以 0 为极限点, 故 $\{x_n\}$ 强收敛于 0.

设 $\{f_n\}$ 在 X^* 中弱* 收敛于 0, 即 $\forall x \in X$, $\lim_{n\to\infty}\langle f_n, x\rangle = 0$, 要证明在 X 的每一有界集上一致收敛于 0. 如若不然, 存在有界集 B, $\epsilon > 0$ 及 B 中序列 $\{x_n\}$, 使 $|\langle f_n, x_n\rangle| > \epsilon$ 对所有 n 成立. 因 B 相对紧, 不妨设 $x_n \to x_0 \in X$, 于是序列 $x'_n \equiv x_n - x_0 \to 0$. 由刚才所证, 它强收敛于 0, 即在 X^* 的任一有界集上一致收敛于 0. 特别取有界集 $\{f_n\}$, 得 $\lim_{n\to\infty}\langle f_n, x'_n\rangle = 0$. 由此推出

$$\lim_{n\to\infty}\langle f_n, x_n\rangle = \lim_{n\to\infty}\langle f_n, x'_n\rangle + \lim_{n\to\infty}\langle f_n, x_0\rangle = 0.$$

这和假定 $|\langle f_n, x_n\rangle| > \epsilon$ 矛盾, 从而定理获证. ∎

系 3.13 在可列 Hilbert 核空间的对偶空间中, 有界集关于强、弱* 拓扑都是相对序列紧的.

证明 有界集相对弱* 紧是空间自反性的推论. 由于序列的强收敛和弱* 收敛等价, 故有界集关于强拓扑相对序列紧. ∎

可以证明 (例如参看 Schaefer[1]), Fréchet 核空间的强对偶空间是完备核空间; 完备核空间 X 的闭子空间 M 以及商空间 X/M 是核空间; 任意多个核空间的直积或投影极限仍是核空间; 可列个核空间的直和或归纳极限仍是核空间.

若某可列 Hilbert 核空间 X 连续、稠密地嵌入某可分 Hilbert 空间 H, 并将 H 与其对偶 H^* 视为同一, 则 H 连续、稠密地嵌入 X 的对偶 X^*. 我们称 $X \hookrightarrow H \hookrightarrow X^*$ 为 **Gel'fand 三元组**.

例 以 $|\cdot|_0$ 表示 Hilbert 空间 $L^2(\mathbb{R})$ 中的 H 范. A 表示算子 $-\frac{d^2}{dt^2} + t^2 + 1$ 的自共轭延拓, 它是 $L^2(\mathbb{R})$ 中的正算子, 以 Hermite 函数

$$e_j(t) \equiv \big((j-1)!\big)^{-1/2}\pi^{-1/4}e^{-t^2/2}H_{j-1}(\sqrt{2}t), \qquad j = 1, 2, \cdots$$

为特征函数, 且

$$Ae_j = 2je_j, \qquad j = 1, 2, \cdots. \tag{3.15}$$

$\{e_j\}_{j\in\mathbb{N}}$ 构成 $L^2(\mathbb{R})$ 之标准正交基.

定义一列 H 范如下:

$$|\varphi|_k \equiv |A^k\varphi|_0 \quad \varphi\in\mathcal{D}(A^k), k=0,1,2,\cdots. \tag{3.16}$$

显然它们相容且单调非减. 令

$$\mathcal{S}_k(\mathbb{R}) \equiv \mathcal{D}(A^k), \qquad k=0,1,2,\cdots, \tag{3.17}$$

则 $\mathcal{S}_k(\mathbb{R})$ 关于范数 $|\cdot|_k$ 为 Hilbert 空间, 且

$$L^2(\mathbb{R}) \equiv \mathcal{S}_0(\mathbb{R}) \supset \mathcal{S}_1(\mathbb{R}) \supset \mathcal{S}_2(\mathbb{R}) \supset \cdots,$$

其投影极限

$$\mathcal{S}(\mathbb{R}) = \varprojlim_k \mathcal{S}_k(\mathbb{R}) \tag{3.18}$$

为可列 Hilbert 空间. 从 $\mathcal{S}_{k+1}(\mathbb{R})$ 到 $\mathcal{S}_k(\mathbb{R})$ 的嵌入算子的 HS 范数为

$$\|A^{-1}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty}(2j)^{-2} = \frac{\pi^2}{24} < \infty. \tag{3.19}$$

故嵌入算子为 Hilbert-Schmidt 算子, 从而 $\mathcal{S}(\mathbb{R})$ 为可列 Hilbert 核空间. 可以证明 (例如见 Simon[1]), 它就是 Schwartz 速降 C^∞ 函数空间.

若在 $L^2(\mathbb{R})$ 上定义一列 H 范:

$$|\varphi|_{-k} \equiv |A^{-k}\varphi|_0, \quad k=0,1,2,\cdots, \tag{3.20}$$

令 $\mathcal{S}_{-k}(\mathbb{R})$ 为 $L^2(\mathbb{R})$ 关于 H 范 $|\cdot|_{-k}$ 完备化所得之 Hilbert 空间, 并将 $L^2(\mathbb{R})$ 和其对偶视为同一空间, 则

$$\mathcal{S}_{-k}(\mathbb{R}) = \mathcal{S}_k(\mathbb{R})^*, \tag{3.21}$$

且

$$L^2(\mathbb{R}) \equiv \mathcal{S}_0(\mathbb{R}) \subset \mathcal{S}_{-1}(\mathbb{R}) \subset \mathcal{S}_{-2}(\mathbb{R}) \subset \cdots,$$

其归纳极限

$$S^*(\mathbb{R}) = \varinjlim_k S_{-k}(\mathbb{R}) \tag{3.22}$$

为 $S(\mathbb{R})$ 之对偶空间，即 Schwartz 缓增广义函数空间．于是我们得到一个 Gel'fand 三元组：

$$S(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow S^*(\mathbb{R}). \tag{3.23}$$

类似地，在 $L^2(\mathbb{R}^d)$ 中考虑算子 $-\Delta+|x|^2+1$ 的自共轭延拓 A，令

$$e_{k_1,\cdots,k_d} \equiv \prod_{j=1}^d e_{k_j}(x_j),\ (k_1,\cdots,k_d)\in \mathbb{N}^d.$$

则

$$A e_{k_1,\cdots,k_d} = \Big(1-d+2\sum_{j=1}^d k_j\Big) e_{k_1,\cdots,k_d}. \tag{3.24}$$

当 $k > d/2$ 时

$$\|A^{-k}\|_{\mathrm{HS}}^2 = \sum_{(k_1,\cdots,k_d)\in\mathbb{N}^d} \Big(1+d+2\sum_{j=1}^d (k_j-1)\Big)^{-2k}. \tag{3.25}$$

逐次利用不等式

$$\sum_{j=0}^\infty (a+j)^{-b} \le \mathrm{const}\, a^{-(b-1)} \quad (a>1, b>1), \tag{3.26}$$

可证 A^{-k} 为 Hilbert-Schmidt 算子，从而 $S(\mathbb{R}^d)$ 为可列 Hilbert 核空间，由此仍得到一个 Gel'fand 三元组：

$$S(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow S^*(\mathbb{R}^d). \tag{3.27}$$

3.3 拓扑张量积、Schwartz 核定理

在 §2 中我们定义了 Hilbert 空间张量积. 对于一般的局部凸空间，我们有以下定义：

定义 3.14 设 X 及 Y 为局部凸空间，X_σ^* 及 Y_σ^* 分别为其弱对偶空间. 对 $x\in X$ 及 $y\in Y$ 定义其张量积 $x\otimes y$ 为 $X_\sigma^*\times Y_\sigma^*$ 上的连续双线性型：

$$x\otimes y(f,g)\equiv\langle f,x\rangle\langle g,y\rangle,\quad f\in X^*,g\in Y^*,\tag{3.28}$$

并将其线性开拓到由 $\{x\otimes y:\ x\in X,y\in Y\}$ 生成的线性空间 $\mathcal{E}$ 上，在 $\mathcal{E}$ 中赋以使映射

$$\chi:X\times Y\ni(x,y)\longmapsto x\otimes y\in\mathcal{E}\tag{3.29}$$

为连续之最强局部凸拓扑，并关于其完备化. 此完备局部凸空间称为 X 与 Y 的 **投影张量积空间**, 记为 $X\tilde{\otimes}Y$.

注 设 $\mathcal{U}$ 及 $\mathcal{V}$ 分别为 X 及 Y 的零点邻域基, 对 $U\in\mathcal{U},V\in\mathcal{V}$, 令 $U\otimes V\equiv\chi(U\times V)=\{(x\otimes y):x\in U,y\in V\}$, $\Gamma(U\otimes V)$ 为其绝对凸壳, 即含 $U\otimes V$ 之最小绝对凸集. 则 $\{\Gamma(U\otimes V):U\in\mathcal{U},V\in\mathcal{V}\}$ 构成投影张量积空间 $X\tilde{\otimes}Y$ 的零点邻域基. 或者等价地，若半范族 $\mathcal{P}$ 及 $\mathcal{Q}$ 分别生成 X 及 Y 之拓扑, 对 $p\in\mathcal{P},q\in\mathcal{Q}$, 在 $\mathcal{E}$ 上定义半范：

$$p\otimes q(z)\equiv\inf\{\textstyle\sum_j p(x_j)q(y_j):z=\sum_j(x_j\otimes y_j)\},\tag{3.30}$$

则半范族 $\{p\otimes q:\ p\in\mathcal{P},q\in\mathcal{Q}\}$ 生成 $X\tilde{\otimes}Y$ 之拓扑. 特别, $p\otimes q(x\otimes y)=p(x)q(y)$.

当 X 及 Y 为赋范空间时，$X\tilde{\otimes}Y$ 也是赋范空间, 且 $\|x\otimes y\|=\|x\|\,\|y\|$. 特别，当 X 及 Y 为 Hilbert 空间时，可以证明投影张量积 $X\tilde{\otimes}Y\cong\mathcal{L}_{(1)}(X^*,Y)$, 而 Hilbert 张量积 $X\otimes Y\cong\mathcal{L}_{(2)}(X^*,Y)$. 此外，当 X 及 Y 为核空间时，$X\tilde{\otimes}Y$ 亦为核空间 (例如，参看 Treves[1]).

引进张量积的好处是，可以将乘积空间上的双线性泛函等同于张量积空间上的线性泛函. 对任意局部凸空间 X 及 Y, 我们以 $\mathcal{B}(X,Y)$ 表示 $X\times Y$ 上连续双线性泛函所构成的线性空间，则有

命题 3.15 $\mathcal{B}(X,Y)$ 代数同构于 $(X\widetilde{\otimes}Y)^*$.

证明 对 $f\in(X\widetilde{\otimes}Y)^*$, 令 $\varphi(x,y)\equiv f(x\otimes y)$. 则映射 $f\longmapsto\varphi$ 为 $(X\widetilde{\otimes}Y)^*\longrightarrow\mathcal{B}(X,Y)$ 之线性单射 (因为若 $f(x\otimes y)=0,\forall x\in X,y\in Y$, 则 $f\equiv 0$); 反之，对 $\varphi\in\mathcal{B}(X,Y), z=\sum_j(x_j\otimes y_j)\in\mathcal{E}$, 令 $f(z)=\sum_j\varphi(x_j,y_j)$. 易见 f 之定义不依赖于 z 的表示，f 为 $\mathcal{E}$ 上线性泛函. 为证其连续性，我们注意到，对任一 $\epsilon>0$, 存在 X 及 Y 之零点邻域 U 及 V, 使 $U\times V\subset\{(x,y):|\varphi(x,y)|<\epsilon\}$, 此即 $U\otimes V\subset\{|f(x\otimes y)|<\epsilon\}$, 但后者为绝对凸集，从而 $\Gamma(U\otimes V)\subset\{|f(x\otimes y)|<\epsilon\}$, 即 f 在 $\mathcal{E}$ 上关于投影张量积拓扑在零点连续，于是可开拓为 $X\widetilde{\otimes}Y$ 上的连续线性泛函，映射 $\varphi\longmapsto f$ 恰好是前述映射之逆. ∎

当 X 及 Y 为赋范空间时，上述代数同构也是等距同构 (即作为 Banach 空间同构), $\mathcal{B}(X,Y)$ 中的范数定义为

$$\|\varphi\|\equiv\sup_{\|x\|\le 1}\sup_{\|y\|\le 1}|\varphi(x,y)|. \tag{3.31}$$

命题 3.16 若 X 及 Y 为赋范空间，则

$$\mathcal{B}(X,Y)\cong(X\widetilde{\otimes}Y)^*. \tag{3.32}$$

证明 考虑命题 3.15 中的线性同构映射: $\varphi=f\circ\chi$. 因

$$|\varphi(x,y)|=|f(x\otimes y)|\le\|f\|\,\|x\otimes y\|=\|f\|\,\|x\|\,\|y\|,$$

故 $\|\varphi\|\le\|f\|$. 反之，设 $z\in\mathcal{E}$ 且 $\|z\|=1$, 由 (3.30) 式知，$\forall\epsilon>0$, 存在 z 的一个表示: $z=\sum_j(x_j\otimes y_j)$, 使 $\sum_j\|x_j\|\|y_j\|<1+\epsilon$. 于是

$$|f(z)|=\Big|\sum_j\varphi(x_j,y_j)\Big|\le\|\varphi\|\sum_j\|x_j\|\,\|y_j\|<\|\varphi\|(1+\epsilon).$$

令 $\epsilon\downarrow 0$. 得证 $\|f\|\leq\|\varphi\|$, 于是此映射为等距映射. ∎

注 当 X 为 Fréchet 核空间, Y 为 Fréchet 空间时, (3.32) 仍成立. 此时 $\mathcal{B}(X,Y)$ 中赋以**双有界收敛拓扑**, 即在 X 的有界集和 Y 的有界集的乘积集合上一致收敛拓扑, 而 $(X\widetilde{\otimes}Y)^*$ 表示强对偶空间. 其证明可参看 Schaefer[1] 或 Treves[1], 证明的关键是对 $X\widetilde{\otimes}Y$ 中任一有界集 B, 构造 X 及 Y 中有界集 B_1 及 B_2, 使 B 含于 $B_1\otimes B_2$ 之闭绝对凸壳 $(\Gamma(B_1\otimes B_2))^-$ 中.

如果在 X 到 Y 的连续线性映射空间 $\mathcal{L}(X,Y)$ 中赋以有界集上一致收敛拓扑, 则我们有以下定理, 它是 Schwartz 核定理的抽象形式.

定理 3.17 设 X 为 Fréchet 核空间, Y 为 Fréchet 空间, 则

$$
\begin{aligned}
X\widetilde{\otimes}Y &\cong \mathcal{L}(X^*,Y), && (3.33)\\
X^*\widetilde{\otimes}Y &\cong \mathcal{L}(X,Y), && (3.34)\\
X^*\widetilde{\otimes}Y^* &\cong \mathcal{L}(X,Y^*)\\
&\cong \mathcal{B}(X,Y)\\
&\cong (X\widetilde{\otimes}Y)^*, && (3.35)
\end{aligned}
$$

其中所有对偶均为强对偶, $\mathcal{B}(X,Y)$ 中为双有界收敛拓扑, $\mathcal{L}(X,Y)$ 中为有界集上一致收敛拓扑.

证明 我们只证明 (3.35) 中后两个同构关系, 且假定 X 为可列 Hilbert 核空间, Y 为可列 Hilbert 空间, 一般情形的证明参看 Treves[1].

由命题 3.16 后面的注, 已知 $\mathcal{B}(X,Y)\cong(X\widetilde{\otimes}Y)^*$. 故对一切 $\varphi\in\mathcal{B}(X,Y)$, 令 $\widetilde{\varphi}$ 表示线性映射

$$
\widetilde{\varphi}:\ X\ni x\longmapsto\varphi(x,\cdot)\in Y^* \tag{3.36}
$$

则 $\widetilde{\varphi}\in\mathcal{L}(X,Y^*)$, 且

$$
\varphi(x,y)=\langle\widetilde{\varphi}x,y\rangle. \tag{3.37}
$$

事实上

$$
\begin{aligned}
\varphi \in \mathcal{B}(X,Y) \Longleftrightarrow & \exists n, m \in \mathbb{N}, C>0\, \forall x \in X, y \in Y \\
& |\varphi(x,y)| \leq C\|x\|_n \|y\|_m \\
\Longleftrightarrow & \exists n, m \in \mathbb{N}, C>0\, \forall x \in X \\
& \|\widetilde{\varphi} x\|_{-m} \leq C\|x\|_n \\
\Longleftrightarrow & \widetilde{\varphi} \in \mathcal{L}(X, Y^*),
\end{aligned}
\tag{3.38}
$$

从而 $\mathcal{B}(X,Y)$ 代数同构于 $\mathcal{L}(X,Y^*)$.

设 $B_1 \subset X, B_2 \subset Y$ 为有界集，则 B_2 之极：

$$
B_2^\circ = \{g \in Y^* : \sup_{y \in B_2} |\langle g, y\rangle| \leq 1\}
$$

为 Y^* 之零点邻域，且当 B_2 跑遍 Y 中一切有界集时，其极构成 Y^* 之零点邻域基. 但

$$
\sup_{x \in B_1} \sup_{y \in B_2} |\varphi(x,y)| \leq 1 \Longleftrightarrow \widetilde{\varphi} B_1 \subset B_2^\circ, \tag{3.39}
$$

从而映射 $\varphi \longmapsto \widetilde{\varphi}$ 为拓扑同胚，即

$$
\mathcal{B}(X,Y) \cong \mathcal{L}(X,Y^*). \qquad \blacksquare
$$

特别，我们有以下推论：

定理 3.18 (Gel'fand-Vilenkin[1]) 设 X 及 Y 为可列 Hilbert 空间且 X 为核空间. 则对一切 $\varphi \in \mathcal{B}(X,Y)$, 存在 $m, n \in \mathbb{N}$ 以及 $A \in \mathcal{L}_{(2)}(X_n, Y_m^*)$ 使

$$
\varphi(x,y) = \langle Ax, y\rangle, \quad \forall x \in X, y \in Y, \tag{3.40}
$$

若取 $\{e_k\}$ 及 $\{f_k\}$ 分别为 X_n 及 Y_m^* 之基，则

$$
\varphi(x,y) = \sum_{k=1}^{\infty} \lambda_k (x, e_k)_n \langle f_k, y\rangle, \quad \forall x \in X, y \in Y, \tag{3.41}
$$

其中 $\lambda_k \geq 0$ 且 $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$.

证明 注意到 (3.38) 式, $\widetilde{\varphi} \in \mathcal{L}(X_n, Y_m^*)$. 但 X 为核空间, 故存在 $n' > n$ 使 $I_{nn'}: X_{n'} \longrightarrow X_n$ 为 Hilbert-Schmidt 算子, 从而 $A \equiv \widetilde{\varphi} \circ I_{nn'} \in \mathcal{L}_{(2)}(X_{n'}, Y_m^*)$. 其余都是显然的推论. ∎

应用到具体空间 $X = \mathcal{S}(\mathbb{R}^m)$ 及 $Y = \mathcal{S}(\mathbb{R}^n)$, 则有

定理 3.19 我们有

$$\begin{aligned} \mathcal{S}^*(\mathbb{R}^m)\widetilde{\otimes}\mathcal{S}^*(\mathbb{R}^n) &\cong \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}^*(\mathbb{R}^n)) \\ &\cong (\mathcal{S}(\mathbb{R}^m)\widetilde{\otimes}\mathcal{S}(\mathbb{R}^n))^* \\ &\cong \mathcal{S}^*(\mathbb{R}^{m+n}). \end{aligned} \tag{3.42}$$

因此, $\forall \widetilde{K} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}^*(\mathbb{R}^n))$, 唯一地对应于一个核 $K \in \mathcal{S}^*(\mathbb{R}^{m+n})$, 使

$$\langle K, \varphi \otimes \psi \rangle = \langle \widetilde{K}\varphi, \psi \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^m), \psi \in \mathcal{S}(\mathbb{R}^n). \tag{3.43}$$

或者形式地写成

$$\widetilde{K}\varphi(x) = \int_{\mathbb{R}^m} K(x,y)\varphi(y)dy, \tag{3.44}$$

这就是 Schwartz 核定理的原来形式.

§4. 拓扑线性空间上的 Borel 测度

4.1 Minlos-Sazanov 定理

设 H 为一实可分 Hilbert 空间, $\mathcal{B}(H)$ 为它的 Borel σ- 代数. 易知 $\mathcal{B}(H)$ 为可分 σ- 代数 (即 $\mathcal{B}(H)$ 是可数生成的). 可测空间 $(H, \mathcal{B}(H))$ 上的测度称为 H 上的 **Borel 测度**. 下面我们只讨论 H 上的有限 Borel 测度.

定义 4.1 设 μ 为 H 上的一有限 Borel 测度. 令

$$\widehat{\mu}(x) = \int_H e^{i(x,y)}\mu(dy), \; x \in H, \tag{4.1}$$

称 $\widehat{\mu}$ 为 μ 的 **Fourier 变换**.

显然，$\widehat{\mu}$ 具有如下几条性质：

(1) $\widehat{\mu}(0)=\mu(H)$;

(2) $\widehat{\mu}$ 在 H 上连续 (甚至关于 H 的弱拓扑连续);

(3) $\widehat{\mu}$ 是 **正定的**, 即对任意自然数 $n\geq 2$ 及 $x_1,\cdots,x_n\in H$ 和复数 $\alpha_1,\cdots,\alpha_n$, 有

$$\sum_{l,k=1}^{n}\widehat{\mu}(x_l-x_k)\alpha_l\bar{\alpha}_k\geq 0. \tag{4.2}$$

事实上，(4.2) 式可由下式推得

$$\sum_{l,k=1}^{n}\widehat{\mu}(x_l-x_k)\alpha_l\bar{\alpha}_k=\int_H\left|\sum_{k=1}^{n}\alpha_k e^{i(x_k,y)}\right|^2\mu(dy).$$

人们自然要问：是否 H 上的任何正定连续泛函都是某一有限 Borel 测度的 Fourier 变换？如果 H 是有穷维欧氏空间，答案是肯定的 (经典的 Bochner 定理说明了这一点). 但对无穷维 Hilbert 空间 H, 答案是否定的. 例如，令 $\varphi(x)=\exp\{-\frac{1}{2}\|x\|^2\}$. 则 φ 是 H 上的正定连续泛函，但 φ 不是 H 上某个有限 Borel 测度的 Fourier 变换. 下面我们将致力于找到有限 Borel 测度的 Fourier 变换的一个刻画. 为此先证明若干引理.

引理 4.2 设 φ 为 H 上一正定泛函. 则

(1) $|\varphi(x)|\leq\varphi(0),\overline{\varphi(x)}=\varphi(-x)$, $\forall x\in H$;

(2)$|\varphi(x)-\varphi(y)|\leq 2\sqrt{\varphi(0)}\sqrt{|\varphi(0)-\varphi(x-y)|}$, $\forall x,y\in H$;

(3) $|\varphi(0)-\varphi(x)|\leq\sqrt{2\varphi(0)(\varphi(0)-\operatorname{Re}\varphi(x))}$, $\forall x\in H$.

证明 设 $x,y\in H$. 令

$$A=\begin{pmatrix}\varphi(0) & \varphi(x)\\ \varphi(-x) & \varphi(0)\end{pmatrix},\quad B=\begin{pmatrix}\varphi(0) & \varphi(x) & \varphi(y)\\ \varphi(-x) & \varphi(0) & \varphi(y-x)\\ \varphi(-y) & \varphi(x-y) & \varphi(0)\end{pmatrix}.$$

由 φ 的正定性推知 A 和 B 为正定矩阵. 特别有 $\bar{A}^\tau = A$, 这里 A^τ 表示 A 的转置. 故有 $\overline{\varphi(x)} = \varphi(-x)$. 此外由 $\det A \geq 0$ 推知 $|\varphi(x)| \leq \varphi(0)$. (1) 得证. 由 (1) 知, 矩阵 B 中的元素 $\varphi(-x)$, $\varphi(-y)$ 及 $\varphi(y-x)$ 可用 $\overline{\varphi(x)}$,$\overline{\varphi(y)}$ 及 $\overline{\varphi(x-y)}$ 替换. 计算 B 的行列式可得

$$\begin{aligned}\det B = &\varphi(0)^3 - \varphi(0)|\varphi(x-y)|^2 \\ &- \varphi(x)[\varphi(0)\overline{\varphi(x)} - \overline{\varphi(x-y)\varphi(y)}] \\ &+ \varphi(y)[\overline{\varphi(x)}\varphi(x-y) - \varphi(0)\overline{\varphi(y)}] \\ = &\varphi(0)^3 - \varphi(0)|\varphi(x-y)|^2 \\ &- \varphi(0)|\varphi(x) - \varphi(y)|^2 \\ &+ 2\mathrm{Re}[\varphi(y)\overline{\varphi(x)}(\varphi(x-y) - \varphi(0))].\end{aligned}$$

因为

$$\varphi(0)^3 - \varphi(0)|\varphi(x-y)|^2 \leq 2\varphi(0)^2|\varphi(0) - \varphi(x-y)|,$$

所以

$$0 \leq \det B \leq 4\varphi(0)^2|\varphi(0) - \varphi(x-y)| - \varphi(0)|\varphi(x) - \varphi(y)|^2,$$

由此推得 (2). (3) 式可由如下不等式推出

$$\begin{aligned}|\varphi(0) - \varphi(x)|^2 &= (\varphi(0) - \varphi(x))(\varphi(0) - \overline{\varphi(x)}) \\ &= \varphi(0)^2 - 2\varphi(0)\mathrm{Re}\,\varphi(x) + |\varphi(x)|^2 \\ &\leq 2\varphi(0)^2 - 2\varphi(0)\mathrm{Re}\,\varphi(x).\end{aligned}$$

引理证毕. ∎

引理 4.3 设 μ 为 H 上的有限 Borel 测度, 则下列断言等价:

(1) $\int_H \|x\|^2\mu(dx) < \infty$;

(2) 存在一正的对称迹算子 S, 使得 $\forall x, y \in H$ 有

$$(Sx, y) = \int_H (x, z)(y, z)\mu(dz). \tag{4.3}$$

如果 (2) 成立，则

$$\operatorname{Tr} S = \int_H \|x\|^2 \mu(dx). \tag{4.4}$$

证明 设 (2) 成立. 令 $\{e_n\}$ 为 H 的一组标准正交基，则有

$$\begin{aligned}\int_H \|x\|^2 \mu(dx) &= \sum_{j=1}^{\infty} \int_H (x, e_j)^2 \mu(dx) \\ &= \sum_{j=1}^{\infty} (Se_j, e_j) = \operatorname{Tr} S.\end{aligned} \tag{4.5}$$

这表明 (1) 成立，并有 (4.4) 式. 反之，设 (1) 成立，则

$$\int_H |(x, z)(y, z)| \mu(dz) \leq \|x\| \, \|y\| \int_H \|z\|^2 \mu(dz).$$

于是存在 H 上一有界线性算子 S, 使得 (4.3) 成立. 显然 S 是正的和对称的. 此外，由 (4.5) 知

$$\operatorname{Tr} S = \int_H \|x\|^2 \mu(dx) < \infty.$$

从而 S 是迹算子. ∎

下一定理是 **Minlos-Sazanov 定理**.

定理 4.4 设 φ 是 H 上的一正定泛函，则下列断言等价：

(1)φ 为 H 上某一有限 Borel 测度 μ 的 Fourier 变换;

(2) $\forall \epsilon > 0$, 存在对称迹算子 S_ϵ, 使得

$$(S_\epsilon x, x) < 1 \Longrightarrow \operatorname{Re}(\varphi(0) - \varphi(x)) < \epsilon; \tag{4.6}$$

(3) 存在 H 上对称迹算子 S, 使得 φ 关于 H 的如下范数 $\|\cdot\|_*$ 连续 (或只在 $x = 0$ 处连续):

$$\|x\|_* = (Sx, x)^{1/2} = \|S^{1/2} x\|. \tag{4.7}$$

证明 (1) $\Longrightarrow$ (2). 设 $\varphi=\widehat{\mu}$. 对一切 $\gamma>0$, 我们有

$$\begin{aligned}\mathrm{Re}(\varphi(0)-\varphi(x)) &= \int_H (1-\cos(x,z))\mu(dz)\\ &\leq \frac{1}{2}\int_{\|z\|\leq\gamma}(x,z)^2\mu(dz)+2\mu(\{z:\ \|z\|>\gamma\}).\end{aligned}$$

令 $\mu_1(A)=\mu(A\cap[\|z\|\leq\gamma])$. 对 μ_1 应用引理 4.3 知，存在一正的对称迹算子 B_γ 使得

$$(B_\gamma z_1,z_2)=\int_{|z|\leq\gamma}(z,z_1)(z,z_2)\mu(dz).$$

对给定 $\epsilon>0$, 先选取 $\gamma>0$ 使得 $\mu([\|z\|>\gamma])<\epsilon/4$, 再令 $S_\epsilon=\epsilon^{-1}B_\gamma$, 则有

$$\mathrm{Re}(\varphi(0)-\varphi(x))<\frac{\epsilon}{2}(S_\epsilon x,x)+\frac{\epsilon}{2}.$$

(2) $\Longrightarrow$ (1). 设 (2) 成立，则 $\mathrm{Re}\,\varphi(x)$ 在 $x=0$ 处连续. 故由引理 4.2 知 φ 在 H 上连续. 现在任意取定 H 上的一组标准正交基 $\{e_n\}$, 并对每个自然数 $n\geq 1$, 令

$$f_{i_1,\cdots,i_n}=\varphi(\omega_1 e_{i_1}+\cdots+\omega_n e_{i_n}),\ \omega_j\in I\!R,\ 1\leq j\leq n, \tag{4.8}$$

则 $f_{i_1,\cdots,i_n}$ 为 $I\!R^n$ 上的一正定函数. 由经典的 Bochner 定理知，$f_{i_1,\cdots,i_n}$ 为 $I\!R^n$ 上一有限 Borel 测度 $\mu_{i_1,\cdots,i_n}$ 的 Fourier 变换. 显然，测度族 $\{\mu_{i_1,\cdots,i_n}\}$ 满足 Kolmogorov 测度扩张定理的相容性条件. 于是存在 $(I\!R^\infty,\mathcal{B}(I\!R^\infty))$ 上唯一的有限测度 ν 使得

$$\mu_{i_1,\cdots,i_n}=\nu\circ(X_{i_1},\cdots,X_{i_n})^{-1}, \tag{4.9}$$

其中 $X_j(\omega)=\omega_j,\omega=(\omega_1,\omega_2,\cdots)\in I\!R^\infty$.

下面我们要证明 $\sum_{k=1}^\infty X_k^2<\infty$, ν-a.e.. 为此，令 P_n 为 $I\!R^n$ 上标准 Gauss 测度. 则

$$\int_{I\!R^n} e^{i(a_1y_1+\cdots+a_ny_n)}P_n(dy)=\exp\Big\{-\frac{1}{2}\sum_{j=1}^n a_j^2\Big\}. \tag{4.10}$$

任给 $\epsilon > 0$, 依据假定，存在正的对称迹算子 S_ϵ 使 (4.6) 成立．于是有

$$\varphi(0) - \operatorname{Re}\varphi(x) \le \epsilon + 2\varphi(0)(S_\epsilon x, x)\ ,\ \forall x \in H. \tag{4.11}$$

由 Fubini 定理得

$$\begin{aligned}
&\varphi(0) - \int_{\mathbb{R}^\infty} \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n} X_{k+j}^2\Big\} d\nu \\
&= \varphi(0) - \int_{\mathbb{R}^\infty} d\nu \int_{\mathbb{R}^n} \exp\Big\{i\sum_{j=1}^{n} y_j X_{k+j}\Big\} P_n(dy) \\
&= \varphi(0) - \int_{\mathbb{R}^n} \varphi\Big(\sum_{j=1}^{n} y_j e_{k+j}\Big) P_n(dy) \\
&= \int_{\mathbb{R}^n} \Big[\varphi(0) - \operatorname{Re}\varphi\Big(\sum_{j=1}^{n} y_j e_{k+j}\Big)\Big] P_n(dy),
\end{aligned}$$

由 (4.11) 式，上式不超过

$$\begin{aligned}
&\epsilon + 2\varphi(0) \int_{\mathbb{R}^n} \Big(S_\epsilon \sum_{j=1}^{n} y_j e_{k+j}, \sum_{j=1}^{n} y_j e_{k+j}\Big) P_n(dy) \\
&= \epsilon + 2\varphi(0) \sum_{j=1}^{n} (S_\epsilon e_{k+j}, e_{k+j}).
\end{aligned}$$

由于 $n \ge 1$ 是任意的，故由上式推知

$$\varphi(0) - \int_{\mathbb{R}^\infty} \exp\Big\{-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2\Big\} d\nu \le \epsilon + 2\varphi(0) \sum_{j=k+1}^{\infty} (S_\epsilon e_j, e_j). \tag{4.12}$$

在 (4.12) 中先令 $k \to \infty$ 再令 $\epsilon \downarrow 0$ 即得 (注意 $\varphi(0) = \nu(\mathbb{R}^\infty)$)

$$\varphi(0) - \lim_{k\to\infty} \int_{\mathbb{R}^\infty} \exp\Big\{-\frac{1}{2}\sum_{j=k+1}^{\infty} X_j^2\Big\} d\nu = 0,$$

这表明 $\sum_{j=1}^{\infty} X_j^2 < \infty, \nu$-a.e..

最后，令 $X(\omega)=\sum_{j=1}^{\infty} X_j(\omega)e_j$, 则 X 在 $\mathbb{R}^{\infty}$ 上 ν-a.e. 有定义，且 X 为 H- 值可测函数．令 $\mu=\nu\circ X^{-1}$, 则 μ 为 H 上的有限 Borel 测度，且由 (4.9) 知

$$\begin{aligned}\widehat{\mu}\Big(\sum_{j=1}^{n}(x,e_j)e_j\Big)&=f_{1,\cdots,n}((x,e_1),\cdots,(x,e_n))\\&=\varphi\Big(\sum_{j=1}^{n}(x,e_j)e_j\Big).\end{aligned}$$

令 $n\to\infty$ 即得 $\widehat{\mu}=\varphi$. (2) $\Longrightarrow$ (1) 证毕.

(2) $\Longleftrightarrow$ (3). 设 (2) 成立．令 $S_{1/k}$ 为与 $\epsilon=1/k$ 相应的正的对称迹算子，选取 $\lambda_k>0$, 使得 $\sum_k\lambda_k\mathrm{Tr}S_{1/k}<\infty$, 令 $S=\sum_k\lambda_kS_{1/k}$. 则 S 为正的对称迹算子．显然有

$$\begin{aligned}(Sx,x)<\lambda_k&\Longrightarrow(S_{1/k}x,x)<1\\&\Longrightarrow \mathrm{Re}(\varphi(0)-\varphi(x))<\frac{1}{k}.\end{aligned}$$

于是 $\mathrm{Re}\,\varphi(x)$ 在 $x=0$ 处关于范数 $\|\cdot\|_*$ 连续，从而由引理 4.2 知 φ 在 H 上关于范数 $\|\cdot\|_*$ 连续．这表明 (2) $\Longrightarrow$ (3). 反之，设 (3) 成立．对给定 $\epsilon>0$, 存在 $\delta>0$, 使得 $|x|_*<\delta\Longrightarrow\mathrm{Re}(\varphi(0)-\varphi(x))<\epsilon$. 令 $S_\epsilon=\delta^{-1}S$, 则 (4.6) 成立．从而 (3) $\Longrightarrow$ (2) 得证. ∎

下面我们将给出 Minlos-Sazanov 定理的一个更加常用形式——Minlos 定理．为此，先引进若干记号和准备一些引理.

设 B 为 H 上一正的对称可逆迹算子．在 H 上引进新的内积 $(\cdot,\cdot)_-$ 及范数 $\|\cdot\|_-$ 如下：

$$(x,y)_-=(Bx,y)\quad,\qquad\|x\|_-=(Bx,x)^{1/2}=\|B^{1/2}x\|.\qquad(4.13)$$

我们用 H_- 表示 H 关于 $\|\cdot\|_-$ 的完备化，则内积 $(\cdot,\cdot)_-$ 可以连续扩张到 H_- 上，且 H_- 关于 $(\cdot,\cdot)_-$ 为一可分 Hilbert 空间．另一方

面，令 H_+ 表示 $B^{-1/2}$ 的定义域，则易知 H_+ 为 $B^{1/2}$ 的值域 (即 $H_+ = B^{1/2}(H)$). 在 H_+ 上引进内积 $(\cdot,\cdot)_+$ 及范数 $\|\cdot\|_+$ 如下：

$$(x,y)_+ = (B^{-1/2}x, B^{-1/2}y)\ ,\quad \|x\|_+ = \|B^{-1/2}x\|\ , x \in H_+. \tag{4.14}$$

则显然有

$$\|Bx\|_+ = \|x\|_-\ ,\quad x \in H, \tag{4.15}$$

$$\|B^{-1}x\|_- = \|x\|_+\ ,\quad x \in B(H), \tag{4.16}$$

$$\|x\| \leq \|B\|^{1/2}\|x\|_+\ ,\quad x \in H_+. \tag{4.17}$$

关于空间 H_- 及 H_+, 我们有如下结果：

引理 4.5 在上述假定及记号下，我们有

(1) H_+ 按内积 $(\cdot,\cdot)_+$ 为一可分 Hilbert 空间；

(2) B 可延拓成为 H_- 到 H_+ 上的保范算子， B^{-1} 可延拓成为 H_+ 到 H_- 上的保范算子；

(3) 作为 H_- 中的线性算子， B 是正的对称迹算子，并且有 $\mathrm{Tr}_-B = \mathrm{Tr}B$. 这里 Tr_-B 表示在 H_- 中计算 B 的迹.

(4) H_+ 与 H_- 互为对偶， $H_+ \times H_-$ 上的典则双线性型 $\langle\cdot,\cdot\rangle$ 为

$$\langle x,y\rangle = (B^{-1}x,y)_-\ ,\quad x \in H_+\ ,\ y \in H_-\ . \tag{4.18}$$

证明 设 $\{x_n\}$ 为 H_+ 中按范数 $\|\cdot\|_+$ 的基本列. 由 (4.17) 知，$\{x_n\}$ 亦为 H 中的基本列，记其极限为 x. 令 $y_n = B^{-1/2}x_n$, 则 $\{y_n\}$ 为 H 中的基本列，其极限为 y. 于是有

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} B^{1/2}y_n = B^{1/2}y.$$

这表明 $x \in H_+$, 且有

$$\|x_n - x\|_+ = \|B^{-1/2}(x_n - x)\| = \|y_n - y\| \to 0.$$

于是， H_+ 按范数 $\|\cdot\|_+$ 是完备的，即 H_+ 按内积 $(\cdot,\cdot)_+$ 为 Hilbert 空间.

(2) 直接由 (4.15) 及 (4.16) 推得.

(3) 作为 H_- 上的线性算子, B 的正性及对称性容易验证. 往证 B 是 H_- 上的迹算子. 设 B 在 H 上的谱分解为

$$Bx = \sum_n \lambda_n (x, e_n) e_n \ , \quad x \in H.$$

由于假定 B 可逆, $\{e_n\}$ 构成 H 的一组基. 令 $f_n = e_n/\sqrt{\lambda_n}$, 则 $(Bf_n, f_m) = (\lambda_n \lambda_m)^{-1/2}(Be_n, e_m) = \delta_{n,m}$. 故 $\{f_n\}$ 为 H_- 的一组基. 我们有

$$\operatorname{Tr}_- B = \sum_{n=1}^{\infty} (Bf_n, f_n)_- = \sum_{n=1}^{\infty} \|Bf_n\|^2 = \sum_{n=1}^{\infty} \lambda_n = \operatorname{Tr} B\,.$$

(4) 由 (2) 知 (4.18) 定义的双线性型 $\langle\cdot,\cdot\rangle$ 有意义, 此外有

$$|\langle x, y\rangle| \le \|B^{-1}x\|_- \|y\|_- = \|x\|_+ \|y\|_-\ .$$

这表明 $\langle\cdot,\cdot\rangle$ 为使 H_+ 和 H_- 相互对偶的典则双线性型. ∎

有了上面的准备以后, 我们可以证明如下的 **Minlos 定理**.

定理 4.6 设 φ 为 H 上一连续正定泛函, B 为 H 上一正的对称可逆迹算子, H_- 如前面所定义. 则存在 H_- 上唯一的有限 Borel 测度 μ, 使得

$$\int_{H_-} e^{i\langle x,z\rangle} \mu(dz) = \varphi(x)\ , \quad \forall x \in H_+\ . \tag{4.19}$$

证明 对 $x \in H_-$, 令 $\psi(x) = \varphi(Bx)$. 则显然 ψ 为 H_- 上的正定泛函. 由引理 4.5 知, B 为 H_- 上的正的对称可逆迹算子. 在 H_- 上定义新范数 $\|\cdot\|_*$ 如下:

$$\|x\|_* = \|B^{1/2}x\|_- = \|Bx\|\ .$$

由 φ 在 H 上连续性推知 ψ 在 H_- 上关于范数 $\|\cdot\|_*$ 的连续性. 故由定理 4.4 知 ψ 为 H_- 上某一有限 Borel 测度 μ 的 Fourier 变换, 即有

$$\int_{H_-} e^{i(y,z)_-}\mu(dz) = \psi(y)\ , \quad \forall y \in H_-\ . \tag{4.20}$$

在 (4.20) 中令 $y = B^{-1}x$, $x \in H_+$, 则由 (4.18) 推得 (4.19). 定理证毕. ∎

设 X 为可列 Hilbert 核空间, X^* 为其拓扑对偶. 由定义 3.8, $\forall n \in \mathbb{N}, \exists m > n$, 使嵌入映射 $I_{nm} : X_m \hookrightarrow X_n$ 为迹算子. 易见, 其对偶映射 $I^*_{nm} : X^*_n \hookrightarrow X^*_m$ 仍为迹算子. 设 φ 为 X 上的连续正定泛函, 则 φ 必在某一 Hilbert 空间 X_n 中连续. 将 X^*_n 与 X_n (通过 Riesz 映射) 等同起来, 并在定理 4.6 中以 X_n 代 H, X^*_m 代 H_-, 即可知 X^*_m 上存在唯一有限 Borel 测度 μ, 以 φ 为其 Fourier 变换. 注意 $X^* = \bigcup_m X^*_m$, 因此我们有

定理 4.7 可列 Hilbert 核空间 X 上任何正定连续泛函必然是其对偶空间 X^* 上某有限 Borel 测度的 Fourier 变换.

特别, 若 $X \hookrightarrow H \hookrightarrow X^*$ 为 Gel′fand 三元组 (见 §3), $\langle\cdot,\cdot\rangle$ 为 $X \times X^*$ 上的典则双线性型. 则对 X 上任一连续正定线性泛函 φ, 存在 X^* 上唯一的有限 Borel 测度 μ, 使得

$$\int_{X^*} e^{i\langle x,z\rangle}\mu(dz) = \varphi(x)\ , \quad x \in X. \tag{4.21}$$

我们称 $\varphi(x)$ 为 μ 的 **特征泛函**.

4.2 Hilbert 空间上的 Gauss 测度

下面我们研究 H 上的一类特殊的 Borel 概率测度 ——Gauss 测度. 首先, 我们对 H 上一般的 Borel 概率测度引进均值向量和协方差算子概念.

定义 4.8 设 μ 为 H 上的一 Borel 概率测度. 如果对一切

$x \in H$, 函数 $z \mapsto (x, z)$ 关于 μ 可积, 且存在 H 的一元素 m, 使得

$$(m, x) = \int_H (x, z)\mu(dz) , \quad x \in H, \tag{4.22}$$

则称 m 为 μ 的 **均值向量**. 如果进一步存在 H 上的一正的对称线性算子 B, 使得

$$(Bx, y) = \int_H (z - m, x)(z - m, y)\mu(dz) , \quad \forall x, y \in H, \tag{4.23}$$

则称 B 为 μ 的 **协方差算子**.

均值向量和协方差算子一般未必存在. 但若 $\int_H \|x\|\mu(dx) < \infty$, 则由 Riesz 表现定理知均值向量 m 存在, 且 $\|m\| \leq \int_H \|x\|\mu(dx)$. 如果进一步有 $\int_H \|x\|^2\mu(dx) < \infty$, 则由前面的引理 4.3 知, 存在一正的对称迹算子 S, 使得

$$(Sx, y) = \int_H (x, z)(y, z)\mu(dz) , \quad \forall x, y \in H. \tag{4.24}$$

令

$$Bx = Sx - (m, x)m . \tag{4.25}$$

容易验证 B 满足 (4.23), 即 B 为 μ 的协方差算子. 这时 B 亦为正的对称迹算子.

定义 4.9 设 μ 为 H 上的一 Borel 概率测度. 如果对每个 $x \in H$, 随机变量 $(x, \cdot)$ 服从 Gauss 分布, 则称 μ 为 **Gauss 测度**.

下面我们将通过 Fourier 变换来刻画 Gauss 测度. 为此, 我们需要一个分析引理.

引理 4.10 设 $\{\alpha_j\}$ 为一列实数, 满足 $\sum_{j=1}^{\infty} \alpha_j^2 = \infty$. 则存在一列实数 $\{\beta_j\}$, 使得 $\alpha_j\beta_j \geq 0$, $\forall j \geq 1$, $\sum_{j=1}^{\infty} \beta_j^2 < \infty$ 且 $\sum_{j=1}^{\infty} \alpha_j\beta_j = \infty$.

证明 令 $n_0 = 0$, 并归纳定义 n_k 如下:

$$n_k \equiv \inf\{l : \sum_{j=n_{k-1}+1}^{l} \alpha_j^2 \geq 1\} , \quad k \geq 1 .$$

显然有 $n_k \uparrow \infty$. 令

$$\beta_j = \frac{\alpha_j}{k+1}\Big(\sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2\Big)^{-1/2}, \ n_k+1 \le j \le n_{k+1}, \quad k=0,1,2,\cdots .$$

则 $\alpha_j\beta_j \ge 0, \forall j \ge 1$, 且有

$$\begin{aligned}\sum_{j=1}^{\infty}\beta_j^2 &= \sum_{k=0}^{\infty}\sum_{j=n_k+1}^{n_{k+1}}\beta_j^2 = \sum_{k=0}^{\infty}\frac{1}{(k+1)^2} < \infty, \\ \sum_{j=1}^{\infty}\alpha_j\beta_j &= \sum_{k=0}^{\infty}\sum_{j=n_k+1}^{n_{k+1}}\alpha_j\beta_j = \sum_{k=0}^{\infty}\frac{1}{k+1}\Big(\sum_{j=n_k+1}^{n_{k+1}}\alpha_j^2\Big)^{1/2} \\ &\ge \sum_{k=0}^{\infty}\frac{1}{k+1} = \infty.\end{aligned}$$

引理证毕. ∎

下一定理给出了 Gauss 测度的一个刻画.

定理 4.11 H 上的 Borel 概率测度 μ 是 Gauss 测度的必要充分条件是其 Fourier 变换 $\widehat{\mu}$ 有如下表达式:

$$\widehat{\mu}(x) = \exp\{i(m,x) - \tfrac{1}{2}(Bx,x)\}, \tag{4.26}$$

其中 $m \in H$, B 为 H 上的一正的对称迹算子. 这时, m 为 μ 的均值向量, B 为 μ 的协方差算子. 此外还有

$$\int_H \|x\|^2\mu(dx) = \operatorname{Tr} B + \|m\|^2. \tag{4.27}$$

证明 必要性. 设 μ 为一 Gauss 测度. 先证 $\int_H \|x\|^2\mu(dx) < \infty$. 依假定, 对每个 x, $(x,\cdot)$ 服从 Gauss 分布, 于是存在实数 m_x 及正数 σ_x, 使得

$$\widehat{\mu}(x) = \int_H e^{i(x,z)}\mu(dz) = \exp\{im_x - \tfrac{1}{2}\sigma_x^2\}. \tag{4.28}$$

令 $\{e_j\}$ 为 H 的标准正交基，则

$$\int_H \|x\|^2\mu(dx) = \sum_{j=1}^{\infty}\int_H (e_j, x)^2\mu(dx) = \sum_{j=1}^{\infty}(\sigma_{e_j}^2 + m_{e_j}^2). \tag{4.29}$$

设 $\{\beta_j\}$ 为一列实数，使得 $\beta_j m_{e_j} \geq 0$, $\sum_{j=1}^{\infty}\beta_j^2 < \infty$. 令

$$\xi(x) = \sum_{j=1}^{\infty}\beta_j(e_j, x) , \tag{4.30}$$

则 ξ 为一 Gauss 随机变量 (因由 Schwarz 不等式，上述级数绝对收敛)，其均值必有限，即 $\sum_{j=1}^{\infty}\beta_j m_{e_j} < \infty$. 于是由引理 4.10 知，必有 $\sum_{j=1}^{\infty} m_{e_j}^2 < \infty$. 因此，为证 $\int_H \|x\|^2\mu(dx) < \infty$, 只需证 $\sum_{j=1}^{\infty}\sigma_{e_j}^2 < \infty$. 由定理 4.4 知，存在正的对称迹算子 S, 使得 $(Sx, x) < 1 \Longrightarrow 1 - \mathrm{Re}\,\widehat{\mu}(x) < \frac{1}{3}$. 于是我们有

$$1 - \exp\{-\frac{1}{2}\sigma_x^2\} \leq 1 - \mathrm{Re}\widehat{\mu}(x) \leq (Sx, x) + \frac{1}{3} , \ \forall x \in H. \tag{4.31}$$

无妨设 S 的零空间为 $\{0\}$. 对 $x \in H$, $x \neq 0$, 令 $y = [3(Sx,x)]^{-1/2}x$, 则 $\sigma_y^2 = [3(Sx,x)]^{-1}\sigma_x^2$, $(Sy, y) = \frac{1}{3}$. 用 y 代替 (4.31) 中的 x, 得到

$$1 - \exp\left\{-\frac{\sigma_x^2}{6(Sx,x)}\right\} \leq \frac{2}{3},$$

即有 $\sigma_x^2 \leq (6\log 3)(Sx, x)$, $\forall x \in H$. 由此推知

$$\sum_{j=1}^{\infty}\sigma_{e_j}^2 \leq (6\log 3)\mathrm{Tr}\, S < \infty .$$

因此，最终证明了 $\int_H \|x\|^2\mu(dx) < \infty$. 由定义 4.8 下面的说明知，$\mu$ 的均值向量 m 及协方差算子 B 存在. 采用前面的记号，我们有

$$\begin{aligned} m_x &= \int_H (x, z)\mu(dz) = (m, x), \\ \sigma_x^2 &= \int_H (x,z)^2\mu(dz) - m_x^2 = \int_H [(x,z)^2 - (m,x)^2]\mu(dz) \\ &= \int_H (x, z-m)^2\mu(dz) = (Bx, x). \end{aligned}$$

故由 (4.28) 推得 (4.26), 由 (4.29) 推得 (4.27).

充分性. 设 $m \in H$, B 为 H 上的一正的对称迹算子,

$$\varphi(x) = \exp\{i(m,x) - \tfrac{1}{2}(Bx,x)\},$$

则容易验证 φ 是 H 上的正定泛函. 令

$$Sx = Bx + (m,x)m,$$

则 S 为 H 上正的对称迹算子. 在 H 上定义范数 $\|\cdot\|_*$ 如下:

$$\|x\|_* = \|S^{1/2}x\| = ((Bx,x) + (m,x)^2)^{1/2}.$$

显然 $\varphi(x)$ 在 $x = 0$ 处关于范数 $\|\cdot\|_*$ 连续, 故由定理 4.4 知 φ 为 H 上某一 Borel 概率测度 μ 的 Fourier 变换. 显然在测度 μ 下, 对一切 $x \in H$, $(x,\cdot)$ 服从均值为 (m,x), 方差为 (Bx,x) 的 Gauss 分布. 于是依定义 μ 为 Gauss 测度. 定理证毕. ∎

4.3 Banach 空间上的 Gauss 测度

现在我们转向研究 Banach 空间上的 Gauss 测度. 首先引进柱集及柱测度等基本概念.

设 X 为一实可分 Banach 空间, X^* 为其对偶空间. 我们用 $\|\cdot\|$ 和 $\|\cdot\|_{X^*}$ 分别表示 X 和 X^* 上的范数, 并用 $\langle\cdot,\cdot\rangle$ 表示 $X \times X^*$ 上的典则双线性型. 令 $\mathcal{F}(X^*)$ 表示 X^* 的有限维线性子空间的全体. 对给定 $K \in \mathcal{F}(X^*)$, 我们称形如

$$C = \{x \in X : (\langle x, y_1\rangle, \cdots, \langle x, y_n\rangle) \in E\} \tag{4.32}$$

的集为 **以 K 为底的柱集**, 这里 $n \geq 1$, E 为 $\mathbb{R}^n$ 的 Borel 子集, $y_1, \cdots, y_n \in K$. 我们用 $\mathcal{C}(K)$ 表示由以 K 为底的柱集在 X 上生成的 σ- 代数. 令

$$\mathcal{R}(X) = \bigcup_{K \in \mathcal{F}(X^*)} \mathcal{C}(K), \tag{4.33}$$

则 $\mathcal{R}(X)$ 为代数.

引理 4.12 设 X 为一实可分 Banach 空间，则 $\sigma(\mathcal{R}(X)) = \mathcal{B}(X)$. 这里 $\mathcal{B}(X)$ 为 X 的 Borel σ- 代数.

证明 首先，显然有 $\sigma(\mathcal{R}(X)) \subset \mathcal{B}(X)$. 由于 X 为可分距离空间. 每个开集可表为可数个闭球的并. 因此, 为证 $\sigma(\mathcal{R}(X)) = \mathcal{B}(X)$, 只需证每个闭球属于 $\sigma(\mathcal{R}(X))$. 设 $S = \{x : \|x - x_0\| \leq r\}$, 其中 $x_0 \in X$, $r > 0$. 令 $\{a_n\}$ 为 X 的可数稠子集. 由 Hahn-Banach 定理, 对每个 $n \geq 1$, 存在 $z_n \in X^*$, 使得 $\langle a_n, z_n\rangle = \|a_n\|$, $\|z_n\|_{X^*} = 1$. 令

$$T = \bigcap_{n=1}^{\infty} \{x \in X : |\langle x - x_0, z_n\rangle| \leq r\} .$$

显然 $S \subset T$, $T \in \sigma(\mathcal{R}(X))$. 往证 $S = T$. 如果 $x \notin S$, 即 $\|x - x_0\| = r_1 > r$, 则存在某个 n, 使得 $\|x - x_0 - a_n\| < (r_1 - r)/2$. 这时必有 $\|a_n\| > (r_1 + r)/2$, 且有

$$\begin{aligned} |\langle x - x_0, z_n\rangle| &\geq |\langle a_n, z_n\rangle| - |\langle x - x_0 - a_n, z_n\rangle| \\ &\geq \|a_n\| - \|x - x_0 - a_n\| > r . \end{aligned}$$

这表明 $x \notin T$. 于是 $T \subset S$. 最终有 $S = T \in \sigma(\mathcal{R}(X))$. 证毕. ∎

定义 4.13 设 μ 为 $\mathcal{R}(X)$ 上的非负集函数. 如果 $\mu(X) = 1$, 且对一切 $K \in \mathcal{F}(X^*)$, μ 限于 σ- 代数 $\mathcal{C}(K)$ 为一测度, 则称 μ 为 X 上的 **柱 (概率) 测度**. X 上的复值函数 f, 如果存在某个 $K \in \mathcal{F}(X^*)$, 使得 f 关于 $\mathcal{C}(K)$ 为可测, 则 f 称为 **柱函数**.

有界柱函数 f 关于柱测度 μ 的积分是有意义的，只要把柱测度看成使 f 可测的 σ- 代数 $\mathcal{C}(K)$ 上的测度. 我们用 $\int_X f(x)\mu(dx)$ 表示这一积分. 特别，对柱测度 μ, 我们可令

$$\widehat{\mu}(z) = \int_X e^{i\langle x,z\rangle}\mu(dx) , \ z \in X^* , \tag{4.34}$$

称 $\widehat{\mu}$ 为 μ 的 **特征泛函**.

显然，柱测度的特征泛函是 X^* 上的连续正定泛函. 反之，设 φ 为 X^* 上的连续正定泛函，且 $\varphi(0)=1$, 则存在唯一的柱测度 μ, 使得 φ 为 μ 的特征泛函.

一个自然的问题是：什么样的柱测度可以扩张成为 X 上的一 Borel 测度？下面我们将对一种特殊情形回答这一问题. 这一特殊情形是： Banach 空间 X 是某个实可分 Hilbert 空间 H 关于某个较弱范数的完备化，而 X 上的柱测度是由 H 上的某个柱测度 "提升" 得到的.

设 H 为一实可分 Hilbert 空间. 我们用 $(\cdot,\cdot)$ 及 $|\cdot|$ 分别表示 H 中的内积及范数. 设 $\|\cdot\|$ 为 H 上的另一范数，满足如下条件：存在一常数 $c>0$, 使得 $\|x\|\le c|x|$. 这时，我们说范数 $\|\cdot\|$ 比范数 $|\cdot|$ 弱. 令 X 为 H 关于范数 $\|\cdot\|$ 的完备化，则 X 为一可分 Banach 空间， H 可视为 X 的一线性子空间. 我们将 H 的对偶 H^* 与 H 等同起来，则 X 的对偶空间 X^* 可以视为 H 的如下子集：

$$X^*=\left\{y\in H:\sup_{x\in H,\|x\|=1}|(x,y)|<\infty\right\}. \tag{4.35}$$

我们用 $\langle\cdot,\cdot\rangle$ 表示 $X\times X^*$ 上的典则双线性型，则 $\langle\cdot,\cdot\rangle$ 在 $H\times X^*$ 上与内积 $(\cdot,\cdot)$ 吻合，即有

$$\langle x,y\rangle=(x,y),\quad \forall x\in H,\ y\in X^*. \tag{4.36}$$

我们用 $\mathcal{F}(X^*)$ 及 $\mathcal{F}(H)$ 分别表示 X^* 及 H 的有限维子空间全体. 由于 $\mathcal{F}(X^*)\subset\mathcal{F}(H)$, 且对每个 $K\in\mathcal{F}(X^*)$, 若以 $\mathcal{C}_X(K)$ 及 $\mathcal{C}_H(K)$ 分别表示以 K 为底的柱集在 X 及 H 上生成的 σ- 代数，则 $\mathcal{C}_X(K)\cap H\subset\mathcal{C}_H(K)$. 因此，我们有 $\mathcal{R}(X)\cap H\subset\mathcal{R}(H)$. 这样一来，对 H 上的每个柱测度 μ, 我们可以定义 X 上的一柱测度 μ^* 如下：

$$\mu^*(C)=\mu(C\cap H),\quad C\in\mathcal{R}(X), \tag{4.37}$$

我们称 μ^* 为 μ 到 X 上的**提升**. 显然，对 $x\in X^*$, 我们有 $\widehat{\mu^*}(x)=\widehat{\mu}(x)$. 这表明 μ^* 的特征泛函是 μ 的特征泛函在 X^* 上的限制. 今

后我们用 (H,X,μ) 表示上面引进的 Hilbert 空间、Banach 空间及 H 上的柱测度，并称它为**基本三元组**. 在回答前面提出的问题之前，我们还需要引进可测范数概念，它是 Gross 在 [1] 中最早提出的.

下面我们用 $\mathcal{P}$ 表示 H 中有限维 (正交) 投影算子全体. 对 $P\in\mathcal{P}$, 令 $f(x)=\|Px\|$, $x\in H$, 则 f 是 H 上的柱函数.

定义 4.14 设 $(H,|\cdot|)$ 为一 Hilbert 空间，μ 为 H 上的柱测度，$\|\cdot\|$ 为 H 上的另一范数，且比范数 $|\cdot|$ 弱. 如果对于每个 $\epsilon>0$, 存在 $P_\epsilon\in\mathcal{P}$, 使得对任何与 P_ϵ 正交的 $P\in\mathcal{P}$ 有

$$\mu\{x\in H:\ \|Px\|>\epsilon\}<\epsilon,$$

则称 $\|\cdot\|$ **关于 μ 可测**.

定义 4.15 设 μ 为 H 上的柱测度. 如果 $\widehat{\mu}(x)=\exp\{-\frac{1}{2}|x|^2\}$, 则称 μ 为 H 上的 **(标准)Gauss 柱测度**.

显然，μ 为 Gauss 柱测度，当且仅当对一切 $P\in\mathcal{P}$, $\mu\circ P^{-1}$ 为 $P(H)$ 上的 Gauss 测度.

下一定理是著名的 **Gross 定理**.

定理 4.16 设 (H,X,μ) 为一基本三元组. 如果 μ 是 Gauss 柱测度，且范数 $\|\cdot\|$ 为 μ- 可测的，则 μ 到 X 上的提升 μ^* 可以扩张成为 X 上的 Borel 测度，称它为 X 上的 Gauss 测度.

证明 下面证明来自 Kallianpur[1]. 令 $\{\xi_n\}$ 为某个概率空间 $(\Omega,\mathcal{F},m)$ 上的一列相互独立的标准正态随机变量. 由范数 $\|\cdot\|$ 的 μ- 可测性，存在 H 的一列有限维正交投影 $\{P_n\}$, 使得 $P_n\uparrow I$(I 为恒等算子) 且对任何与 P_n 正交的 $P\in\mathcal{P}$ 有

$$\mu\{x\in H:\ \|Px\|>2^{-n}\}<2^{-n}\ .$$

我们可以选取 H 的一组标准正交基 $\{e_n\}$, 使得 $\{e_1,\cdots,e_{n_k}\}$ 为 $P_k(H)$ 的标准正交基. 令

$$\eta_k(\omega)=\sum_{j=1}^{n_k}\xi_j(\omega)e_j\ ,$$

则我们有

$$\eta_{k+1}-\eta_k=\sum_{j=n_k+1}^{n_{k+1}}\xi_j(\omega)e_j\ .$$

由于 $P_{k+1}x-P_kx=\sum_{j=n_k+1}^{n_{k+1}}(x,e_j)e_j$，且 $\forall E\in\mathcal{B}(\mathbb{R}^{n_{k+1}-n_k})$，

$$\begin{aligned}&m\{\omega:(\xi_{n_k+1}(\omega),\cdots,\xi_{n_{k-1}}(\omega))\in E\}\\&=\mu\{x\in H:\big((e_{n_k+1},x),\cdots(e_{n_{k+1}},x)\big)\in E\},\end{aligned}$$

于是有

$$m(\|\eta_{k+1}-\eta_k\|>2^{-k})=\mu\{x\in H\ :\ \|P_{k+1}x-P_kx\|>2^{-k}\}<2^{-k}\ .$$

因此，$\{\eta_k\}$ 依概率收敛于一 X- 值随机元 η. 令 ν 为 η 的分布，即 $\nu=m\circ\eta^{-1}$, 则对每个 $z\in X^*$,

$$\begin{aligned}\widehat{\nu}(z)&=\int_X e^{i(x,z)}\nu(dx)=\int_\Omega e^{i(\eta(\omega),z)}m(d\omega)\\&=\lim_{k\to\infty}\int_\Omega\exp\Big\{i\langle\sum_{j=1}^{n_k}\xi_j(\omega)e_j,z\rangle\Big\}m(d\omega)\\&=\lim_{k\to\infty}\prod_{j=1}^{n_k}e^{-\langle e_j,z\rangle^2/2}=e^{-|z|^2/2}=\widehat{\mu}^*(z)\ .\end{aligned}$$

这表明 μ^* 与 ν 在 $\mathcal{R}(X)$ 上一致，即 ν 为 μ^* 的扩张. ∎

定义 4.17 设 X 为一实可分 Banach 空间，μ 为 X 上的 Borel 概率测度，如果对一切 $z\in X^*$, $\langle\cdot,z\rangle$ 为 X 上的零均值正态随机变量，则称 μ 为 X 上的 **对称 Gauss 测度**. 这时称 $(X,\mathcal{B}(X),\mu)$ 为 **Gauss 测度空间**.

设 $(X,\mathcal{B}(X),\mu)$ 为 Gauss 测度空间. H 为一 Hilbert 空间，它在 X 中稠，X 的范数 $\|\cdot\|$ 限于 H 比 H 的 Hilbert 范数 $|\cdot|$ 弱，这时将 H 的对偶空间与 H 等同，则 X 的对偶空间 X^* 可视为 H 的子集. 若 μ 的特征泛函 $\widehat{\mu}(z)=\exp\{-\frac{1}{2}|z|^2\}$, $z\in X^*$, 则称三元组 (H,X,μ) 为一 **抽象 Wiener 空间**.

可以证明：对任一实可分 Banach 空间 X, 至少可以构造 X 的一稠线性子集 H, 它按某个 Hilbert 范数为 Hilbert 空间，使得 X 的范数限于 H 比 H 的 Hilbert 范数弱，且关于 H 上的标准 Gauss 柱测度可测. 于是，由定理 4.16 知：存在 X 上的唯一 Gauss 测度 μ(它是 H 上柱测度到 X 上的提升的扩张), 使得 (H,X,μ) 为一抽象 Wiener 空间. 进一步可以证明 (参看 Kuo[1]):

定理 4.18 若 (H,X,μ) 为抽象 Wiener 空间，则存在另一 Banach 空间 Y, 使 $Y \hookrightarrow X$ 为紧，且 (H,Y,μ) 仍为抽象 Wiener 空间.

下一定理表明：经典 Wiener 空间连同 Cameron-Martin 空间构成一抽象 Wiener 空间.

定理 4.19 令 $X = C_0([0,1];\mathbb{R}^d)$ 为 $[0,1]$ 上在 0 处为 0 的 $\mathbb{R}^d$- 值连续函数全体，$\|\cdot\|$ 为 X 上的上确界范数，即对 $h = (h_1,\cdots,h_d) \in X$, $\|h\| \equiv \sum_{i=1}^d \sup_{0\le t\le 1} | h_i(t) |$. 令

$$H \equiv \left\{h \in X : \text{每个 } h_i \text{ 绝对连续，且} \sum_{i=1}^d \int_0^1 \dot{h}_i(s)^2 ds < \infty\right\},$$

(H 称为 **Cameron-Martin 空间**.) 在 H 上定义如下内积：

$$(h,g) = \sum_{i=1}^d \int_0^1 \dot{h}_i(s)\dot{g}_i(s)ds\,, \quad h,g \in H\,. \tag{4.38}$$

H 上的范数记为 $|\cdot|$. 设 μ 为 X 上的标准 Wiener 测度 (即 d- 维标准 Brown 运动的分布), 则 (H,X,μ) 为一抽象 Wiener 空间.

证明 首先，易证 X 的范数 $\|\cdot\|$ 限于 H 比 H 的 Hilbert 范数 $|\cdot|$ 弱，且 H 在 X 中稠. 下面我们将把 X 的对偶空间 X^* 按 (4.35) 与 H 的一子集等同起来. 由 Riesz 表现定理，X^* 为 $(0,1]$ 上有 d 个分量的有限符号测度全体，$X \times X^*$ 上的典则双线性型为

$$\langle x,\nu\rangle = \sum_{i=1}^d \int_0^1 x_i(s)d\nu_i(s)\,. \tag{4.39}$$

令

$$g_i^\nu(t) = \nu_i((0,1])t - \int_0^t \nu_i((0,s])ds \ , \tag{4.40}$$

则 $g^\nu = (g_1^\nu, \cdots, g_d^\nu) \in H$, 且有

$$\dot{g}_i^\nu(t) = \nu_i((0,1]) - \nu_i((0,t]) \ , \ 0 \le t \le 1 . \tag{4.41}$$

特别 $\dot{g}_i^\nu(1) = 0$. 于是由有限变差函数的分部积分公式得

$$\begin{aligned}\langle h, \nu\rangle &= -\sum_{i=1}^d \int_0^1 h_i(s) d\dot{g}_i^\nu(t) = \sum_{i=1}^d \int_0^1 \dot{g}_i^\nu(s)\dot{h}_i(s)ds \\ &= (h, g^\nu), \qquad \forall h \in H .\end{aligned} \tag{4.42}$$

这表明, 我们可以把 X^* 中的 ν 与 H 中的 g^ν 等同起来. 另一方面, 设 $B_t(x) = x(t)$, $x \in X$, 则在 μ 下, $(B_t, 0 \le t \le 1)$ 为 d- 维标准 Brown 运动. 由随机积分的分部积分公式得

$$\sum_{j=1}^d \int_0^1 \dot{g}_j^\nu(s) dB_s^j(x) = -\sum_{j=1}^d \int_0^1 x_j(s) d\dot{g}_j^\nu(s) = \langle x, \nu\rangle \ , \ \mu\text{-a.e. } x .$$

于是我们有

$$\begin{aligned}\widehat{\mu}(\nu) &= \int_X e^{i\langle x,\nu\rangle}\mu(dx) = \mathbb{E}_\mu\Big[\exp\Big\{i\int_0^1 \dot{g}^\nu(s)dB_s\Big\}\Big] \\ &= \exp\Big\{-\frac{1}{2}\sum_{j=1}^d \int_0^1 \dot{g}_j^\nu(s)^2 ds\Big\} = \exp\{-\tfrac{1}{2}|g^\nu|^2\} .\end{aligned}$$

因此, 依定义 (H, X, μ) 为一抽象 Wiener 空间. ∎

最后我们以 Fernique 关于 Banach 空间中对称 Gauss 测度的一个重要结果结束这一节.

定理 4.20 设 E 为一实可分 Banach 空间, μ 为 $(E, \mathcal{B}(E))$ 上的对称 Gauss 测度. 则存在 $\lambda > 0$, 使得

$$\int_E e^{\lambda\|x\|^2}\mu(dx) < \infty . \tag{4.43}$$

证明 设 X,Y 为某个概率空间 $(\Omega,\mathcal{F},I\!P)$ 上的两个独立 E- 值随机元，其分布都是 μ. 令

$$\widetilde{X}=\frac{1}{\sqrt{2}}(X+Y),\quad \widetilde{Y}=\frac{1}{\sqrt{2}}(X-Y).$$

容易看出，$\widetilde{X}$ 与 $\widetilde{Y}$ 相互独立，且其分布仍为 μ. 设 $t\ge s\ge 0$, 则有

$$\begin{aligned}
I\!P(\|X\|\le s)I\!P(\|X\|>t)&=I\!P(\|\widetilde{Y}\|\le s)I\!P(\|\widetilde{X}\|>t)\\
&=I\!P\left(\frac{\|X-Y\|}{\sqrt{2}}\le s\right)I\!P\left(\frac{\|X+Y\|}{\sqrt{2}}>t\right)\\
&=I\!P\left(\frac{\|X-Y\|}{\sqrt{2}}\le s,\ \frac{\|X+Y\|}{\sqrt{2}}>t\right)\\
&\le I\!P(|\,\|X\|-\|Y\|\,|\le\sqrt{2}s,\ \|X\|+\|Y\|>\sqrt{2}t)\\
&\le I\!P\left(\|X\|>\frac{t-s}{\sqrt{2}},\ \|Y\|>\frac{t-s}{\sqrt{2}}\right)\\
&=\left[I\!P\left(\|X\|>\frac{t-s}{\sqrt{2}}\right)\right]^2.
\end{aligned}\tag{4.44}$$

固定 $r>0$. 令 $t_0=r$, $t_{n+1}=r+\sqrt{2}t_n, n\ge 1$, 定义

$$\alpha_n(r)=\frac{I\!P(\|X\|>t_n)}{I\!P(\|X\|\le r)},\quad n=0,1,2,\cdots.$$

则由 (4.44) 得到

$$\begin{aligned}
\alpha_{n+1}(r)&=\frac{I\!P(\|X\|>r-\sqrt{2}t_n)}{I\!P(\|X\|\le r)}\\
&\le\left[\frac{I\!P(\|X\|>t_n)}{I\!P(\|X\|\le r)}\right]^2=\alpha_n(r)^2\ ,n=0,1,2,\cdots.
\end{aligned}$$

于是 $\alpha_n(r)\le\exp\{2nlog\alpha_0(r)\}$, $n=0,1,\cdots$. 此外，由于 $(\sqrt{2})^{n+4}r>t_n$,

$$\begin{aligned}
I\!P(\|X\|>(\sqrt{2})^{n+4}r)&\le I\!P(\|X\|>t_n)=\alpha_n(r)I\!P(\|X\|\le r)\\
&\le\exp\{2n\log\alpha_0(r)\}\ ,\quad n=0,1,2,\cdots.
\end{aligned}$$

因此，对 $\lambda > 0$, 令

$$\Sigma_n = \{x \in E : (\sqrt{2})^{n+4} r < \|x\| \le (\sqrt{2})^{n+5} r\},$$

我们有

$$\begin{aligned}\int_{\|x\|>4r} e^{\lambda\|x\|^2}\mu(dx) &= \sum_{n=0}^{\infty}\int_{\Sigma_n} e^{\lambda\|x\|^2}\mu(dx)\\ &\le \sum_{n=0}^{\infty} \mathbb{P}(\|X\| > (\sqrt{2})^{n+4} r)\exp\{\lambda r^2 2^{n+5}\}\\ &\le \sum_{n=0}^{\infty}\exp\{2n(\log\alpha_0(r) + 32\lambda r^2)\}.\end{aligned}$$

先取 r 充分大，使 $\mathbb{P}(\|X\| > r) < e^{-1}\mathbb{P}(\|X\| \le r)$, 再取 λ 充分小，使得

$$\log\frac{\mathbb{P}(\|X\| > r)}{\mathbb{P}(\|X\| \le r)} + 32\lambda r^2 \le -1,$$

由于 $2n \le 2^n$, 故有

$$\int_E e^{\lambda\|x\|^2}\mu(dx) \le e^{16\lambda r^2} + \frac{e^2}{e^2-1}.$$

定理证毕. ∎

第二章　Malliavin 随机变分学

Malliavin 随机变分学是 Wiener 空间上的一种无穷维微分分析. 自从 1923 年 N. Wiener 构造了 Brown 运动的数学模型, 即连续函数空间上的 Wiener 测度以来, 人们试图对 Wiener 泛函建立一套分析理论. 但不幸的是, 许多常见的泛函, 例如 Itô 积分或 Itô 随机微分方程的解, 作为 Wiener 空间上的泛函, 未必都是连续的, 当然更谈不上存在 Fréchet 导数了. 直到 1976 年, P. Malliavin 利用 Wiener 测度的拟不变性质建立了一套对 Wiener 泛函的弱微分运算, 使这些重要泛函在弱微分意义下是光滑的, 从而获得了重大突破. 特别是他应用这种运算, 研究了 Wiener 泛函分布密度的光滑性, 首次用概率方法证明了偏微分方程理论中关于亚椭圆算子的著名的 Hörmander 定理, 引起了数学界的广泛重视. 因为这种对 Wiener 泛函的微分运算是对 Wiener 过程的轨道的"扰动"来定义的, 所以 Malliavin 称之为 **随机变分学**, 而现在则被广泛地称之为 **Malliavin Calculus**.

Wiener 空间上的这种微分结构完全取决于其 Cameron-Martin 子空间. K. Itô[3,4] 的研究工作表明, 可以仅从一个可分 Hilbert 空间出发, 建立不依赖于其它任何附加结构的随机变分学. 这一基本框架的突出优点是: 人们可以根据实际问题的需要, 选择不同的模型. Malliavin[5] 称这一基本框架为 Gauss 概率空间. 在本书中, 我们将在这一基本框架下来阐述无穷维随机分析的基本理论.

§ 1. Gauss概率空间与 Wiener 混沌分解

1.1 Gauss概率空间及其上的泛函

假定 H 为实、可分 Hilbert 空间，其中内积和范数分别记为 $(\cdot,\cdot)_H$ 和 $\|\cdot\|_H$. 由 Kolmogorov 定理，存在概率空间 $(\Omega,\mathcal{F},\mu)$ 及其上 Gauss 随机变量族 $\mathcal{H}=\{W_h,h\in H\}$，满足：

$$\mathbb{E}[W_h]=0;\ \mathbb{E}[W_hW_g]=(h,g)_H,\quad \forall h,g\in H. \tag{1.1}$$

于是映射 $h\mapsto W_h$ 为 H 到 $L^2(\Omega,\mathcal{F},\mu)$ 中的线性等距，从而 H 同构于 $L^2(\Omega,\mathcal{F},\mu)$ 的闭子空间 $\mathcal{H}$.

定义 1.1 设 $(\Omega,\mathcal{F},\mu)$ 为一完备概率空间，H 为一实、可分 Hilbert 空间，$\mathcal{H}=\{W_h,h\in H\}$ 为一族 Gauss 随机变量，满足 (1.1) 式，则 $(\Omega,\mathcal{F},\mu;H)$ 称为 **Gauss概率空间**.

下面是几个常见的例子.

例 1 (有限维 Gauss 空间) 设 $\mu=\gamma^n$ 为 $\mathbb{R}^n$ 上的标准 Gauss 测度：

$$\mu(dx)=(2\pi)^{-n/2}\exp\{-\tfrac{1}{2}|x|^2\}dx, \tag{1.2}$$

$\mathcal{F}$ 为 $\mathcal{B}(\mathbb{R}^n)$ 关于 μ 的完备化 σ- 代数，则 $(\mathbb{R}^n,\mathcal{F},\mu)$ 为一完备概率空间. 设 $H=\mathbb{R}^n,\forall h\in H$, 令 $W_h(x)\equiv h\cdot x=\sum_{k=1}^n h_kx_k$, 则 $\mathcal{H}=\{W_h,h\in H\}$ 为一族 Gauss 随机变量满足 (1.1) 式，因此 $(\mathbb{R}^n,\mathcal{F},\mu;\mathbb{R}^n)$ 为一 Gauss 概率空间.

例 2 (经典 Wiener 空间) 设 $W=C_0[0,1]$ 为定义在区间 [0,1] 上、满足 $w(0)=0$ 的实值连续函数 w 全体构成的 Banach 空间，其范数为

$$\|w\|_W\equiv\sup_{0\le t\le 1}|w(t)|, \tag{1.3}$$

μ 为其上 Wiener 测度，$\mathcal{F}$ 为 $\mathcal{B}(W)$ 关于 μ 的完备化. 设 $H=L^2[0,1],\forall h\in H$, 令

$$W_h(w)\equiv\int_0^1 h(t)dw(t) \tag{1.4}$$

为 Wiener随机积分， 则 $\mathcal{H}=\{W_h, h\in H\}$ 满足 (1.1)式，$(W,\mathcal{F},\mu;\ H)$ 为 Gauss概率空间.

对 $h\in H$, 令 $\widetilde{h}(t)=\int_0^t h(s)ds\ (0\leq t\leq 1)$, 则 $\widetilde{h}\in W$, 且

$$\begin{aligned}\|\widetilde{h}\|_W &= \sup_{0\leq t\leq 1}\left|\int_0^t h(s)ds\right| \\ &\leq \sup_{0\leq t\leq 1}\left(t\int_0^t |h(s)|^2 ds\right)^{1/2} \\ &\leq \left(\int_0^1 |h(s)|^2 ds\right)^{1/2} = \|h\|_H.\end{aligned}$$

于是映射 $J: h\mapsto \widetilde{h}$ 为 $H\to W$ 中的连续线性单射，且 $\widetilde{H}\equiv J(H)$ 在 W 中稠密. $\widetilde{H}$ 称为 W 的 **Cameron-Martin子空间**.

设 W^* 为 W 的对偶空间，将 H^* 与 H 视为等同，我们有 (见第一章定理 4.19)

$$W^*\hookrightarrow H^*\cong H\hookrightarrow W. \tag{1.5}$$

例 3 (抽象 Wiener 空间) 设 X 为一可分 Banach 空间， H 为一可分 Hilbert 空间且连续、稠密地嵌入 X. 若 $J: H\to X$ 为嵌入映射，则其对偶映射 J^* 将 X^* 连续、稠密地嵌入 $H^*\cong H$, 于是有

$$X^*\hookrightarrow H^*\cong H\hookrightarrow X. \tag{1.6}$$

设 μ 为 X 上的 Gauss 测度，满足

$$\int_X \exp\{i\langle l,x\rangle\}\mu(dx)=\exp\left\{-\tfrac{1}{2}\|J^*l\|_H^2\right\},\quad \forall l\in X^*, \tag{1.7}$$

其中 $\langle\cdot,\cdot\rangle$ 表示 $X^*\times X$ 上典则双线性型. (H,X,μ) 称为 **抽象Wiener空间**(见定义 I.4.17). 例 2 的经典 Wiener 空间是抽象 Wiener 空间的特殊情形.

设 $\mathcal{F}$ 为 $\mathcal{B}(X)$ 关于 μ 的完备化. $\forall l\in X^*$, 令 $W_l(x)\equiv\langle l,x\rangle$, 由 (1.7) 式可知， $\{W_l, l\in X^*\}$ 为 $(X,\mathcal{F},\mu)$ 上一族 Gauss 随机变量，满足

$$\mathbb{E}[W_l]=0;\ \mathbb{E}[W_lW_{l'}]=(J^*l,J^*l')_H,\quad \forall l,l'\in X^*.$$

于是映射 $J^*l \mapsto W_l$ 为 $J^*(X^*)$ 到 $L^2(X,\mathcal{F},\mu)$ 中的线性等距，由于 $J^*(X^*)$ 在 H 中稠密，可开拓为 H 到 $L^2(X,\mathcal{F},\mu)$ 中的线性等距且满足 (1.1) 式，从而 $(X,\mathcal{F},\mu;H)$ 为一 Gauss 概率空间.

例 4 (白噪声空间) 设 $H=L^2(\mathbb{R}),\mathcal{S}(\mathbb{R})$ 及 $\mathcal{S}^*(\mathbb{R})$ 分别为 Schwartz 的速降 C^∞ 函数及缓增广义函数空间，我们有

$$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^*(\mathbb{R}).$$

因为 $\mathcal{S}(\mathbb{R})$ 是 Fréchet 核空间 (参看第一章 §3 的例), 由 Minlos 定理，在 $\mathcal{B}(\mathcal{S}^*(\mathbb{R}))$ 上存在唯一 Gauss 测度 μ, 使得 $\forall\xi\in\mathcal{S}(\mathbb{R})$ 有

$$\int_{\mathcal{S}^*(\mathbb{R})}\exp\{i\langle\omega,\xi\rangle\}\mu(d\omega)=\exp\{-\tfrac{1}{2}\|\xi\|_H^2\}, \tag{1.8}$$

其中 $\langle\omega,\xi\rangle$ 为 $\mathcal{S}^*(\mathbb{R})\times\mathcal{S}(\mathbb{R})$ 上的典则双线性型 (参看第一章定理 4.7). 令 $\mathcal{F}$ 为 $\mathcal{B}(\mathcal{S}^*(\mathbb{R}))$ 关于 μ 的完备化. $\forall\xi\in\mathcal{S}(\mathbb{R})$, 令 $W_\xi(\omega)\equiv\langle\omega,\xi\rangle$, 如同例 3 一样，映射 $\xi\mapsto W_\xi$ 可开拓为 $L^2(\mathbb{R})\to L^2(\mathcal{S}^*(\mathbb{R}),\mathcal{F},\mu)$ 的线性等距, 从而$(\mathcal{S}^*(\mathbb{R}),\mathcal{F},\mu;L^2(\mathbb{R}))$ 为一 Gauss 概率空间，此即所谓 **白噪声空间**, 而 μ 称为 **白噪声测度**. 我们将在第四章中讨论建立在这个空间上的白噪声分析.

定义 1.2 设 $(\Omega,\mathcal{F},\mu;H)$ 为 Gauss 概率空间，$\mathcal{N}$ 为 μ- 零集全体，$\mathcal{F}^0$ 为由 $\mathcal{H}=\{W_h,h\in H\}$ 生成的 σ- 代数. 若 $\mathcal{F}=\sigma(\mathcal{F}^0\cup\mathcal{N})$, 我们称此空间 **不可约**.

考虑如下形式的泛函:

$$F(\omega)=f(W_{h_1}(\omega),\cdots,W_{h_n}(\omega)),\quad n\in\mathbb{N},h_1,\cdots,h_n\in H. \tag{1.9}$$

若 f 为 n 个变量的多项式，F 称为 **多项式泛函**, 其总体记为 $\mathcal{P}$, 即由 $\mathcal{H}$ 生成的代数; 若 f 为缓增 C^∞ 函数 (即本身以及各阶导数增长的阶均不超过多项式的无穷次可微函数), F 称为 **光滑泛函**, 其总体记为 $\mathcal{S}_M$.

若 E 为可分 Hilbert 空间，具有内积 $(\cdot,\cdot)_E$ 和范数 $|\cdot|_E$. 对 $p\in[1,\infty)$, 以 $L^p(E)\equiv L^p(\Omega,\mathcal{F},\mu;E)$ 表示 E 值 p 次幂可积的随

机元 (等价类) 所构成的 Banach 空间，其中范数为

$$\|F\|_p \equiv \left(\int_\Omega |F(\omega)|_E^p \mu(d\omega)\right)^{1/p}. \tag{1.10}$$

当 $E = \mathbb{R}$ 时简记 $L^p(E)$ 为 L^p.

考虑如下形式的 E 值泛函：

$$F(\omega) = \sum_{k=1}^m F_k(\omega) e_k, \quad m \in \mathbb{N}, e_1, \cdots, e_m \in E. \tag{1.11}$$

若 $F_1, \cdots, F_m \in \mathcal{P}$(或 $\mathcal{S}_M$), 上述泛函称为 **E值多项式**(或 **E 值光滑**)**泛函**, 其总体记为 $\mathcal{P}(E)$(或 $\mathcal{S}_M(E)$).

注 在 (1.9) 式中，通过 Schmidt 正交化手续，适当改变 f 的形式，总可以假定

$$(h_i, h_j)_H = \delta_{ij} \quad (i, j = 1, \cdots, n),$$

此时多项式的阶数由 F 唯一确定，而 $W_{h_1}, \cdots, W_{h_n}$ 之联合分布为 $\mathbb{R}^n$ 上的标准 Gauss 分布. 类似地，在 (1.11) 式中，也不妨假定

$$(e_i, e_j)_E = \delta_{ij} \quad (i, j = 1, \cdots, m).$$

命题 1.3 若 $(\Omega, \mathcal{F}, \mu; H)$ 为不可约 Gauss 概率空间， E 为可分 Hilbert 空间，则

$$\mathcal{P}(E) \subset \mathcal{S}_M(E) \subset L^p(E) \quad (1 \le p < \infty),$$

且 $\mathcal{P}(E)$ 在 $L^p(E)$ 中稠密.

证明 由 E 可构造 Gauss 概率空间 $(\Omega', \mathcal{F}', \mu'; E)$, 而形如 (1.11) 式的 E 值泛函可以看成乘积空间 $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mu \times \mu')$ 上的纯量值泛函

$$F'(\omega, \omega') = \sum_{k=1}^m F_k(\omega) W_{e_k}(\omega'), \tag{1.12}$$

于是只需就 $E=\mathbb{R}$ 的情形予以证明.

选择 H 的一组基 $\{h_j\}$, 则 $\{W_{h_j}\}$ 为独立标准 Gauss 变量序列, 它们生成 σ- 代数 $\mathcal{F}$, 且对任意 $n\in\mathbb{N},t_1,\cdots,t_n\in\mathbb{R}$ 及 $c>0$ 有

$$\mathbb{E}\left[\exp\left\{c\sum_{j=1}^{n}|t_jW_{h_j}|\right\}\right]<\infty. \tag{1.13}$$

由此容易看出, $\mathcal{P}\subset\mathcal{S}_M\subset L^p\,(1\le p<\infty)$.

若 $\mathcal{P}$ 不在某 L^p 中稠密, 则存在非零随机变量 $\xi\in L^q\,(q^{-1}+p^{-1}=1)$, 使

$$\mathbb{E}[\xi F]=0,\quad \forall F\in\mathcal{P}. \tag{1.14}$$

因 $q>1$, 由 (1.13) 式可知

$$\mathbb{E}\left[|\xi|\exp\left\{\sum_{j=1}^{n}|t_jW_{h_j}|\right\}\right]<\infty,$$

故由 (1.14) 式

$$\mathbb{E}\left[\xi\exp\left\{i\sum_{j=1}^{n}t_jW_{h_j}\right\}\right]=\sum_{m=0}^{\infty}\frac{i^m}{m!}\mathbb{E}\left[\xi\Big(\sum_{j=1}^{n}t_jW_{h_j}\Big)^m\right]=0. \tag{1.15}$$

令 $\xi_n\equiv\mathbb{E}[\xi|W_{h_1},\cdots,W_{h_n}]$. 则存在 $\mathbb{R}^n$ 上可测函数 g, 使 $\xi_n=g(W_{h_1},\cdots,W_{h_n})$. 由 (1.15) 式

$$\mathbb{E}\left[\xi_n\exp\left\{i\sum_{j=1}^{n}t_jW_{h_j}\right\}\right]=\mathbb{E}\left[\xi\exp\left\{i\sum_{j=1}^{n}t_jW_{h_j}\right\}\right]=0,$$

此即

$$\int_{\mathbb{R}^n}g(x)\exp\left\{i\sum_{j=1}^{n}t_jx_j\right\}\gamma^n(dx)=0.$$

由 Fourier 变换唯一性可知 $g=0$ a.e. $[\gamma^n]$, 于是 $\xi_n=0$ a.s.. 但由鞅收敛定理, $\xi_n\to\xi$ a.s., 从而 $\xi=0$ a.s., 这和假定矛盾. 所以 $\mathcal{P}$ 在一切 L^p 中稠密. ∎

1.2 数值模型

设 $(\mathbb{R},\mathcal{B}(\mathbb{R}),\gamma)$ 为一维 Gauss 空间， $\mathcal{P}$ 为实多项式全体，则 $\mathcal{P}$ 在 $L^2(\mathbb{R},\gamma)$ 中稠密. 在 $\mathcal{P}$ 上定义算子:

$$\partial \equiv \frac{d}{du}, \quad \partial^* \equiv -\frac{d}{du}+u\cdot, \tag{1.16}$$

则

$$\partial^*\partial = -\frac{d^2}{du^2}+u\frac{d}{du}. \tag{1.17}$$

由分部积分公式， $\forall \varphi,\psi\in\mathcal{P}$ 有

$$(\partial\varphi,\psi)_{L^2(\mathbb{R},\gamma)} = (\varphi,\partial^*\psi)_{L^2(\mathbb{R},\gamma)}, \tag{1.18}$$

从而 ∂,∂^* 及 $\partial^*\partial$ 可延拓为 $L^2(\mathbb{R},\gamma)$ 中的闭算子， ∂ 与 ∂^* 共轭， $\partial^*\partial$ 自共轭 (即一维计数算子). 由附录 A 之 (A.6) 式知， Hermite 多项式 $H_n(u)$ 为 $\partial^*\partial$ 之特征函数:

$$\partial^*\partial H_n = nH_n, \quad n\in\mathbb{N}_0. \tag{1.19}$$

由 (A.5) 及 (A.7) 式可得递推公式:

$$H_{n+1} = \partial^* H_n, \quad n\in\mathbb{N}_0. \tag{1.20}$$

因此

$$H_n = (\partial^*)^n 1, \quad n\in\mathbb{N}_0. \tag{1.21}$$

而

$$\begin{aligned}(H_n,H_m)_{L^2(\mathbb{R},\gamma)} &= (H_n,(\partial^*)^m 1)_{L^2(\mathbb{R},\gamma)}\\ &= (\partial^m H_n,1)_{L^2(\mathbb{R},\gamma)}.\end{aligned}$$

当 $m>n$ 时，由于 H_n 为 n 阶多项式，故上式为 0; 当 $m=n$ 时，由 (A.5) 式可知上式等于 $n!$. 这样，我们重新得到了 (A.10) 式，即 $\{(n!)^{-1/2}H_n\}$ 为 $L^2(\mathbb{R},\gamma)$ 之正交基.

设 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$ 为 $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$ 的无穷维乘积空间. 对非负整数列 $\alpha = \{\alpha_j\}_{j\in\mathbb{N}}$, 记 $|\alpha| \equiv \sum_j \alpha_j, \alpha! \equiv \prod_j(\alpha_j!)$, 并以 Λ 表示使 $|\alpha|$ 为有限数值的序列 α 的集合, 对 $\alpha = \{\alpha_j\} \in \Lambda$ 及 $x = \{x_j\} \in \mathbb{R}^\infty$ 定义

$$H_\alpha(x) \equiv \prod_j H_{\alpha_j}(x_j). \tag{1.22}$$

注意 $\{\alpha_j\}$ 中只含有限个非 0 项且 $H_0(u) \equiv 1$, 故此乘积实际上只是有限项的乘积. 类似于一维情况, 我们有

定理 1.4 对 $\alpha, \beta \in \Lambda$, 有

$$\int_{\mathbb{R}^\infty} H_\alpha(x) H_\beta(x) \gamma^\infty(dx) = \alpha! \delta_{\alpha\beta}, \tag{1.23}$$

从而 $\{(\alpha!)^{-1/2} H_\alpha : \alpha \in \Lambda\}$ 构成 $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$ 之正交基, 且有乘法公式:

$$H_\alpha(x) H_\beta(x) = \sum_{\substack{\kappa \leq \alpha\wedge\beta \\ \kappa\in\Lambda}} \kappa! \binom{\alpha}{\kappa}\binom{\beta}{\kappa} H_{\alpha+\beta-2\kappa}(x), \tag{1.24}$$

其中 $\binom{\alpha}{\kappa} \equiv \prod_j \binom{\alpha_j}{\kappa_j}, \alpha + \beta - 2\kappa \equiv \{\alpha_j + \beta_j - 2\kappa_j\}_{j\in\mathbb{N}}, \kappa \leq \alpha \wedge \beta$ 表示 $\kappa_j \leq \alpha_j \wedge \beta_j, \forall j \in \mathbb{N}$. 若记 $i = \sqrt{-1}, (x \pm iy)^\alpha \equiv \prod_j (x_j \pm iy_j)^{\alpha_j}$, 则

$$H_\alpha(x) = \int_{\mathbb{R}^\infty} (x \pm iy)^\alpha \gamma^\infty(dy). \tag{1.25}$$

证明 由各坐标分量的独立性, (1.23) 和 (1.25) 式分别是 (A.10) 和 (A.11) 式的推论. (1.24) 式可由乘法公式 (A.8) 直接计算而得到. ∎

对 $j \in \mathbb{N}$, 若以 ∂_j 和 ∂_j^* 分别表示对第 j 个坐标分量的微分及其共轭算子, 令

$$\mathcal{L} \equiv -\sum_{j=1}^\infty \partial_j^* \partial_j \tag{1.26}$$

为无穷维 Ornstein-Uhlenbeck 算子. 对 $\alpha \in \Lambda$, 记 $\partial_\alpha^* \equiv \prod_j (\partial_j^*)^{\alpha_j}$, 类似于 (1.21) 式, 我们有

$$H_\alpha = \partial_\alpha^* 1, \quad \alpha \in \Lambda, \tag{1.27}$$

且 H_α 为 $\mathcal{L}$ 的特征函数, 其特征方程为

$$\mathcal{L} H_\alpha = -|\alpha| H_\alpha, \quad \alpha \in \Lambda. \tag{1.28}$$

记 $\Lambda_n \equiv \{\alpha \in \Lambda : |\alpha| = n\}, \mathcal{H}_n$ 为 $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$ 中由 $\{H_\alpha : \alpha \in \Lambda_n\}$ 生成的闭子空间, 则有如下正交分解:

$$L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty) = \bigoplus_{n=0}^\infty \mathcal{H}_n. \tag{1.29}$$

现在设 $(\Omega, \mathcal{F}, \mu; H)$ 为任一不可约 Gauss 概率空间. 取定 H 的一组基 $\{h_j\}_{j \in \mathbb{N}}$. 则 H 同构于平方可和序列空间 l^2. 对 $\omega \in \Omega$, 令 $T\omega \equiv \{W_{h_j}(\omega)\}_{j \in \mathbb{N}}$, 则 $T : \Omega \to \mathbb{R}^\infty$ 为 $\mathcal{F}/\mathcal{B}^\infty$ 可测, 且为保测变换: $\gamma^\infty = \mu \circ T^{-1}$. 对 $1 \le p \le \infty, \varphi \in L^p(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$, 令 $T_* \varphi(\omega) \equiv \varphi(T\omega)$, 则 $T_* : L^p(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty) \to L^p(\Omega, \mathcal{F}, \mu)$ 为同构映射, 限制于多项式泛函空间 $\mathcal{P}$ 时, 它是代数同态. 我们称 Gauss 概率空间 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$ 为 $(\Omega, \mathcal{F}, \mu; H)$ 的一个 **数值模型**.

显然, 数值模型依赖于 H 中基的选取. 然而 Gauss 概率空间中有许多性质是不依赖于基的选取的, 我们称之为 **内蕴性质**. 当我们研究这些性质时, 采用数值模型是十分方便的. 例如所谓混沌分解, 就是一种内蕴性质.

定理 1.5 设 $(\Omega, \mathcal{F}, \mu; H)$ 为不可约 Gauss 概率空间, 任取 H 之一组基 $\{h_j\}$, 对 $\alpha \in \Lambda$, 令

$$H_\alpha(\omega) \equiv \prod_j H_{\alpha_j}(W_{h_j}(\omega)), \tag{1.30}$$

则 $\{(\alpha!)^{-1/2} H_\alpha : \alpha \in \Lambda\}$ 构成 $L^2(\Omega, \mathcal{F}, \mu)$ 之基. 令 $\mathcal{H}_0 \equiv \mathbb{R}$; 对 $n \ge 1$, 令 $\mathcal{H}_n$ 为由 $\{H_\alpha : \alpha \in \Lambda_n\}$ 生成之闭子空间, 则

$$L^2(\Omega, \mathcal{F}, \mu) = \bigoplus_{n=0}^\infty \mathcal{H}_n, \tag{1.31}$$

此分解不依赖于 H 中基的选取，且

$$L^2(\Omega,\mathcal{F},\mu)\cong\Gamma(H),\tag{1.32}$$

即同构于 H 上的对称 Fock 空间.

证明 考虑数值模型 $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$，由 $L^2(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty)$ 与 $L^2(\Omega,\mathcal{F},\mu)$ 的同构关系及 (1.29) 式，上述分解的存在性是显然的，现在要证明此分解不依赖于 H 中基的选取.

设 $\{\widetilde{h}_j\}$ 为 H 中另一组基，由 (1.30) 式定义相应的多项式泛函 $\{\widetilde{H}_\alpha\}$ 并得到分解:

$$L^2(\Omega,\mathcal{F},\mu)=\bigoplus_{n=0}^{\infty}\widetilde{\mathcal{H}}_n.$$

若 $\widetilde{h}_j=\sum_k a_{kj}h_k$(在 H 中收敛)，由等距关系，$W_{\widetilde{h}_j}=\sum_k a_{kj}W_{h_k}$ (在 L^2 中收敛).

注意到对于中心化 Gauss 变量 ξ_n 及 ξ 有

$$\mathbb{E}|\xi_n-\xi|^{2p}=(2p-1)!!(\mathbb{E}|\xi_n-\xi|^2)^p,\quad \forall p\in\mathbb{N}.$$

因此 $\{\xi_n\}$ 的 L^2 收敛性可推出其 L^{2p} 收敛，进而可推出其任一多项式的 L^2 收敛性. 于是

$$\bigoplus_{k=0}^{n}\widetilde{\mathcal{H}}_k\subset\bigoplus_{k=0}^{n}\mathcal{H}_k,\quad \forall n\in\mathbb{N}.$$

由对称性，相反包含关系也成立，从而 $\widetilde{\mathcal{H}}_n=\mathcal{H}_n,\forall n\in\mathbb{N}$.

若令

$$\widehat{h}_\alpha\equiv\widehat{\bigotimes_j}h_j^{\otimes\alpha_j}\quad \alpha\in\Lambda,\tag{1.33}$$

则由第一章命题 2.7 可知，$\{(\alpha!)^{-1/2}\widehat{h}_\alpha:\alpha\in\Lambda_n\}$ 构成 $H^{\widehat{\otimes}n}$ 之基，于是

$$\mathcal{H}_n\cong H^{\widehat{\otimes}n},\quad n\in\mathbb{N}_0,\tag{1.34}$$

得证 (1.32) 式. ∎

以上分解称为 **Wiener-Itô-Segal 混沌 (chaos) 分解**. (特别, $\mathcal{H}_1$ 就是 $\mathcal{H}$.) 它表明 Gauss 概率空间上的平方可积泛函空间和对称 Fock 空间同构, 这是一个十分重要的结论.

1.3 多重 Wiener-Itô 积分表示

下面考虑一个重要的特殊情形, 即 $H = L^2(T, \mathcal{B}, \lambda)$, 其中 $(T, \mathcal{B})$ 为一可测空间, λ 为其上 σ- 有限无原子测度. 前面提到的经典 Wiener 空间和白噪声空间都属于这种情形. 我们将证明, 此时混沌分解表现为多重 Wiener-Itô 随机积分.

首先给出多重 Wiener-Itô 随机积分的一般定义.

设 $\mathcal{B}_0 \equiv \{A \in \mathcal{B} : \lambda(A) < \infty\}$, 在 $\mathcal{B}_0$ 上定义随机集函数

$$W(A) \equiv W_{\mathbf{1}_A}, \quad A \in \mathcal{B}_0. \tag{1.35}$$

由 (1.1) 式可知 $W(\cdot)$ 为取值于 $L^2(\Omega, \mathcal{F}, \mu)$ 的随机集函数, 且对 $A, B \in \mathcal{B}_0$ 有

1° $W(A) \sim N(0, \lambda(A))$;

2° $\mathbb{E}[W(A)W(B)] = \lambda(A \cap B)$.

由此推出, 若 $A_1, \cdots, A_n, \cdots$ 互不相交, 则 $W(A_1), \cdots, W(A_n), \cdots$ 相互独立, 且若 $\cup_n A_n \in \mathcal{B}_0$, 则

$$W\Big(\bigcup_n A_n\Big) = \sum_n W(A_n) \quad (L^2 \text{收敛}).$$

这样的随机集函数称为 $(T, \mathcal{B})$ 上以 λ 为构成测度的 **Gauss 正交随机测度**. 对 $h \in H, W_h$ 恰好是 h 关于此随机测度的随机积分:

$$W_h = \int_T h(t) W(dt). \tag{1.36}$$

现在我们来构造多重随机积分. 设 $n \geq 1, f \in L^2(T^n, \mathcal{B}^n, \lambda^n)$, 具有形式:

$$f = \sum_{j_1, \cdots, j_n = 1}^{N} a_{j_1 \cdots j_n} \mathbf{1}_{A_{j_1} \times \cdots \times A_{j_n}}, \tag{1.37}$$

其中 $A_1,\cdots,A_n\in\mathcal{B}_0$, 两两不相交, 且若 $j_1,\cdots,j_n$ 中有两指标相同时, $a_{j_1\cdots j_n}=0$. 我们定义其 n 重随机积分为

$$I_n(f)\equiv\sum_{j_1,\cdots,j_n=1}^{N}a_{j_1\cdots j_n}W(A_{j_1})\cdots W(A_{j_n}). \tag{1.38}$$

容易看出, 此定义不依赖于 f 的具体表示且关于 f 为线性. 考虑其对称化:

$$\widetilde{f}(t_1,\cdots,t_n)\equiv\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}f(t_{\sigma(1)},\cdots,t_{\sigma(n)}).$$

由定义可知, $I_n(f)=I_n(\widetilde{f})$. 从而不妨假定 $f\in H^{\widehat{\otimes}n}$, 即关于 $t_1,\cdots,t_n$ 为对称:

$$a_{j_1\cdots j_n}=a_{\sigma(j_1)\cdots\sigma(j_n)},\quad\forall\sigma\in\mathfrak{S}_n. \tag{1.39}$$

命题 1.6 在上述条件下, $\forall n\in\mathbb{N},I_n$ 可唯一开拓为 $H^{\widehat{\otimes}n}$ 到 $L^2(\Omega,\mathcal{F},\mu)$ 中的线性等距, 且当 $m\neq n$ 时, $\mathcal{R}(I_m)\perp\mathcal{R}(I_n)$.

我们称 $I_n(f)$ 为函数 f 关于随机测度 W 的 n 重 **Wiener-Itô 随机积分**, 记为

$$I_n(f)=\int_{T^n}f(t_1,\cdots,t_n)W(dt_1)\cdots W(dt_n). \tag{1.40}$$

证明 设 $f\in H^{\widehat{\otimes}n},g\in H^{\widehat{\otimes}m}$, 且有形式 (1.37). 设 $n>m$, 且它们有相同的分划 $\{A_1,\cdots,A_N\}$. 因 $W(\cdot)$ 在不相交集合上取独立值且期望为 0, 故 $\mathbb{E}[I_n(f)I_m(g)]=0$.

现设 $m=n$, g 具形式 (1.37), 但其中 $a_{j_1\cdots j_n}$ 代之以 $b_{j_1\cdots j_n}$. 因 $\mathbb{E}[W(A_j)^2]=\lambda(A_j)$, $1\leq j\leq n$, 故

$$\begin{aligned}\mathbb{E}[I_n(f)I_n(g)]&=(n!)^2\sum_{j_1<\cdots<j_n}a_{j_1\cdots j_n}b_{j_1\cdots j_n}\lambda(A_{j_1})\cdots\lambda(A_{j_n})\\&=n!(f,g)_{H^{\otimes n}}.\end{aligned} \tag{1.41}$$

为证形如 (1.37) 的函数类在 $H^{\otimes n}$ 中稠密，只要考虑逼近示性函数 1_A，其中 $A=A_1\times\cdots\times A_n, A_j\in\mathcal{B}_0, 1\le j\le n$. 因 λ 无原子，故 $\forall\epsilon>0$, 可将每个 $A_j(1\le j\le n)$ 表为 $\mathcal{B}_0$ 中有限个其测度小于 ϵ 的互不相交集合之并，设这些集合总体为 $\{B_j\}_{1\le j\le M}$, 于是

$$\mathbf{1}_A=\sum_{j_1,\cdots,j_n=1}^{M}\epsilon_{j_1\cdots j_n}\mathbf{1}_{B_{j_1}\times\cdots\times B_{j_n}},$$

其中 $\epsilon_{j_1\cdots j_n}=0$ 或 1. 因为 $j_1,\cdots,j_n$ 中含有相同指标的集合 $B_{j_1}\times\cdots\times B_{j_n}$ 的测度之和可随 ϵ 任意小，所以 $\mathbf{1}_A$ 可用形如 (1.37) 的函数在 $H^{\otimes n}$ 中逼近. 由 (1.41) 式可知，I_n 可唯一开拓为 $H^{\widehat{\otimes}n}$ 到 $L^2(\Omega,\mathcal{F},\mu)$ 中的线性等距映射. ∎

对 $f\in H^{\widehat{\otimes}n}, g\in H^{\widehat{\otimes}m}, 1\le r\le m\wedge n$, 我们定义

$$\begin{aligned}&f\otimes_r g(t_1,\cdots,t_{m+n-2r})\\&\equiv\int_{T^r}f(t_1,\cdots,t_{n-r},s)g(t_{n+1-r},\cdots,t_{n+m-2r},s)\lambda^r(ds),\end{aligned}\qquad(1.42)$$

则 $f\otimes_r g\in H^{\otimes(m+n-2r)}$, 其对称化记为 $f\widehat{\otimes}_r g$, 当 $r=0$ 时即 $f\widehat{\otimes}g$.

命题 1.7 设 $f\in H^{\widehat{\otimes}n}, g\in H$, 则

$$I_n(f)I_1(g)=I_{n+1}(f\otimes g)-nI_{n-1}(f\otimes_1 g).\qquad(1.43)$$

证明 由 I_n 的线性及连续性，我们不妨设 $f=\widetilde{\mathbf{1}}_{A_1\times\cdots\times A_n}$, 其中 $A_1,\cdots,A_n\in\mathcal{B}_0$ 且互不相交；$g=\mathbf{1}_A, A\in\mathcal{B}_0$, 或与 $A_1,\cdots,A_n$ 均不相交，或等于其中某个 $A_j(1\le j\le n)$.

在第一种情形，$f\otimes_1 g=0$. (1.43) 式两边均等于

$$W(A_1)\cdots W(A_n)W(A).$$

下面考虑第二种情形，不妨设 $A=A_1$, 此时

$$f\otimes_1 g=\frac{1}{n}\lambda(A_1)\widetilde{\mathbf{1}}_{A_2\times\cdots\times A_n},$$

$$I_{n-1}(f \otimes_1 g) = \frac{1}{n}\lambda(A_1)W(A_2)\cdots W(A_n),$$

且存在 A_1 之分割序列 $\{B_{m1}, B_{m2}, \cdots, B_{mn_m}\}_{m\in\mathbf{N}}$，使在 $L^2(T^2)$ 中有

$$\lim_{m\to\infty}\sum_{j\neq k}\mathbf{1}_{B_{mj}\times B_{mk}} = \mathbf{1}_{A_1\times A_1}.$$

于是在 $L^2(\Omega)$ 中有

$$\begin{aligned}
&\lim_{m\to\infty}\sum_{j\neq k}W(B_{mj})W(B_{mk}) \\
&= \lim_{m\to\infty}\left\{\Big(\sum_j W(B_{mj})\Big)^2 - \sum_j W(B_{mj})^2\right\} \\
&= W(A_1)^2 - \lambda(A_1),
\end{aligned}$$

由此可得

$$I_{n+1}(f\otimes g) = (W(A_1)^2 - \lambda(A_1))W(A_2)\cdots W(A_n),$$

从而 (1.43) 式成立. ∎

更一般地，我们有

命题 1.8 设 $f \in H^{\widehat{\otimes} n}, g \in H^{\widehat{\otimes} m}$，则

$$I_n(f)I_m(g) = \sum_{r=0}^{n\wedge m} r!\binom{n}{r}\binom{m}{r} I_{n+m-2r}(f\widehat{\otimes}_r g). \tag{1.44}$$

证明 对 m 用归纳法，假定 $n \geq m$. 当 $m=1$ 时，归结为 (1.43) 式. 设 m 代之以 $m-1$ 时 (1.44) 式成立，利用线性组合逼近，不妨设 $g = g_1\widehat{\otimes} g_2$，其中 $g_1 \in H^{\widehat{\otimes}(m-1)}, g_2 \in H$，且 $g_1 \otimes_1 g_2 = 0$. 由 (1.43) 式

$$I_m(g) = I_{m-1}(g_1)I_1(g_2).$$

于是由归纳法假设

$$
\begin{aligned}
I_n(f)I_m(g) &= \sum_{r=0}^{m-1} r!\binom{n}{r}\binom{m-1}{r} I_{n+m-1-2r}(f\otimes_r g_1)I_1(g_2) \\
&= \sum_{r=0}^{m-1} r!\binom{n}{r}\binom{m-1}{r} [I_{n+m-2r}((f\widehat{\otimes}_r g_1)\otimes g_2) \\
&\quad + (n+m-1-2r)I_{n+m-2r-2}((f\widehat{\otimes}_r g_1)\otimes_1 g_2)] \\
&= \sum_{r=0}^{m-1} r!\binom{n}{r}\binom{m-1}{r} I_{n+m-2r}((f\widehat{\otimes}_r g_1)\otimes g_2) \\
&\quad + \sum_{r=1}^{m} (r-1)!\binom{n}{r-1}\binom{m-1}{r-1}(n+m+1-2r) \\
&\quad \times I_{n+m-2r}((f\widehat{\otimes}_{r-1} g_1)\otimes_1 g_2).
\end{aligned}
$$

可以验证，当 $1\le r\le m$ 时

$$
m(f\widehat{\otimes}_r g) = \tfrac{r(n+m-2r+1)}{n-r+1}((f\widehat{\otimes}_{r-1}g_1)\otimes_1 g_2) + (m-r)(f\widehat{\otimes}_r g_1)\widehat{\otimes} g_2,
$$

将此式代入上一式子即得 (1.44) 式. ∎

下一定理表明多重随机积分和 Hermite 多项式有着密切的关系.

定理 1.9 设 $h\in H=L^2(T)$, 且 $\|h\|=1$, 则

$$
H_n(W_h) = I_n(h^{\otimes n}), \quad \forall n\in \mathbb{N}. \tag{1.45}
$$

若 $\{h_j\}_{j\in\mathbb{N}}$ 为 H 之基，则

$$
H_\alpha = I_{|\alpha|}(\widehat{h}_\alpha), \quad \forall \alpha\in\Lambda, \tag{1.46}
$$

其中 H_α 由 (1.30) 式定义，$\widehat{h}_\alpha$ 由 (1.33) 式定义.

证明 先用归纳法证明 (1.45) 式． $n=1$ 时即 (1.36) 式．设 $n\le m$ 时 (1.45) 式成立，利用递推公式 (A.7) 及 (1.43) 即有

$$
\begin{aligned}
I_{m+1}(h^{\otimes(m+1)}) &= I_m(h^{\otimes m})I_1(h) - mI_{m-1}(h^{\otimes(m-1)}) \\
&= H_m(W_h)W_h - mH_{m-1}(W_h) \\
&= H_{m+1}(W_h),
\end{aligned}
$$

于是当 $n=m+1$ 时 (1.45) 式仍成立.

因为

$$H_\alpha=\prod_j H_{\alpha_j}(W_{h_j})=\prod_j I_{\alpha_j}(h_j^{\otimes\alpha_j}),$$

注意当 $j\neq k, r>0$ 时, $h_j^{\otimes\alpha_j}\otimes_r h_k^{\otimes\alpha_k}=0$, 由 (1.44) 式可知

$$I_{\alpha_j}(h_j^{\otimes\alpha_j})I_{\alpha_k}(h_k^{\otimes\alpha_k})=I_{\alpha_j+\alpha_k}(h_j^{\otimes\alpha_j}\widehat{\otimes}h_k^{\otimes\alpha_k}),$$

故

$$H_\alpha=I_{|\alpha|}(\widehat{\otimes}_j h_j^{\otimes\alpha_j})=I_{|\alpha|}(\widehat{h}_\alpha),$$

(1.46) 式得证. ∎

我们看到, 当 $H=L^2(T,\mathcal{B},\lambda)$ 时, 同构关系 (1.34), 从而 (1.32) 可以通过多重随机积分来表示.

定理 1.10 设 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间, 其中 $H=L^2(T,\mathcal{B},\lambda)$, λ 为 $(T,\mathcal{B})$ 上的 σ- 有限无原子测度. 则一切 $F\in L^2(\Omega,\mathcal{F},\mu)$, 存在唯一正交分解:

$$F=\sum_{n=0}^\infty I_n(f_n), \tag{1.47}$$

其中 $f_n\in H^{\widehat{\otimes}n}, n\geq 1, I_0(f_0)=\mathbb{E}[F]$, 且

$$\|F\|^2=(\mathbb{E}[F])^2+\sum_{n=1}^\infty n!\|f_n\|^2, \tag{1.48}$$

其中 $\|f_n\|$ 表示在 $L^2(T^n,\mathcal{B}^n,\lambda^n)$ 中的范数.

证明 由定理 1.5, 命题 1.6 及定理 1.9 即得. ∎

例 指数泛函

$$\mathcal{E}(h)\equiv\exp\Big\{W_h-\tfrac{1}{2}\|h\|^2\Big\},\quad h\in H \tag{1.49}$$

具有正交分解:

$$\mathcal{E}(h)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h^{\otimes n}). \tag{1.50}$$

事实上，由 (1.45) 式可知

$$I_n(h^{\otimes n})=\|h\|^n H_n(\|h\|^{-1}W_h).$$

在 (A.2) 式中令 $t=\|h\|, u=\|h\|^{-1}W_h$, 即得 (1.50) 式. 注意在同构关系 (1.32) 式中，指数泛函对应于指数向量 (I.2.32). 由命题 I.2.14 可知，指数泛函族 $\{\mathcal{E}(h), h\in H\}$ 为线性独立集，所生成的线性空间在 $L^2(\Omega)$ 中稠密.

当 $T=[0,1], \lambda$ 为 Lebesgue 测度时，我们得到经典 Wiener 空间，此时 $W_t=W([0,t]), t\geq 0$ 为 Brown 运动,

$$I_n(f)=n!\int_0^1\int_0^{t_n}\cdots\int_0^{t_2}f(t_1,\cdots,t_n)dW_{t_1}\cdots dW_{t_n} \tag{1.51}$$

为 n 次叠代 Itô 积分.

§2. 泛函的微分运算、梯度与散度

2.1 有限维 Gauss 概率空间

为研究无穷维 Gauss 概率空间上泛函 (随机变量) 的微分运算，我们先对多项式或光滑泛函定义梯度、散度及 Ornstein-Uhlenbeck 算子，然后借助于这些算子定义一系列的 Sobolev 范数，并将这些算子的定义域延拓到相应的 Sobolev 空间上去. 其中关键是利用了无穷维 Gauss 测度的拟不变性质以及 Ornstein-Uhlenbeck 半群的所谓超压缩性质.

然而对多项式或光滑泛函的分析只涉及有限多个变量，实质上只是有限维的. 因此，我们首先讨论有限维 Gauss 概率空间.

如同上节例 1. 设 $\mu=\gamma^n$ 为 $\mathbb{R}^n$ 上的标准 Gauss 测度，$(\mathbb{R}^n,\mathcal{F},\mu;\mathbb{R}^n)$ 为有限维 Gauss 概率空间，若 E 为任一可分 Hilbert 空间，则 $\mathcal{S}_M(E)$ 表示 E 值光滑泛函总体，特别当 $E=\mathbb{R}$ 时简记为 $\mathcal{S}_M$.

设 $\varphi\in\mathcal{S}_M$ 为 $\mathbb{R}^n$ 上的光滑泛函，其梯度 $D\varphi\equiv\{\partial_j\varphi\}_{1\le j\le n}$ 为 $\mathbb{R}^n$ 值光滑泛函，即 $D\varphi\in\mathcal{S}_M(\mathbb{R}^n)$. 对 $x,h\in\mathbb{R}^n,\varphi$ 在 x 点沿 h 方向的导数为

$$\begin{aligned}D_h\varphi(x)&=\lim_{\epsilon\to0}\epsilon^{-1}[\varphi(x+\epsilon h)-\varphi(x)]\\&=\sum_{j=1}^n h_j\partial_j\varphi(x)\\&=(D\varphi(x),h).\end{aligned}\tag{2.1}$$

若 $\psi\in\mathcal{S}_M(E)$ 为 $\mathbb{R}^n$ 上的 E 值光滑泛函，则其 **梯度**$D\psi\in\mathcal{S}_M(\mathbb{R}^n\otimes E)$, 它可由下式唯一确定:

$$\begin{aligned}(D\psi(x),h\otimes e)_{\mathbb{R}^n\otimes E}=\frac{\partial}{\partial\epsilon}\Big[(\psi(x+\epsilon h),e)_E\Big]_{\epsilon=0},\\x,h\in\mathbb{R}^n,e\in E.\end{aligned}\tag{2.2}$$

特别，对 $\varphi\in\mathcal{S}_M,D^2\varphi\equiv D(D\varphi)\in\mathcal{S}_M(\mathbb{R}^n\otimes\mathbb{R}^n),\cdots\cdots,D^m\varphi\equiv D(D^{m-1}\varphi)\in\mathcal{S}_M((\mathbb{R}^n)^{\otimes m})$.

我们知道，若 $\psi\in\mathcal{S}_M(\mathbb{R}^n)$, 则 $D\psi\in\mathcal{S}_M(\mathbb{R}^n\otimes\mathbb{R}^n)$ 为其 Jacobi 矩阵，其对角线元素之和为

$$\operatorname{div}\psi(x)\equiv\operatorname{Tr}(D\psi(x)),\tag{2.3}$$

称为 ψ 的散度. 显然， $\operatorname{div}\psi\in\mathcal{S}_M$. 然而在 Gauss 概率空间，更方便的是考虑如下算子:

$$\begin{aligned}\delta\psi(x)&=\sum_{j=1}^n(x_j\psi_j(x)-\partial_j\psi_j(x))\\&=(\psi(x),x)-\operatorname{Tr}(D\psi(x)).\end{aligned}\tag{2.4}$$

如用上节 (1.16) 式的符号，则 $\delta\psi = \sum_{j=1}^{n} \partial_j^* \psi_j$，其中 ∂_j^* 为 ∂_j 在 $L^2(\mathbb{R},\gamma)$ 中的共轭.

一般地，若 $\psi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$，则其 **散度**$\delta\psi \in \mathcal{S}_M(E)$ 由 (2.4) 式定义. 我们有以下分部积分公式.

命题 2.1 若 $\varphi \in \mathcal{S}_M(E), \psi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$，则 $\delta\psi \in \mathcal{S}_M(E)$，$D\varphi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$，且

$$\int_{\mathbb{R}^n} (D\varphi(x), \psi(x))_{\mathbb{R}^n \otimes E} \mu(dx) = \int_{\mathbb{R}^n} (\varphi(x), \delta\psi(x))_E \mu(dx). \tag{2.5}$$

证明 只需就 $E = \mathbb{R}$ 的情形加以证明. 由各分量的独立性，(2.5) 式归结为一维分部积分公式 (1.18). ∎

对 $\varphi \in \mathcal{S}_M(E)$，定义

$$\begin{aligned}\mathcal{L}\varphi(x) \equiv -\delta D\varphi(x) &= -\sum_{j=1}^{n} \partial_j^* \partial_j \varphi(x) \\ &= \Delta\varphi(x) - \sum_{j=1}^{n} x_j \partial_j \varphi(x).\end{aligned} \tag{2.6}$$

$\mathcal{L}$ 称为 **Ornstein-Uhlenbeck算子**. 由 (2.5) 式可知，D, δ 及 $\mathcal{L}$ 均可闭. 若其闭包仍分别记为 D, δ 及 $\mathcal{L}$，则 δ 与 D 共轭，$\mathcal{L}$ 为 $L^2(\mathbb{R}^n, \mu)$ 中自共轭算子.

由扩散过程理论，$\mathcal{L}$- 扩散过程为以下随机微分方程之唯一强解：

$$dX_t = -X_t dt + \sqrt{2} dW_t, \quad X_0 = x \in \mathbb{R}^n. \tag{2.7}$$

此解有明显的表达式：

$$X_t = e^{-t}x + \sqrt{2}\int_0^t e^{-(t-s)} dW_s, \quad t \in \mathbb{R}_+. \tag{2.8}$$

因为随机向量 $(X_t - e^{-t}x)(1 - e^{-2t})^{-1/2}$ 服从 n 维标准 Gauss 分

布，故由 $\mathcal{L}$ 生成的转移半群有以下形式：

$$
\begin{aligned}
(T_t\varphi)(x) &\equiv \mathbb{E}_x[\varphi(X_t)] \\
&= \int_{\mathbb{R}^n} \varphi(e^{-t}x+\sqrt{1-e^{-2t}}y)\mu(dy) \\
&\qquad t\geq 0, x\in\mathbb{R}^n, \varphi\in L^2(\mu).
\end{aligned} \tag{2.9}
$$

值得注意的是，(2.9) 式右边对 $\varphi\in L^p(\mu), p\geq 1$ 仍有意义，而且定义了 $L^p(\mu)$ 中的压缩算子半群.

命题 2.2 一切 $p\geq 1$, 由 (2.9) 式右边定义的 $\{T_t, t\geq 0\}$ 为 $L^p(\mathbb{R}^n,\mu)$ 中的压缩算子半群.

证明 考虑乘积空间 $\mathbb{R}^n\times\mathbb{R}^n$ 到 $\mathbb{R}^n$ 的变换 $Q_t:(x,y)\longmapsto e^{-t}x+\sqrt{1-e^{-2t}}y$. 由 Gauss 测度旋转不变性可知： $\mu=(\mu\times\mu)\circ Q_t^{-1}$, 从而 $\forall p\geq 1$,

$$
\begin{aligned}
\int_{\mathbb{R}^n\times\mathbb{R}^n} &|\varphi(e^{-t}x+\sqrt{1-e^{-2t}}y)|^p\mu(dx)\mu(dy) \\
&= \int_{\mathbb{R}^n} |\varphi(x)|^p\mu(dx).
\end{aligned} \tag{2.10}
$$

由此推出 $\|T_t\varphi\|_p\leq\|\varphi\|_p$. ∎

下面我们要证明， Ornstein-Uhlenbeck 半群具有一种更强的压缩性质，称为 **超压缩性 (hyper-contractivity)**.

定理 2.3(Nelson[1]) 设 $\{T_t, t\geq 0\}$ 为由 (2.9) 定义的 Ornstein-Uhlenbeck 半群. 对 $p>1, t>0$, 令 $q(t)=e^{2t}(p-1)+1>p$, 则对一切 $\varphi\in L^p(\mathbb{R}^n,\mu)$ 有

$$
\|T_t\varphi\|_{q(t)}\leq\|\varphi\|_p. \tag{2.11}
$$

证明 任意固定 $t>0$, 令 $q=q(t), q^*$ 为 q 的共轭指数，即 $q^{-1}+(q^*)^{-1}=1, a=e^{-t}$. 为证 (2.11) 式，只需证明 $\forall\psi\in L^{q^*}(\mathbb{R}^n,\mu)$ 我们有

$$
\left|\int_{\mathbb{R}^n}\psi T_t\varphi d\mu\right|\leq\|\varphi\|_p\|\psi\|_{q^*}. \tag{2.12}
$$

由于 T_t 非负, $|T_t\varphi| \le T_t(|\varphi|)$, 故不妨假定 φ 及 ψ 非负; 利用有界函数逼近, 不妨假定 φ 及 ψ 有上界及正的下界.

设 $\{B_t, 0 \le t \le 1\}$ 及 $\{\widetilde{B}_t, 0 \le t \le 1\}$ 为任一概率空间 $(\Omega, \mathcal{F}, P)$ 上的相互独立的两个 Brown 运动, 生成 σ- 代数流 $\{\mathcal{F}_t, 0 \le t \le 1\}$. 作正交变换:

$$\begin{cases} B'_t = aB_t + \sqrt{1-a^2}\widetilde{B}_t & (0 \le t \le 1), \\ \widetilde{B}'_t = \sqrt{1-a^2}B_t - a\widetilde{B}_t & (0 \le t \le 1), \end{cases}$$

则 B'_t 及 $\widetilde{B}'_t$ 关于 $\mathcal{F}_t$ 仍为相互独立的两个 Brown 运动. 注意 $\mathbb{E}[\varphi(B'_1)^p] = \|\varphi\|_p^p, \mathbb{E}[\psi(B_1)^{q^*}] = \|\psi\|_{q^*}^{q^*}$, 由随机积分表现定理, 存在 n 维循序可测过程 X_t 及 Y_t 使

$$\varphi(B'_1)^p = \|\varphi\|_p^p + \int_0^1 (X_s, dB'_s),$$

$$\psi(B_1)^{q^*} = \|\psi\|_{q^*}^{q^*} + \int_0^1 (Y_s, dB_s).$$

应用 Itô 公式于如下有界正鞅

$$M_t \equiv \|\varphi\|_p^p + \int_0^t (X_s, dB'_s), \quad 0 \le t \le 1$$

及

$$N_t \equiv \|\psi\|_{q^*}^{q^*} + \int_0^t (Y_s, dB_s), \quad 0 \le t \le 1$$

得

$$
\begin{aligned}
\varphi(B_1')\psi(B_1) &= M_1^{1/p}N_1^{1/q^*} \\
&= M_0^{1/p}N_0^{1/q^*} + \int_0^1 \frac{1}{p}M_s^{1/p-1}N_s^{1/q^*}dM_s \\
&\quad + \int_0^1 \frac{1}{q^*}M_s^{1/p}N_s^{1/q^*-1}dN_s \\
&\quad + \int_0^1 \frac{1}{2p}\Big(\frac{1}{p}-1\Big)M_s^{1/p-2}N_s^{1/q^*}|X_s|^2ds \\
&\quad + \int_0^1 \frac{a}{pq^*}M_s^{1/p-1}N_s^{1/q^*-1}(X_s,Y_s)ds \\
&\quad + \int_0^1 \frac{1}{2q^*}\Big(\frac{1}{q*}-1\Big)M_s^{1/p}N_s^{1/q^*-2}|Y_s|^2ds.
\end{aligned}
$$

于是

$$
\begin{aligned}
\mathbb{E}[\varphi(B_1')\psi(B_1)] &= \|\varphi\|_p\|\psi\|_{q^*} \\
&\quad - \frac{1}{2}\mathbb{E}\Big[\int_0^1 M_s^{1/p-2}N_s^{1/q^*-2}\Big\{\frac{1}{p}\Big(1-\frac{1}{p}\Big)N_s^2|X_s|^2 \\
&\quad - \frac{2a}{pq^*}M_sN_s(X_s,Y_s) + \frac{1}{q^*}\Big(1-\frac{1}{q^*}\Big)M_s^2|Y_s|^2\Big\}ds\Big].
\end{aligned}
$$

因为 $a^2=(p-1)/(q-1)=(p-1)(q^*-1)$, 且 $(X_s,Y_s)^2\le|X_s|^2|Y_s|^2$, 上式右边花括号内关于 M_s 和 N_s 的二次三项式非负，从而有

$$
\mathbb{E}[\varphi(B_1')\psi(B_1)]\le\|\varphi\|_p\|\psi\|_{q^*},
$$

但

$$
\begin{aligned}
\mathbb{E}[\varphi(B_1')\psi(B_1)] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varphi(ax+\sqrt{1-a^2}y)\psi(x)\mu(dx)\mu(dy) \\
&= \int_{\mathbb{R}^n}\psi(x)T_t\varphi(x)\mu(dx),
\end{aligned}
$$

故 (2.12) 式得证. ∎

2.2 光滑泛函的梯度与散度

现在我们要将算子 $D, \delta, \mathcal{L}$ 的定义推广到无穷维 Gauss 概率空间情形. 我们知道, 有限维空间中微分的定义利用了 Lebesgue 测度的平移不变性质. 既然有限维 Gauss 测度与 Lebesgue 测度等价 (相互绝对连续), 因此在有限维 Gauss 空间定义微分运算不会遇到什么困难. 但是在无穷维空间, 却不存在平移不变的测度. 我们定义微分运算是利用 Gauss 测度的所谓拟不变性质, 即相对于某个稠密线性子空间中的"方向"而言, 平移算子将零测度集变为零测度集, 这是 Cameron-Martin 定理的一个结论.

设 $(\Omega, \mathcal{F}, \mu; H)$ 为一无穷维 Gauss 概率空间. 我们定义

$$L^{\infty -} \equiv \bigcap_{1<p<\infty} L^p(\Omega, \mathcal{F}, \mu), \tag{2.13}$$

$$L^{1+} \equiv \bigcup_{1<p<\infty} L^p(\Omega, \mathcal{F}, \mu) \tag{2.14}$$

分别为投影极限和归纳极限, 则 $L^{\infty-}$ 为赋可列范空间, L^{1+} 为其拓扑对偶空间. 利用 Hölder 不等式容易证明, $L^{\infty-}$ 为一代数且乘积运算连续.

在上节例 3 的抽象 Wiener 空间 (H, X, μ) 中, H 是 X 的线性稠密子空间, 对 X 上的泛函 f, 可以定义沿 $h \in H$ 方向的平移算子: $\tau_h f(x) \equiv f(x+h)$. 对于一般的 Gauss 概率空间, 借助于数值模型, 也可以得到 H 中加群的表现.

定义 2.4 设 $(\Omega, \mathcal{F}, \mu; H)$ 为不可约 Gauss 概率空间, 任选 H 之一组基, 得同构映射 $J: H \to l^2$, 数值模型 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$ 及同构映射

$$T_*: L^{1+}(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty) \longrightarrow L^{1+}(\Omega, \mathcal{F}, \mu).$$

对 $h \in H$, 定义 $L^{1+}(\Omega, \mathcal{F}, \mu)$ 中的算子:

$$\rho(h) \equiv T_* \circ \tau_{J(h)} \circ T_*^{-1}, \tag{2.15}$$

其中 τ 为作用于 $\mathbb{R}^\infty$ 上泛函的平移算子. 易见, $\rho(h)$ 限制于 $L^{\infty-}$ 为自同态映射, 且

$$\rho(h+g)=\rho(h)\rho(g),\quad h,g\in H. \tag{2.16}$$

此外, 定义 (2.15) 不依赖于基的选取, 从而是内蕴的. 称 ρ 为 H 中加群的 **典则表现**.

定理 2.5 (Cameron-Martin) 设 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间, ρ 为 H 中加群的典则表现, $\mathcal{E}(h)=\exp\{W_h-\frac{1}{2}\|h\|^2\}$ $(h\in H)$ 为指数泛函, 则 $\mathcal{E}(h)\in L^{\infty-}$ 且

$$\|\mathcal{E}(h)\|_p\le\exp\left\{\frac{p-1}{2}\|h\|^2\right\},\quad 1<p<\infty. \tag{2.17}$$

对 $f\in L^{1+}$ 有

$$\mathbb{E}[\rho(h)f]=\mathbb{E}[\mathcal{E}(h)f],\quad h\in H. \tag{2.18}$$

此外

$$\lim_{t\to 0}t^{-1}(\mathcal{E}(th)-1)=W_h. \tag{2.19}$$

证明 由典则表现的内蕴性质, 我们不妨考虑数值模型 $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$. 对 $h=\{h_j\}\in l^2, x=\{x_j\}\in\mathbb{R}^\infty$ 及 $n\in\mathbb{N}$, 令

$$\mathcal{E}_n(h)(x)\equiv\exp\left\{\sum_{j=1}^n h_jx_j-\frac{1}{2}\sum_{j=1}^n h_j^2\right\},$$

$\mathcal{F}_n$ 为 x 之前 n 个坐标函数生成的 σ- 代数. 容易看出, $\{\mathcal{E}_n(h),\mathcal{F}_n: n\in\mathbb{N}\}$ 为鞅, 且 $\mathbb{E}[\mathcal{E}_n(h)]=1$. 若 f 只依赖于前 m 个坐标, 则当 $n\ge m$ 时, 由 $\mathbb{R}^m$ 中积分变量替换得

$$\mathbb{E}[\rho(h)f]=\mathbb{E}[\tau_hf]=\mathbb{E}[\mathcal{E}_n(h)f]. \tag{2.20}$$

对 $p\in(1,\infty)$. 由 $\mathbb{E}[\mathcal{E}_n(ph)]=1$ 可知

$$\mathbb{E}\left[\exp\left\{p\sum_{j=1}^n h_jx_j\right\}\right]\le\exp\left\{\frac{p^2}{2}\|h\|^2\right\},$$

从而 $\{\mathcal{E}_n(h),\mathcal{F}_n;n\in\mathbb{N}\}$ 为 L^p 鞅, 在 L^p 中收敛于 $\mathcal{E}(h)$, 且

$$\|\mathcal{E}(h)\|_p\le\exp\left\{\frac{p-1}{2}\|h\|^2\right\}.$$

在 (2.20) 中取极限即可证明 (2.18) 式, (2.19) 式是显然的. ∎

(2.18) 式表明, Gauss 测度 μ 在 "平移" $\rho(h)$ 下具有某种不变性质, $\rho(h)^*\mu$ 关于 μ 绝对连续, 其 Radon-Nikodym 导数恰为 $\mathcal{E}(h)$. 有了这种拟不变性质, 我们才有可能定义无穷维 Gauss 概率空间上的微分运算.

定义 2.6 设 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间, ρ 为 H 中加群的典则表现. 对光滑泛函 $F\in\mathcal{S}_M$, 其 **梯度** $DF\in\mathcal{S}_M(H)$ 由下式确定:

$$(DF,h)_H=\lim_{\epsilon\to0}\epsilon^{-1}[\rho(\epsilon h)F-F],\quad h\in H.\tag{2.21}$$

一般地, 若 E 为可分 Hilbert 空间, $F\in\mathcal{S}_M(E)$, 其 **梯度** $DF\in\mathcal{S}_M(H\otimes E)$ 由下式确定:

$$(DF,h\otimes e)_{H\otimes E}=\lim_{\epsilon\to0}\epsilon^{-1}(\rho(\epsilon h)F-F,e)_E,\quad h\in H,e\in E.\tag{2.22}$$

下一命题给出了光滑泛函的梯度的具体表达式, 从而验证了上述定义的合理性.

命题 2.7 若 $F\in\mathcal{S}_M$ 具有形式:

$$F=f(W_{h_1},\cdots,W_{h_n}),\quad h_1,\cdots,h_n\in H,$$

则 $\forall h\in H$ 有

$$(DF,h)=\sum_{j=1}^n\partial_jf(W_{h_1},\cdots,W_{h_n})(h_j,h),\tag{2.23}$$

且

$$\mathbb{E}[(DF,h)]=\mathbb{E}[FW_h].\tag{2.24}$$

证明 借助正交化手续，不妨设 $(h_i,h_j)=\delta_{ij}(i,j=1,\cdots,n)$ 并将其扩充为 H 之一组基 $\{h_j\}_{j\in\mathbb{N}}$. 于是，$h=\sum_j(h_j,h)h_j$. 考虑数值模型，由定义可知

$$
\begin{aligned}
(DF,h)&=\frac{\partial}{\partial\epsilon}\Big[f(W_{h_1}+\epsilon(h_1,h),\cdots,W_{h_n}+\epsilon(h_n,h))\Big]_{\epsilon=0}\\
&=\sum_{j=1}^{n}\partial_j f(W_{h_1},\cdots,W_{h_n})(h_j,h).
\end{aligned}
$$

由分部积分公式得

$$
\begin{aligned}
\mathbb{E}[FW_h]&=\mathbb{E}\Big[F\sum_j(h_j,h)W_{h_j}\Big]\\
&=\sum_{j=1}^{n}(h_j,h)\int_{\mathbb{R}^n}x_j f(x_1,\cdots,x_n)\gamma^n(dx)\\
&=\sum_{j=1}^{n}(h_j,h)\int_{\mathbb{R}^n}\partial_j f(x_1,\cdots,x_n)\gamma^n(dx)\\
&=\mathbb{E}[(DF,h)].
\end{aligned}
$$

∎

注 对光滑泛函的微分运算，实质上是有限维的，利用有限维 Gauss 概率空间的结果容易得到梯度的如下性质：

$$D(FG)=FDG+GDF,\quad G,F\in\mathcal{S}_M. \tag{2.25}$$

若 $F=f(\varphi_1,\cdots,\varphi_n),\varphi_1,\cdots,\varphi_n\in\mathcal{S}_M$, f 为 $\mathbb{R}^n$ 上的光滑泛函，则

$$DF=\sum_{j=1}^{n}\partial_j f(\varphi_1,\cdots,\varphi_n)D\varphi_j. \tag{2.26}$$

若 $F\in\mathcal{S}_M(E)$ 具有形式 (1.11), 则

$$DF=\sum_{k=1}^{m}DF_k\otimes e_k. \tag{2.27}$$

(2.24) 式可推广为

$$\mathbb{E}[(DF, h\otimes e)] = \mathbb{E}[(F,e)W_h],\ h\in H, e\in E. \tag{2.28}$$

又若 $G\in\mathcal{S}_M$, 则 $GF\in\mathcal{S}_M(E)$, 且

$$(D(GF), h\otimes e) = G(DF, h\otimes e) + (DG, h)(F, e). \tag{2.29}$$

从而

$$\mathbb{E}[G(DF, h\otimes e)] = \mathbb{E}[GW_h(F,e)] - \mathbb{E}[(DG,h)(F,e)]. \tag{2.30}$$

读者试自行证明之.

E 值泛函梯度的定义 (2.22) 蕴含了高阶微分: 设 $F\in\mathcal{S}_M(E)$, 则 $DF\in\mathcal{S}_M(H\otimes E)$. 定义 $D^2F\equiv D(DF)\in\mathcal{S}_M(H\otimes H\otimes E),\cdots$, 如此下去, 可对 $k\geq 1$ 定义 $D^kF = D(D^{k-1}F)\in\mathcal{S}_M(H^{\otimes k}\otimes E)$.

现在来讨论 D 的共轭算子. 因梯度 DF 为 H 值泛函, 故 H 相当于切空间, H 值泛函即向量场, 而 D 的共轭算子 δ 相当于向量场的散度.

定义 2.8 对光滑向量场 $V\in\mathcal{S}_M(H)$, 其 **散度** $\delta V\in\mathcal{S}_M$ 由下式确定:

$$\mathbb{E}[G\delta V] = \mathbb{E}[(DG, V)_H],\quad \forall G\in\mathcal{S}_M. \tag{2.31}$$

一般地, 若 E 为可分 Hilbert 空间, 对 $V\in\mathcal{S}_M(H\otimes E)$, 其 **散度** $\delta V\in\mathcal{S}_M(E)$ 由下式确定:

$$\mathbb{E}[(G,\delta V)_E] = \mathbb{E}[(DG, V)_{H\otimes E}],\quad \forall G\in\mathcal{S}_M(E). \tag{2.32}$$

下一命题给出了光滑向量场的散度的具体表达式.

命题 2.9 若 $V\in\mathcal{S}_M(H)$ 具有形式:

$$V = \sum_{k=1}^m F_k h_k,\quad F_k\in\mathcal{S}_M, h_k\in H, k = 1,\cdots,m,$$

则

$$\delta V = \sum_{k=1}^m (F_k W_{h_k} - (DF_k, h_k)). \tag{2.33}$$

若 $F \in \mathcal{S}_M, V \in \mathcal{S}_M(H)$, 则 $FV \in \mathcal{S}_M(H)$, 且

$$\delta(FV) = F\delta V - (DF, V). \tag{2.34}$$

证明 根据 (2.30) 式, $\forall G \in \mathcal{S}_M$ 及 $k = 1, \cdots, m$ 有

$$\mathbb{E}[G(DF_k, h_k)] = \mathbb{E}[GF_k W_{h_k}] - \mathbb{E}[F_k(DG, h_k)].$$

对 k 求和即可看出由 (2.33) 式给出的 δV 满足定义式 (2.31). 在 (2.33) 式中以 FF_k 代 F_k, 并注意到微分公式 (2.25), 即得 (2.34) 式. ∎

注 若 $V \equiv h \in H$ 为常向量场, 由 (2.33) 式可知, $\delta h = W_h$. 因此 (2.24) 式是 (2.31) 的特殊情形. 由 (2.34) 式, $\forall F \in \mathcal{S}_M$ 有

$$\delta(Fh) = FW_h - (DF, h). \tag{2.35}$$

若记 $D_h F \equiv (DF, h), \delta_h F \equiv \delta(Fh)$, 上式可写成

$$\delta_h + D_h = W_h \cdot, \tag{2.36}$$

其中 $W_h \cdot$ 表示乘以 W_h 的算子. 在 (2.31) 式中令 $V = Fh$ 即得

$$\mathbb{E}[G\delta_h F] = \mathbb{E}[FD_h G], \quad F, G \in \mathcal{S}_M. \tag{2.37}$$

因为 $\mathcal{S}_M$ 在 L^2 中稠密, 由定理 I.1.5 可知, D_h 及 δ_h 均可闭, 其闭包相互共轭.

同样道理, 由 (2.32) 式, 对任意可分 Hilbert 空间 E, D 作为 $L^2(E)$ 到 $L^2(H \otimes E)$ 中的稠定线性算子, δ 作为 $L^2(H \otimes E)$ 到 $L^2(E)$ 中稠定线性算子, 它们都是可闭的, 其闭包相互共轭. 我们将其闭包仍记为 D 和 δ, 下面就来确定它们的定义域.

2.3 泛函的 Sobolev 空间

为了记号简明, 在不致混淆的情况下, $L^p(\Omega; E)$ 中的范数一律记为 $\|\cdot\|_p$ 而不论 E 是哪一个 Hilbert 空间.

定义 2.10 对 $k \in \mathbb{N}, 1 \leq p < \infty, F \in \mathcal{S}_M(E)$, 令

$$\|F\|_{k,p} \equiv \left(\|F\|_p^p + \sum_{j=1}^{k} \|D^j F\|_p^p\right)^{1/p}. \tag{2.38}$$

以 $\mathbb{D}_k^p(E)$ 表示 $\mathcal{S}_M(E)$ 关于范数 $\|\cdot\|_{k,p}$ 完备化所得的 Banach 空间. 当 $E = \mathbb{R}$ 时简记为 $\mathbb{D}_k^p$.

特别, 当 $p = 2$ 时, $\mathbb{D}_k^2$ 为 Hilbert 空间, 具有内积:

$$(F, G)_{k,2} \equiv (F, G)_2 + \sum_{j=1}^{k} (D^j F, D^j G)_2. \tag{2.39}$$

容易看出, 作为 L^2 到 $L^2(H)$ 中的算子, $\mathbb{D}_1^2$ 就是梯度的定义域.

命题 2.11 范数族 $\{\|\cdot\|_{k,p}; k \in \mathbb{N}, 1 \leq p < \infty\}$ 具有以下性质:

1° **单调性:** 若 $1 \leq p \leq q < \infty, k \leq l$, 则 $\|\cdot\|_{k,p} \leq \|\cdot\|_{l,q}$;

2° **相容性:** 对一切 $p, q \in [1, \infty), k, l \in \mathbb{N}$, 若 $\{F_n\} \subset \mathcal{S}_M$, $\lim_{n\to\infty} \|F_n\|_{k,p} = 0$ 且同时为 $\|\cdot\|_{l,q}$- 基本列, 则 $\lim_{n\to\infty} \|F_n\|_{l,q} = 0$.

证明 单调性很明显, 现证其相容性. 在所给情形下, 对 $j \in \mathbb{N}, j \leq l, \{D^j F_n\}$ 为 $L^q(H^{\otimes j})$ 中的基本列, 由 $L^q(H^{\otimes j})$ 的完备性, 存在极限 G_j. 为证 $G_j = 0$, 我们用归纳法: 显然 $G_0 = 0$; 设 $G_1 = \cdots = G_{j-1} = 0$, 则对一切有界光滑泛函 F 及 $h_1, \cdots, h_j \in H$, 由 (2.30) 式有

$$\begin{aligned}
&\mathbb{E}[F(G_j, \otimes_{i=1}^{j} h_i)] \\
&= \lim_{n\to\infty} \mathbb{E}[F(D^j F_n, \otimes_{i=1}^{j} h_i)] \\
&= \lim_{n\to\infty} \{\mathbb{E}[F W_{h_j}(D^{j-1} F_n, \otimes_{i=1}^{j-1} h_i)] \\
&\qquad - \mathbb{E}[(DF, h_j)(D^{j-1} F_n, \otimes_{i=1}^{j-1} h_i)]\} \\
&= \mathbb{E}[F W_{h_j}(G_{j-1}, \otimes_{i=1}^{j-1} h_i)] \\
&\qquad - \mathbb{E}[(DF, h_j)(G_{j-1}, \otimes_{i=1}^{j-1} h_i)] \\
&= 0,
\end{aligned}$$

从而 $G_j = 0$. ∎

根椐范数族的相容性，我们可以定义 $\{\mathbb{D}_k^p(E)\}$ 的投影极限.

定义 2.12 对 $k \in \mathbb{N}$, 令

$$\mathbb{D}_k^\infty(E) \equiv \bigcap_{1<p<\infty} \mathbb{D}_k^p(E), \tag{2.40}$$

$$\mathbb{D}^\infty(E) \equiv \bigcap_{k\in\mathbb{N}} \bigcap_{1<p<\infty} \mathbb{D}_k^p(E) \tag{2.41}$$

为拓扑投影极限. 它们都是赋可列范空间. 当 $E = \mathbb{R}$ 时简记为 $\mathbb{D}_k^\infty$ 和 $\mathbb{D}^\infty$.

我们约定，当 $k = 0$ 时 $\mathbb{D}_0^p(E) \equiv L^p(E), \mathbb{D}_0^\infty(E) \equiv L^{\infty-}(E)$.

由定义 (2.38) 式及 $\mathcal{S}_M(E)$ 在一切 $L^p(E)$ 中的稠密性可知，$\forall k \in \mathbb{N}$ 及 $p \geq 1$, 梯度算子 D 可开拓为 $\mathbb{D}_k^p(E)$ 到 $\mathbb{D}_{k-1}^p(H \otimes E)$ 中的连续线性算子，从而也是 $\mathbb{D}_k^\infty(E)$ 到 $\mathbb{D}_k^\infty(H \otimes E)$ 和 $\mathbb{D}^\infty(E)$ 到 $\mathbb{D}^\infty(H \otimes E)$ 中的连续线性算子. 特别，公式 (2.24)–(2.26) 及 (2.28)–(2.30) 均可推广到 $\mathbb{D}_1^\infty$ 及 $\mathbb{D}_1^\infty(E)$ 上去，并由 (2.25) 式可知 $\mathbb{D}_1^\infty$ 为一代数.

现在假定 $H = L^2(T, \mathcal{B}, \lambda), \lambda$ 为 $(T, \mathcal{B})$ 上 σ- 有限无原子测度. 此时 H 值泛函可以看作以 T 为参数集的随机过程 (将 $L^2(\Omega; H)$ 等同于 $L^2(T \times \Omega)$). 特别，若 $F \in \mathbb{D}_1^2$, 则 $DF \in L^2(H), DF$ 可以看作随机过程 $\{D_t F; t \in T\}$, 因此

$$(DF, h) = \int_T (D_t F) h(t) dt. \tag{2.42}$$

若 $F \in \mathcal{S}_M$ 具有形式 $F = f(W_{h_1}, \cdots, W_{h_n}), h_1, \cdots, h_n \in H$, 则

$$D_t F = \sum_{j=1}^n \partial_j f(W_{h_1}, \cdots, W_{h_n}) h_j(t). \tag{2.43}$$

一般地，$D^k F$ 可以看作 k 参数过程:

$$D^k_{t_1 \cdots t_k} F = D_{t_1} \cdots D_{t_k} F, \quad (t_1, \cdots, t_k) \in T^k.$$

当然，它只对 a.e. $(t_1,\cdots,t_k,\omega)$ $[\lambda^k\times\mu]$ 有定义.

我们已经知道， $F\in L^2$ 具有分解

$$F=\sum_{n=0}^{\infty}I_n(f_n),\quad f_n\in H^{\widehat{\otimes}n}. \tag{2.44}$$

下一命题告诉我们，当 $\{f_n\}$ 满足什么条件时 F 的导数存在以及如何计算它的导数.

命题 2.13 设 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间， $H=L^2(T,\mathcal{B},\lambda),F\in L^2(\Omega)$ 具有分解式 (2.44), 则 $F\in\mathbb{D}_1^2$ 的充要条件是

$$\sum_{n=1}^{\infty}nn!\|f_n\|^2<\infty, \tag{2.45}$$

此时

$$D_tF=\sum_{n=1}^{\infty}nI_{n-1}(f_n(\cdot,t)) \tag{2.46}$$

对 a.e.t 在 $L^2(\Omega)$ 中收敛，其中 $f_n(\cdot,t)$ 表示固定其中一个变元 t 作为其余 $n-1$ 个变元的对称函数，且

$$\begin{aligned}\mathbb{E}[\|DF\|^2]&=\mathbb{E}\left[\int_T(D_tF)^2\lambda(dt)\right]\\&=\sum_{n=1}^{\infty}nn!\|f_n\|^2.\end{aligned} \tag{2.47}$$

证明 先设 $F=I_n(h^{\otimes n})=H_n(W_h)$, 其中 $h\in H$ 且 $\|h\|=1$. 由 (2.43) 式

$$\begin{aligned}D_tF&=H_n'(W_h)h(t)=nH_{n-1}(W_h)h(t)\\&=nI_{n-1}(h^{\otimes(n-1)})h(t),\qquad \text{a.e.}t,\end{aligned}$$

即 (2.46) 式对 $F=I_n(f_n),f_n(t_1,\cdots,t_n)=h(t_1)\cdots h(t_n)$ 成立，且

$$\begin{aligned}\mathbb{E}\left[\int_T(D_tF)^2\lambda(dt)\right]&=n^2(n-1)!\|h\|^{2n}\\&=nn!\|f_n\|^2.\end{aligned}$$

由极化公式 (I.2.12) 可知，由 $\{h^{\otimes n}: h \in H\}$ 生成的线性子空间在 $H^{\widehat{\otimes} n}$ 中稠密，因此命题对于任一多重 Wiener 积分 $F = I_n(f_n)$, $f_n \in H^{\widehat{\otimes} n}$ 成立.

现设 F 具有分解式 (2.44). 对 $m \in \mathbb{N}$ 令 $F^{(m)} = \sum_{n=0}^{m} I_n(f_n)$, 则 $F^{(m)} \in \mathbb{D}_1^2$ 且

$$\mathbb{E}[\|DF^{(m)}\|^2] = \sum_{n=1}^{m} nn!\|f_n\|^2.$$

若条件 (2.45) 满足，则 $\{F^{(m)}\}_{m\in\mathbb{N}}$ 在 $\mathbb{D}_1^2$ 中收敛于 F, 从而 $F \in \mathbb{D}_1^2$; 反之，若 $F \in \mathbb{D}_1^2, G = I_n(h^{\otimes n}), h \in H$, 取 H 之基 $\{e_j\}$, 由分部积分公式 (2.30) 得

$$\begin{aligned}\lim_{m\to\infty} \mathbb{E}[G(DF^{(m)}, e_j)] &= \lim_{m\to\infty}(\mathbb{E}[GW_{e_j}F^{(m)}] - \mathbb{E}[F^{(m)}(DG, e_j)]) \\ &= \mathbb{E}[GW_{e_j}F] - \mathbb{E}[F(DG, e_j)] \\ &= \mathbb{E}[G(DF, e_j)], \qquad j = 1, 2, \cdots.\end{aligned}$$

当 $m > n$ 时

$$\mathbb{E}[G(DF^{(m)}, e_j)] = \mathbb{E}\left[(n+1)GI_n\left(\int_T f_{n+1}(\cdot, t)e_j(t)\lambda(dt)\right)\right],$$

即 (DF, e_j) 在 $\mathcal{H}_n$ 的投影为 $(n+1)I_n\left(\int_T f_{n+1}(\cdot, t)e_j(t)\lambda(dt)\right)$, 由此推出

$$\begin{aligned}\sum_{n=1}^{\infty} nn!\|f_n\|^2 &= \sum_{j=1}^{\infty} \mathbb{E}[(DF, e_j)^2] \\ &= \mathbb{E}[\|DF\|^2] < \infty.\end{aligned}$$ ∎

下面再计算向量场的散度. 设 $V \in L^2(\Omega; H)$, 即随机过程 $V = \{V_t, t \in T\}$ 满足

$$\int_T \mathbb{E}[V_t^2]\lambda(dt) < \infty.$$

因 D 的定义域为 $\mathbb{D}_1^2$, 由 (2.31) 式可知，其共轭算子 δ 的定义域为

$$\mathcal{D}(\delta)=\Big\{V\in L^2(\Omega;H):\ \exists c\geq 0\text{ 使 }\forall G\in\mathbb{D}_1^2$$
$$\Big|\mathbb{E}\Big[\int_T(D_tG)V_t\lambda(dt)\Big]\Big|\leq c\|G\|_2\text{ 成立}\Big\}. \tag{2.48}$$

若 $V\in L^2(\Omega;H)$, 则对 λ-a.e.t, $V_t\in L^2$ 有如下分解:

$$V_t=\sum_{n=0}^{\infty}I_n(f_{n+1}(\cdot,t)),\quad \text{a.e.}t, \tag{2.49}$$

式中 $f_{n+1}(\cdot,t)\in H^{\widehat{\otimes}n}$, 它关于前 n 个变量对称且依赖于参数 t, 我们总可选取 f_{n+1} 使之关于 $n+1$ 个变量可测，从而其对称化

$$\widetilde{f}_{n+1}(t_1,\cdots,t_n,t)=\frac{1}{n+1}\Big[f_{n+1}(t_1,\cdots,t_n,t)$$
$$+\sum_{j=1}^{n}f_{n+1}(t_1,\cdots,t_{j-1},t,t_{j+1},\cdots,t_n,t_j)\Big] \tag{2.50}$$

属于 $H^{\widehat{\otimes}(n+1)}$, 且

$$\mathbb{E}\Big[\int_T V_t^2\lambda(dt)\Big]=\sum_{n=1}^{\infty}n!\|\widetilde{f}_{n+1}\|^2<\infty. \tag{2.51}$$

命题 2.14 设 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间， $H=L^2(T,\mathcal{B},\lambda)$, $V\in L^2(\Omega;H)$ 具有分解式 (2.49), 则 $V\in\mathcal{D}(\delta)$ 之充要条件是

$$\sum_{n=1}^{\infty}n!\|\widetilde{f}_n\|^2<\infty. \tag{2.52}$$

此时

$$\delta V = \sum_{n=1}^{\infty} I_n(\widetilde{f}_n), \tag{2.53}$$

且

$$\mathbb{E}[(\delta V)^2] = \sum_{n=1}^{\infty} n!\|\widetilde{f}_n\|^2. \tag{2.54}$$

证明 记 (2.49) 式中前 $m+1$ 项和为 $V_t^{(m)}$,(2.53) 式右边级数前 m 项和为 $F^{(m)}$, 则 (2.52) 式前 m 项和为 $\mathbb{E}[(F^{(m)})^2]$. 容易看出, 当且仅当条件 (2.52) 满足时 (2.53) 式右边级数在 L^2 中收敛.

设 $G \in \mathbb{D}_1^2, G = \sum_n I_n(g_n), g_n \in H^{\widehat{\otimes} n}, n \in \mathbb{N}$. 由 (2.46) 式

$$\begin{aligned}\left| \mathbb{E}\left[\int_T (D_tG)V_t^{(m)}\lambda(dt)\right] \right| &= \left| \sum_{n=1}^{m} n!(\widetilde{f}_n, g_n) \right| \\ &\le \|F^{(m)}\|_2\|G\|_2.\end{aligned}$$

由 (2.48) 式可知, 当且仅当条件 (2.52) 满足时 $V \in \mathcal{D}(\delta)$, 此时 δV 由 (2.53) 式给出. ∎

注 若 $V \equiv h \in H$ 为常向量场, 则

$$\delta h = W_h = \int_T h(t)W(dt).$$

一般地, 可以将 δV 记为 $\int_T V_t W(dt)$. 在第三章中我们将看到, 它重合于 **Skorohod 积分**, 是 Itô 积分的推广.

§ 3. Meyer 不等式及其推论

3.1 Ornstein-Uhlenbeck 半群

设 $(\Omega, \mathcal{F}, \mu; H)$ 为不可约 Gauss 概率空间, E 为可分 Hilbert 空间. E 值平方可积泛函空间 $L^2(\Omega, \mathcal{F}, \mu; E) \cong L^2(\Omega, \mathcal{F}, \mu) \otimes E$

仍然具有混沌分解 (1.31). 为叙述简单起见, 本节只就 $E=I\!R$ 的情形予以证明, 但所有结果均可推广到一般 E 值泛函情形. 我们以 J_n 表示向子空间 $\mathcal{H}_n$ 的正交投影. 仿照有限维情形, 我们来构造无穷维空间上的 Ornstein-Uhlenbeck 半群.

定义 3.1 对 $F\in\mathcal{S}_M$, 定义

$$\mathcal{L}F\equiv-\delta DF. \tag{3.1}$$

由 δ 和 D 的共轭性可知其为本性自共轭, 其闭包为 L^2 中的自共轭算子, 称为 **Ornstein-Uhlenbeck 算子**(简称 **OU 算子**).

命题 3.2 若 $F\in\mathcal{S}_M$ 具有形式

$$F=f(W_{h_1},\cdots,W_{h_n}),\quad h_1,\cdots,h_n\in H,$$

则

$$\begin{aligned}\mathcal{L}F=&\sum_{j,k=1}^{n}\partial_k\partial_j f(W_{h_1},\cdots,W_{h_n})(h_k,h_j)\\&-\sum_{j=1}^{n}\partial_j f(W_{h_1},\cdots,W_{h_n})W_{h_j}.\end{aligned} \tag{3.2}$$

一般地我们有

$$\mathcal{L}=-\sum_{n=0}^{\infty}nJ_n, \tag{3.3}$$

$$\mathcal{D}(\mathcal{L})=\{F\in L^2:\textstyle\sum_n n^2\|J_nF\|^2<\infty\}. \tag{3.4}$$

证明 由 (2.23) 式

$$DF=\sum_{j=1}^{n}\partial_j f(W_{h_1},\cdots,W_{h_n})h_j,$$

再应用公式 (2.33) 即得 (3.2) 式. 特别考虑数值模型及 $I\!R^\infty$ 上的 Hermite 多项式 (1.22), 由 (1.28) 式可知

$$\mathcal{L}H_\alpha=-|\alpha|H_\alpha,\quad\alpha\in\Lambda. \tag{3.5}$$

既然子空间 $\mathcal{H}_n$ 是由 $\{H_\alpha : \alpha \in \Lambda, |\alpha| = n\}$ 所生成，且混沌分解是 Gauss 概率空间的内蕴性质，故 $\mathcal{L}$ 具有形式 (3.3), 其在 L^2 中定义域由 (3.4) 给出. ∎

注 $N \equiv -\mathcal{L} = \sum_n nJ_n$ 称为 **计数算子**, 它是 L^2 中正的自共轭算子.

下面考察由 $\mathcal{L}$ 生成的半群.

定义 3.3 L^2 上的压缩算子半群

$$T_t \equiv \sum_{n=0}^{\infty} e^{-nt} J_n, \quad t \geq 0 \tag{3.6}$$

称为 **Ornstein-Uhlenbeck 半群**(简称 **OU** 半群).

半群 $\{T_t, t \geq 0\}$ 满足:

1° 保正性: $F \geq 0 \Longrightarrow T_t(F) \geq 0$,(参看 (3.9) 式);

2° 对称性: $\mathbb{E}[GT_tF] = \mathbb{E}[FT_tG]$.

容易看出, $\mathcal{L}$ 是 $\{T_t, t \geq 0\}$ 的生成算子. 事实上, 若 $F \in \mathcal{D}(\mathcal{L})$, 则

$$\mathbb{E}[|t^{-1}(T_tF - F) - \mathcal{L}F|^2] = \sum_{n=0}^{\infty}(t^{-1}(e^{-nt} - 1) + n)^2 \mathbb{E}[|J_nF|^2].$$

由于 $|t^{-1}(e^{-nt} - 1)| \leq n$ 且当 $t \downarrow 0$ 时 $t^{-1}(e^{-nt} - 1) + n \longrightarrow 0$, 故上式趋于零. 反之, 若 $t^{-1}(T_tF - F)$ 在 L^2 中收敛于某泛函 G, 则 $\forall n$

$$J_nG = \lim_{t \downarrow 0} t^{-1}(T_tJ_nF - J_nF) = -nJ_nF,$$

故 $F \in \mathcal{D}(\mathcal{L})$ 且 $\mathcal{L}F = G$.

利用数值模型和有限维逼近, 我们可以证明 OU 半群在一切 $L^p(1 \leq p < \infty)$ 空间中的压缩性. 考虑数值模型 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$,

对 $h=\{h_j\}\in l^2, x=\{x_j\}\in \mathbb{R}^\infty$ 及 $n\in\mathbb{N}$ 令

$$\begin{aligned}\mathcal{E}_n(h)(x) &\equiv \exp\Big\{\sum_{j=1}^n h_jx_j-\frac{1}{2}\sum_{j=1}^n h_j^2\Big\}\\ &=\prod_{j=1}^n\sum_{n=0}^\infty\frac{h_j^n}{n!}H_n(x_j)\\ &=\sum_{\alpha\in\mathbb{N}_0^n}\prod_{j=1}^n\frac{h_j^{\alpha_j}}{\alpha_j!}H_{\alpha_j}(x_j).\end{aligned}$$

在定理 2.5 中已证明，当 $n\to\infty$ 时 $\mathcal{E}_n(h)$ 在 L^p 中收敛于指数泛函 $\mathcal{E}(h)$，于是

$$\mathcal{E}(h)(x)=\sum_{\alpha\in\Lambda}\frac{h^\alpha}{\alpha!}H_\alpha(x), \tag{3.7}$$

其中 $h^\alpha\equiv\prod_j h_j^{\alpha_j}, \alpha=\{\alpha_j\}\in\Lambda$. 因此

$$J_n\mathcal{E}(h)=\sum_{\alpha\in\Lambda_n}\frac{h^\alpha}{\alpha!}H_\alpha,\quad n\in\mathbb{N}_0, \tag{3.8}$$

其中 $\Lambda_n=\{\alpha\in\Lambda:|\alpha|=n\}$.

命题 3.4 在数值模型 $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$ 中，对 $t\geq 0, F\in L^2$ 有

$$(T_tF)(x)=\int_{\mathbb{R}^\infty}F(e^{-t}x+\sqrt{1-e^{-2t}}y)\gamma^\infty(dy). \tag{3.9}$$

从而对一切 $p\geq 1, T_t$ 在 L^p 中为压缩算子.

证明 由于指数泛函的线性组合在 L^2 中稠密，只要对 $F=\mathcal{E}(h), h\in l^2$ 验证 (3.9) 式. 先设 $F=\mathcal{E}_n(h)$, 直接计算 (3.9) 式右边的积分得 $\exp\{\sum_{j=1}^n x_je^{-t}h_j-\frac{1}{2}\sum_{j=1}^n(e^{-t}h_j)^2\}$, 令 $n\to\infty$ 并注意 (3.8) 式得

$$\begin{aligned}\mathcal{E}(e^{-t}h)(x)&=\sum_{n=0}^\infty e^{-nt}J_n\mathcal{E}(h)(x)\\ &=T_t\mathcal{E}(h)(x),\end{aligned} \tag{3.10}$$

从而 (3.9) 式成立. 如同有限维情形一样 (参看命题 2.2) 可证 T_t 在一切 L^p 中的压缩性. ∎

由于 OU 半群的定义不依赖于数值模型的选取, 我们也可将 (3.9) 式作为它的定义. 它对 $F \in L^1$ 仍有意义. 采用这种定义, 利用有限维逼近, 立即得到 OU 半群的超压缩性 (参看定理 2.3).

定理 3.5 设 $\{T_t, t \geq 0\}$ 为 OU 半群. 对 $p > 1, t > 0$, 令 $q(t) = e^{2t}(p-1) + 1 > p$, 则对一切 $F \in L^p$ 有

$$\|T_t F\|_{q(t)} \leq \|F\|_p. \tag{3.11}$$

由超压缩性可以得到两个有用的推论:

系 3.6 对一切 $p \in (1, \infty)$, Wiener 混沌分解中的每个子空间 $\mathcal{H}_n$ 是 L^p 的闭子空间, 在其中一切 L^p 范数均等价.

证明 任给 $q > p > 1, \exists t > 0$ 使 $q = e^{2t}(p-1) + 1$. 于是任给 n 及 $F \in \mathcal{H}_n$ 有

$$e^{-nt}\|F\|_q = \|T_t F\|_q \leq \|F\|_p,$$

即在 $\mathcal{H}_n$ 中 L^p 范数与 L^q 范数等价. ∎

系 3.7 对一切 $p \in (1, \infty)$ 及 $n \in \mathbb{N}_0, J_n$ 为 L^p 中有界线性算子.

证明 当 $p = 2$ 时结论显然成立. 若 $p > 2$, 取 $t > 0$ 使 $p = e^{2t}+1$, 则对 $F \in L^p$ 有

$$\begin{aligned}\|J_n F\|_p &= e^{nt}\|T_t J_n F\|_p \leq e^{nt}\|J_n F\|_2 \\ &\leq e^{nt}\|F\|_2 \leq e^{nt}\|F\|_p;\end{aligned}$$

若 $p < 2$, 令 q 为其共轭指数, 则 $q > 2$. 取 $t > 0$ 使 $q = e^{2t} + 1$, 则对 $F, G \in L^2$ 有

$$\begin{aligned}|\mathbb{E}[G J_n F]| &= |\mathbb{E}[F J_n G]| \\ &\leq \|F\|_p \|J_n G\|_q \\ &\leq \|F\|_p e^{nt}\|G\|_q,\end{aligned}$$

即

$$\|J_n F\|_p \le e^{nt}\|F\|_p,$$

从而 J_n 可连续延拓到 L^p 上. ∎

3.2 L^p 乘子定理

一般地说，给定一个实数序列 $\rho = \{\rho_n\}_{n\in \mathbb{N}_0}$，在 $\mathcal{P}$ 上可定义一个线性算子：

$$T_\rho \equiv \sum_{n=0}^{\infty} \rho_n J_n. \tag{3.12}$$

例如，$\rho_n = e^{-nt}, -n$ 时，T_ρ 分别为 OU 半群及 OU 算子. 那么，对于什么样的序列 ρ, T_ρ 可延拓为 L^p 上的有界线性算子？下面的 L^p **乘子定理** 给出了答案：

定理 3.8 (Meyer) 设 $\rho = \{\rho_n\}$ 为一实数序列，$T_\rho \equiv \sum_n \rho_n J_n$. 若 $\exists n_0 \in \mathbb{N}$ 及 $\beta > 0$ 使

$$\rho_n = \sum_{k=0}^{\infty} a_k (n^{-\beta})^k, \quad n \ge n_0, \tag{3.13}$$

其中实数列 $\{a_k\}$ 满足

$$\sum_{k=0}^{\infty} |a_k|(n_0^{-\beta})^k < \infty, \tag{3.14}$$

则 $\forall p \in (1,\infty), T_\rho$ 可唯一延拓为 L^p 上的有界线性算子.

注 此条件等价于：存在于 0 点某邻域解析的函数 φ 及 $\beta > 0$ 使

$$\rho_n = \varphi(n^{-\beta}). \tag{3.15}$$

我们这里采用 Shigekawa[4] 的简化证明. 为此先证明一个引理：

引理 3.9 若 I 为恒等算子，$n \in \mathbb{N}$. 令

$$I_n \equiv I - J_0 - \cdots - J_{n-1} = \sum_{k=n}^{\infty} J_k,$$

$$R_n \triangleq \int_0^\infty T_t I_n dt = \sum_{k=n}^\infty \frac{1}{k} J_k.$$

则 $\forall p \in (1,\infty)$ 及 $n \in \mathbb{N}, \exists c = c(p,n) > 0$, 使 $\forall F \in L^p, k \in \mathbb{N}$ 有

$$1^\circ \qquad \|T_t I_n F\|_p \le c e^{-nt}\|F\|_p \quad t \ge 0; \tag{3.16}$$

$$2^\circ \qquad \|R_n^k F\|_p \le c n^{-k}\|F\|_p. \tag{3.17}$$

证 1° $p=2$ 时显然成立. 若 $p>2$, 取 $t_0>0$ 使 $p=e^{2t_0}+1$, 则 $\forall t \ge 0$ 由 OU 半群超压缩性有

$$\begin{aligned}\|T_{t+t_0} I_n F\|_p &= \|T_{t_0} T_t I_n F\|_p \le \|T_t I_n F\|_2 \\ &= \left(\sum_{k=n}^\infty \|e^{-kt} J_k F\|_2^2\right)^{1/2} \\ &\le e^{-nt}\left(\sum_{k=n}^\infty \|J_k F\|_2^2\right)^{1/2} \\ &\le e^{-nt}\|F\|_2 \le e^{-nt}\|F\|_p,\end{aligned}$$

即对 $t \ge t_0$ 证明了 (3.16) 式. 当 $t < t_0$ 时，由压缩性可得

$$\begin{aligned}\|T_t I_n F\|_p &\le \|I_n F\|_p \le \|I_n\|\|F\|_p \\ &\le e^{nt_0}\|I_n\| e^{-nt}\|F\|_p,\end{aligned}$$

于是 (3.16) 式仍成立.

若 $p<2$, 考虑其共轭指数，类似于系 3.7 之证明可得 (3.16) 式.

2° 由 (3.16) 式可得 $\|R_n F\|_p \le c n^{-1}\|F\|_p$,

$$\begin{aligned}\|R_n^2 F\|_p &= \left\|\int_0^\infty \int_0^\infty T_t I_n T_s I_n F dt ds\right\|_p \\ &= \left\|\int_0^\infty \int_0^\infty T_{t+s} I_n F dt ds\right\|_p \\ &\le \int_0^\infty \int_0^\infty \|T_{t+s} I_n F\|_p dt ds \\ &\le \|F\|_p \int_0^\infty \int_0^\infty c e^{-n(t+s)} dt ds \\ &= c n^{-2}\|F\|_p.\end{aligned}$$

依此类推，可证 (3.17) 式.

定理 3.8 之证明 只须证 $\beta \leq 1$ 的情形. 先设 $\beta = 1$, 记 $T_\rho = T_1 + T_2$, 其中

$$T_1 \equiv \sum_{n=0}^{n_0-1} \rho_n J_n, \quad T_2 \equiv \sum_{n=n_0}^{\infty} \rho_n J_n.$$

由于 J_n 为 L^p 上有界线性算子，故 T_1 有界. 设 $n \geq n_0, F_n \in \mathcal{H}_n$, 因 $R_{n_0} F_n = n^{-1} F_n$, 故

$$\sum_{k=0}^{\infty} a_k R_{n_0}^k F_n = \sum_{k=0}^{\infty} a_k n^{-k} F_n = \rho_n F_n,$$

从而

$$T_2 = \sum_{k=0}^{\infty} a_k R_{n_0}^k.$$

但由 (3.17) 式及条件 (3.14) 可知，此算子级数依 L^p 中算子范数收敛，从而为 L^p 中有界算子.

对 $\beta < 1$, 设 $\lambda_t^\beta(ds)$ 为 $\mathcal{B}(\mathbb{R}_+)$ 上的概率测度，满足

$$\int_0^\infty e^{-us} \lambda_t^\beta(ds) = \exp\{-u^\beta t\}, \quad u \geq 0,$$

即 λ_t^β 为 β 阶单边稳定分布. 再令

$$T_t^\beta \equiv \int_0^\infty T_s \lambda_t^\beta(ds), \quad t \geq 0. \tag{3.18}$$

因为 $\lambda_t^\beta * \lambda_s^\beta = \lambda_{t+s}^\beta$, 可以证明 $\{T_t^\beta, t \geq 0\}$ 为 L^p 中强连续压缩半群，且对 $F_n \in \mathcal{H}_n$ 有

$$\begin{aligned} T_t^\beta F_n &= \int_0^\infty T_s F_n \lambda_t^\beta(ds) \\ &= \int_0^\infty e^{-ns} F_n \lambda_t^\beta(ds) \\ &= \exp\{-n^\beta t\} F_n. \end{aligned}$$

以 T_t^β 代 T_t, 类似地可证明定理结论. ∎

作为 L^p 乘子定理的推论，我们有

命题 3.10 对 $s \in \mathbb{R}$ 定义

$$(I - \mathcal{L})^{s/2} \equiv \sum_n (1+n)^{s/2} J_n, \tag{3.19}$$

则当 $s \le 0$ 时，$(I-\mathcal{L})^{s/2}$ 可延拓为 $L^p(1<p<\infty)$ 上的有界线性算子.

证明 因 $\varphi(x) \equiv (x/(1+x))^{-s/2}$ 在 $x=0$ 附近解析，此时 $\rho_n = (1+n)^{s/2} = \varphi(n^{-1})$, 由定理 3.8 命题得证. ∎

3.3 Meyer 不等式

下面我们来证明重要的 Meyer 不等式 (参看 Meyer[1]). 它是无穷维 Sobolev 空间理论的基础. 我们采用 Pisier[1] 的简化证明. 证明中要利用如下的 **Hilbert 变换**

$$Tf(x) \equiv \int_{\mathbb{R}} \frac{f(x+t)-f(x-t)}{t} dt \tag{3.20}$$

的性质，即：变换 T 在一切 $L^p(\mathbb{R})(1<p<\infty)$ 中为有界线性算子 (参看 Stein[1], Dunford - Schwartz[1] 或 Bass[1]).

我们先给出两个简单的引理

引理 3.11 在 $\mathcal{P}$ 上有

$$DJ_0 = 0, \quad DJ_n = J_{n-1}D, \quad n \ge 1. \tag{3.21}$$

对 $\rho = \{\rho_n\}$, 令 $\rho^+ \equiv \{\rho_{n+1}\}$, 则

$$DT_\rho = T_{\rho^+}D, \tag{3.22}$$

其中 T_ρ 由 (3.12) 定义.

证明 注意 Hermite 多项式 $\{h_n(x)\}$ 满足 $h_n'(x) = nh_{n-1}(x)$, 即可验证上述等式. ∎

引理 3.12 记

$$Q \equiv (I-\mathcal{L})^{-1/2} \equiv \sum_{n=0}^{\infty}(1+n)^{-1/2}J_n, \tag{3.23}$$

则

$$Q = \pi^{-1/2}\int_0^{\infty} t^{-1/2}e^{-t}T_t dt. \tag{3.24}$$

证明 投影到每个子空间 $\mathcal{H}_n$, 此式归结为以下的恒等式:

$$(n+1)^{-1/2} = \pi^{-1/2}\int_0^{\infty} t^{-1/2}e^{-(n+1)t}dt. \qquad \blacksquare$$

在以下证明中, C_p 和 $\widetilde{C}_p$ 表示仅依赖于 p 的常数, 但在不同场合可能不相等.

命题 3.13 设 $p\in(1,\infty)$, 则 $\exists C_p>0$, 使对一切 $F\in\mathcal{P}$ 有

$$\|QF\|_{1,p} \le C_p\|F\|_p. \tag{3.25}$$

证明 考虑数值模型, 对 $0\le\theta<\pi/2$ 作变换 $t=|\log\cos\theta| = -\log\cos\theta, e^{-t}=\cos\theta$, 则 $T_t=e^{-tN}=(\cos\theta)^N$,

$$Q = \pi^{-1/2}\int_0^{\pi/2} |\log\cos\theta|^{-1/2}(\cos\theta)^N\sin\theta d\theta. \tag{3.26}$$

对 $x,y\in\mathbb{R}^{\infty}, F\in L^p$, 令

$$R_\theta F(x,y) = F(x\cos\theta + y\sin\theta),$$

则由 (3.9) 式可知, $\forall F\in\mathcal{P}$

$$(T_tF)(x) = \mathbb{E}^y[R_\theta F(x,y)], \tag{3.27}$$

其中 $\mathbb{E}^y$ 表示关于 $\gamma^{\infty}(dy)$ 的积分. 若以 D_h^y 表示关于 y 的导数算子, 则对 $h\in l^2$

$$D_h^y R_\theta F(x,y) = \sin\theta R_\theta D_h F(x,y),$$

而

$$
\begin{aligned}
\mathbb{E}^y[D_h^y R_\theta F] &= \sin\theta\,\mathbb{E}^y[R_\theta D_h F] \\
&= \sin\theta(\cos\theta)^N D_h F \\
&= \sum_n \sin\theta\cos^n\theta J_n D_h F.
\end{aligned} \tag{3.28}
$$

令

$$
\varphi(\theta) \equiv \frac{1}{2}|\pi\log\cos\theta|^{-1/2}\cos\theta\,\mathrm{sgn}\,\theta. \tag{3.29}
$$

因 φ 在 0 点附近的阶为 θ^{-1}, 故可表示为

$$
\varphi(\theta) = \left(\frac{1}{2\sqrt{\pi}\theta} + r(\theta)\right)\mathrm{sgn}\,\theta,
$$

其中 $r(\theta)$ 为有界函数. 令

$$
\begin{aligned}
\int_{-\pi/2}^{\pi/2}\varphi(\theta)R_\theta F d\theta &\equiv \lim_{\epsilon\downarrow 0}\int_\epsilon^{\pi/2}(\varphi(\theta)R_\theta F + \varphi(-\theta)R_{-\theta}F)d\theta \\
&= \lim_{\epsilon\_0}\int_\epsilon^{\pi/2}\frac{(R_\theta F - R_{-\theta}F)}{2\sqrt{\pi|\log\cos\theta|}}\cos\theta d\theta,
\end{aligned} \tag{3.30}
$$

我们要证明上述极限存在.

由 Gauss 测度旋转不变性及 (3.27) 式,

$$
\mathbb{E}^x\mathbb{E}^y[|R_\theta F(x,y)|^p] = \|F\|_p^p < \infty
$$

为不依赖于 θ 的常数, 从而

$$
\int_{-\pi/2}^{\pi/2}|R_\theta F(x,y)|^p d\theta < \infty \quad \text{a.e. } (\gamma^\infty\times\gamma^\infty).
$$

但

$$
\int_{-\epsilon}^{\epsilon}\varphi(\theta')R_{\theta+\theta'}F d\theta' \approx \int_0^\epsilon \frac{R_{\theta+\theta'}F - R_{\theta-\theta'}F}{\theta'}d\theta',
$$

由 Hilbert 变换 (3.20) 在 $L^p(\mathbb{R})$ 中的有界性, $\exists C_p>0$ 使

$$\int_{-\pi/2}^{\pi/2}\Big|\int_{-\pi/2}^{\pi/2}\varphi(\theta')R_{\theta+\theta'}Fd\theta'\Big|^p d\theta\le C_p\int_{-\pi/2}^{\pi/2}|R_\theta F|^p d\theta.$$

于是由旋转不变性得

$$\begin{aligned}&\mathbb{E}^x\mathbb{E}^y\Big[\Big|\int_{-\pi/2}^{\pi/2}\varphi(\theta)R_\theta F(x,y)d\theta\Big|^p\Big]\\&=\mathbb{E}^x\mathbb{E}^y\Big[\Big|\int_{-\pi/2}^{\pi/2}\varphi(\theta')R_{\theta+\theta'}F(x,y)d\theta'\Big|^p\Big]\\&\le C_p\mathbb{E}^x\mathbb{E}^y[|R_\theta F(x,y)|^p]\\&=C_p\|F\|_p^p,\end{aligned}\tag{3.31}$$

故 (3.30) 式中极限在 $L^p(\gamma^\infty\times\gamma^\infty)$ 中存在.

设 $h\in l^2$. 由 (2.24) 及 (3.28) 式得

$$\begin{aligned}&\mathbb{E}^y\Big[W_h\int_{-\pi/2}^{\pi/2}\varphi(\theta)R_\theta Fd\theta\Big]=\mathbb{E}^y\Big[D_h^y\int_{-\pi/2}^{\pi/2}\varphi(\theta)R_\theta Fd\theta\Big]\\&=\int_{-\pi/2}^{\pi/2}\varphi(\theta)\mathbb{E}^y[D_h^yR_\theta F]d\theta\\&=\sum_n\Big(\int_{-\pi/2}^{\pi/2}\varphi(\theta)\sin\theta\cos^n\theta d\theta\Big)J_nD_hF\\&=\sum_n(n+2)^{-1/2}J_nD_hF=D_hQF,\end{aligned}$$

其中最后的等式由引理 3.11 而得. 因 $D_hQF=(DQF,h)_{l^2}$, 上式表明: 对 a.e.$x\,[\gamma^\infty]$,

$$DQF(x)=J_1^y\Big[\int_{-\pi/2}^{\pi/2}\varphi(\theta)R_\theta F(x,y)d\theta\Big],$$

其中 J_1^y 表示关于变量 y 而言到 $\mathcal{H}_1$ 的投影. 由 J_1^y 在 L^p 中有界性 (系 3.7), $\exists C_p>0$ 使得

$$\|DQF(x)\|_{l^2}^p\le C_p\mathbb{E}^y\Big[\Big|\int_{-\pi/2}^{\pi/2}\varphi(\theta)R_\theta F(x,y)d\theta\Big|^p\Big].$$

关于 x 积分并由 (3.31) 式得

$$\|DQF\|_p \le C_p\|F\|_p. \tag{3.32}$$

最后，由命题 3.10, Q 可延拓为 L^p 中有界算子，故 $\exists C_p > 0$ 使 $\|QF\|_p \le C_p\|F\|_p$, 结合 (3.32) 式即得 (3.25) 式. ∎

利用对偶性，容易得到相反的不等式.

命题 3.14 设 $p \in (1,\infty)$, 则 $\exists \widetilde{C}_p > 0$ 对一切 $F \in \mathcal{P}$ 有

$$\|F\|_p \le \widetilde{C}_p\|QF\|_{1,p}. \tag{3.33}$$

证明 因为 $Q^{-2} = I - \mathcal{L} = I + \delta D$, 故对 $F,G \in \mathcal{P}$ 有

$$\begin{aligned}(F,G) &= (Q^{-1}QF, Q^{-1}QG) = (Q^{-2}QF, QG)\\ &= (QF,QG) + (\delta DQF, QG)\\ &= (QF,QG) + (DQF, DQG),\end{aligned}$$

从而 $\exists C_{pq} > 0$, 对 $p^{-1} + q^{-1} = 1$ 有

$$\begin{aligned}|(F,G)| &\le \|QF\|_p\|QG\|_q + \|DQF\|_p\|DQG\|_q\\ &\le C_{pq}\|QF\|_{1,p}\|QG\|_{1,q}.\end{aligned}$$

但由 (3.25) 式,

$$\begin{aligned}\|F\|_p &= \sup\{|(F,G)| : \|G\|_q \le 1\}\\ &\le \sup\{|(F,G)| : \|QG\|_{1,q} \le C_q\}\\ &\le C_{pq}C_q\|QF\|_{1,p},\end{aligned}$$

由此立得 (3.33) 式. ∎

注 由于 $Q\mathcal{P} = \mathcal{P}$ 在 $I\!D_1^p$ 及 L^p 中稠密，故由命题 3.13 及命题 3.14 可知， $Q: L^p \longrightarrow I\!D_1^p$ 为 Banach 空间的同构.

下面将此结果推广到高阶导数. 为使记号简化，若对某线性空间中两个范数 $\|\cdot\|_1$ 和 $\|\cdot\|_2$, 存在常数 c 使 $\|\cdot\|_1 \le c\|\cdot\|_2$, 则记为 $\|\cdot\|_1 \preceq \|\cdot\|_2$, 若 $\|\cdot\|_1 \preceq \|\cdot\|_2$ 且 $\|\cdot\|_2 \preceq \|\cdot\|_1$, 则此二范数等价，记为 $\|\cdot\|_1 \sim \|\cdot\|_2$.

定理 3.15 (Meyer 不等式) 对 $p \in (1,\infty)$ 及 $k \in \mathbb{N}, \exists C_{k,p} > 0$ 及 $\widetilde{C}_{k,p} > 0$, 使 $\forall F \in \mathcal{P}$ 有

$$\widetilde{C}_{k,p}\|F\|_p \leq \|Q^k F\|_{k,p} \leq C_{k,p}\|F\|_p. \tag{3.34}$$

证明 用归纳法. 定义算子

$$\Gamma : \mathbb{D}_1^p(E) \longrightarrow L^p(E) \oplus L^p(H \otimes E)$$

为

$$\Gamma F \equiv \begin{pmatrix} F \\ DF \end{pmatrix},$$

显然 $\forall k \geq 1$ 有 $D\Gamma^k = \Gamma^k D$, 且由 (3.25) 及 (3.33) 式可知, $\Gamma Q : L^p(E) \longrightarrow L^p(E) \oplus L^p(H \otimes E)$ 为有界算子且具有有界逆. 由定义

$$\|\Gamma F\|_p \sim \|F\|_{1,p}.$$

对 k 用归纳法可证明

$$\begin{aligned}\|\Gamma^k F\|_p &\sim \|F\|_{k-1,p} + \|DF\|_{k-1,p} \\ &\sim \|F\|_{k,p}.\end{aligned} \tag{3.35}$$

令

$$M \equiv \left(\frac{2-\mathcal{L}}{1-\mathcal{L}}\right)^{1/2} = \sum_n \left(\frac{2+n}{1+n}\right)^{1/2} J_n.$$

由于 $\varphi(x) = \left(\frac{1+2x}{1+x}\right)^{1/2}$ 及 $\varphi^{-1}(x)$ 均在零点附近解析, 令 $\rho_n = \left(\frac{n+2}{n+1}\right)^{1/2} = \left(\frac{1+2/n}{1+1/n}\right)^{1/2}$, 由定理 3.8 可知 M 及 M^{-1} 均可延拓为 $L^p(E)$ 中有界算子. 对 $k \geq 1$ 令

$$A_k \equiv \begin{pmatrix} I & 0 \\ 0 & M^{-k} \end{pmatrix}.$$

则 A_k 及 A_k^{-1} 在 $L^p(E) \oplus L^p(H \otimes E)$ 中有界, 且

$$\Gamma Q^k = Q^k A_k \Gamma.$$

若令 $B_k \equiv \Gamma^k Q^k$, 则 $B_1 = \Gamma Q$ 有界，且

$$\begin{aligned} B_k &= \Gamma^{k-1}\Gamma Q^{k-1} Q \\ &= \Gamma^{k-1} Q^{k-1} A_{k-1} B_1 \\ &= B_{k-1} A_{k-1} B_1. \end{aligned}$$

对 k 用归纳法可证明 B_k 及 B_k^{-1} 均有界，于是由 (3.35) 式得

$$\begin{aligned} \|Q^k F\|_{k,p} &\sim \|\Gamma^k Q^k F\|_p \\ &= \|B_k F\|_p \\ &\sim \|F\|_p. \end{aligned}$$

此即 Meyer 不等式 (3.34). ∎

3.4 Meyer-Watanabe 广义泛函

S.Watanabe[1] 引进了如下 Sobolev 空间:

定义 3.16 对 $p \in (1, \infty)$ 及 $s \in \mathbb{R}$, 在 $\mathcal{P}(E)$ 上定义一族范数:

$$\|F\|_{s,p}^{\sim} \equiv \|(I - \mathcal{L})^{s/2} F\|_p. \tag{3.36}$$

以 $\widetilde{\mathbb{D}}_s^p(E)$ 表示 $\mathcal{P}(E)$ 关于 $\|\cdot\|_{s,p}^{\sim}$ 完备化而得的 Banach 空间.

注 当 $s \in \mathbb{N}$ 时，由 Meyer 不等式可知，$\|\cdot\|_{s,p}^{\sim} \sim \|\cdot\|_{s,p}$, 从而 $\widetilde{\mathbb{D}}_s^p(E) = \mathbb{D}_s^p(E)$. 以后我们均略去 $\sim$, 记为 $\mathbb{D}_s^p(E)$. 显然，此族范数具有单调性及相容性，从而当 $p \le p'$ 及 $s \le s'$ 时，

$$\mathbb{D}_{s'}^{p'}(E) \subset \mathbb{D}_s^p(E).$$

设 $p^{-1} + q^{-1} = 1$, 若将 $L^2(E)$ 与其对偶空间 $L^2(E)^*$ 视为同一，则 $L^p(E)^* = L^q(E)$. $\forall F, G \in \mathcal{P}(E)$, 有

$$\mathbb{E}[(F, G)] = \mathbb{E}[((I - \mathcal{L})^{-s/2} F, (I - \mathcal{L})^{s/2} G)],$$

从而

$$\|F\|_{-s,q}^{\sim} = \sup\{|\mathbb{E}[(F, G)]| : \|G\|_{s,p}^{\sim} \le 1\}, \tag{3.37}$$

即 $\|\cdot\|^{\sim}_{-s,q}$ 为 $\|\cdot\|^{\sim}_{s,p}$ 的对偶范数，且

$$\mathbb{D}^p_s(E)^* = \mathbb{D}^q_{-s}(E). \tag{3.38}$$

于是，当 $1 < p \leq q < \infty, 0 \leq r \leq s < \infty$ 时有

$$\begin{array}{ccccccccc}
\mathbb{D}^p_s(E) & \subset & \mathbb{D}^p_r(E) & \subset & L^p(E) & \subset & \mathbb{D}^p_{-r}(E) & \subset & \mathbb{D}^p_{-s}(E) \\
\cup & & \cup & & \cup & & \cup & & \cup \\
\mathbb{D}^q_s(E) & \subset & \mathbb{D}^q_r(E) & \subset & L^q(E) & \subset & \mathbb{D}^q_{-r}(E) & \subset & \mathbb{D}^q_{-s}(E)
\end{array}$$

由此可见，当 $s < 0$ 时， $\mathbb{D}^p_s(E)$ 中元素未必属于 $L^p(E)$, 令

$$\mathbb{D}^{\infty}(E) \equiv \bigcap_{s>0} \bigcap_{1<p<\infty} \mathbb{D}^p_s(E), \tag{3.39}$$

赋以投影极限拓扑，而

$$\mathbb{D}^{-\infty}(E) \equiv \bigcup_{s>0} \bigcup_{1<p<\infty} \mathbb{D}^p_{-s}(E), \tag{3.40}$$

赋以归纳极限拓扑，则 $\mathbb{D}^{\infty}(E)$ 与 (2.41) 式定义一致， $\mathbb{D}^{-\infty}(E)$ 为其对偶空间，其中元素自然地称为**广义泛函**.

由 Meyer 不等式，我们有以下推论：

命题 3.17 算子 D 可唯一延拓为 $\mathbb{D}^{-\infty}(E)$ 到 $\mathbb{D}^{-\infty}(H \otimes E)$ 中的算子，使 $\forall p \in (1, \infty)$ 及 $s \in \mathbb{R}$,

$$D : \mathbb{D}^p_{s+1}(E) \longrightarrow \mathbb{D}^p_s(H \otimes E)$$

为连续. 特别

$$D : \mathbb{D}^{\infty}(E) \longrightarrow \mathbb{D}^{\infty}(H \otimes E)$$

为连续.

作为共轭算子， $\delta = D^*$, 有

命题 3.18 算子 δ 可唯一延拓为 $\mathbb{D}^{-\infty}(H \otimes E)$ 到 $\mathbb{D}^{-\infty}(E)$ 中的算子，使 $\forall p \in (1, \infty)$ 及 $s \in \mathbb{R}$,

$$\delta : \mathbb{D}^p_{s+1}(H \otimes E) \longrightarrow \mathbb{D}^p_s(E)$$

为连续. 特别
$$\delta : \mathbb{D}^{\infty}(H \otimes E) \longrightarrow \mathbb{D}^{\infty}(E)$$
为连续.

因为 $\mathcal{L} = -\delta D$, 于是有

命题 3.19 算子 $\mathcal{L}$ 可唯一延拓为 $\mathbb{D}^{-\infty}(E)$ 到 $\mathbb{D}^{-\infty}(E)$ 中的算子, 使 $\forall p \in (1, \infty)$ 及 $s \in \mathbb{R}$,
$$\mathcal{L} : \mathbb{D}^p_{s+2}(E) \longrightarrow \mathbb{D}^p_s(E)$$
为连续. 特别
$$\mathcal{L} : \mathbb{D}^{\infty}(E) \longrightarrow \mathbb{D}^{\infty}(E)$$
为连续.

命题 3.20 若 E_1, E_2 为实可分 Hilbert 空间, $p, q \in (1, \infty)$, $k \in \mathbb{N}$ 且 $p^{-1} + q^{-1} = r^{-1} < 1$, 则 $\exists C = C(p, q, k) > 0$, 对一切 $F \in \mathcal{P}(E_1), G \in \mathcal{P}(E_2)$ 有
$$\|F \otimes G\|^{\sim}_{k,r} \leq C\|F\|^{\sim}_{k,p}\|G\|^{\sim}_{k,q}. \tag{3.41}$$
从而映射
$$(F, G) \longmapsto F \otimes G$$
可延拓为 $\mathbb{D}^p_k(E_1) \times \mathbb{D}^q_k(E_2) \longrightarrow \mathbb{D}^r_k(E_1 \otimes E_2)$ 的连续双线性映射. 特别, 若 $E_1 = E_2 = \mathbb{R}, F, G \in \mathbb{D}^{\infty}$, 则 $FG \in \mathbb{D}^{\infty}$, 从而 $\mathbb{D}^{\infty}$ 为一拓扑代数. 更一般地, $\forall k \in \mathbb{N}$, 由 (2.40) 式定义的 $\mathbb{D}^{\infty}_k$ 均为拓扑代数 (当 $k = 0$ 时, $\mathbb{D}^{\infty}_0 \equiv L^{\infty -}$).

证明 容易验证:
$$D(F \otimes G) = (DF) \otimes G + F \otimes (DG).$$
且一般地对 $k \in \mathbb{N}$ 有
$$D^k(F \otimes G) = \sum_{j=0}^{k} \binom{k}{j} (D^j F) \otimes (D^{k-j} G). \tag{3.42}$$

由 Meyer 不等式,

$$
\begin{aligned}
\|F\otimes G\|^{\sim}_{k,r} &\preceq \sum_{j=0}^{k}\|D^j(F\otimes G)\|_r \\
&\preceq \left(\sum_{j=0}^{k}\|D^jF\|_p\right)\left(\sum_{j=0}^{k}\|D^jG\|_q\right) \\
&\preceq \|F\|^{\sim}_{k,p}\|G\|^{\sim}_{k,q},
\end{aligned}
$$

(3.41) 式得证. ∎

由对偶性, 我们有

系 3.21 在命题 3.20 条件下, $\exists\widetilde{C}=\widetilde{C}(p,q,k)>0$ 使

$$
\|F\otimes G\|^{\sim}_{-k,r}\le \widetilde{C}\|F\|^{\sim}_{k,p}\|G\|^{\sim}_{-k,q}. \tag{3.43}
$$

特别, $\mathbb{D}^{\infty}(E)$ 和 $\mathbb{D}^{-\infty}(E)$ 均为 $\mathbb{D}^{\infty}$- 模.

复合泛函微分公式 (2.25),(2.26),(2.34) 及 (3.2) 等均可推广到相应的 Sobolev 空间上去. 特别当 $\mathcal{S}_M$ 和 $\mathcal{S}_M(E)$ 分别代之以 $\mathbb{D}^{\infty}$ 和 $\mathbb{D}^{\infty}(E)$ 时这些公式均成立.

命题 3.22 设 f 为 $\mathbb{R}^n$ 上光滑泛函, $\varphi_1,\cdots,\varphi_n\in\mathbb{D}^{\infty}$, 则 $F\equiv f(\varphi_1,\cdots,\varphi_n)\in\mathbb{D}^{\infty}$, 且

$$
DF=\sum_{j=1}^{n}\partial_j f(\varphi_1,\cdots,\varphi_n)D\varphi_j, \tag{3.44}
$$

$$
\begin{aligned}
\mathcal{L}F=&\sum_{j=1}^{n}\partial_j f(\varphi_1,\cdots,\varphi_n)\mathcal{L}\varphi_j \\
&+\sum_{j,k=1}^{n}\partial_j\partial_k f(\varphi_1,\cdots,\varphi_n)(D\varphi_j,D\varphi_k)_H.
\end{aligned} \tag{3.45}
$$

特别, 若 $F,G\in\mathbb{D}^{\infty}$, 则

$$
\mathcal{L}(FG)=F\mathcal{L}G+G\mathcal{L}F+2(DF,DG)_H. \tag{3.46}
$$

设 $F,G\in \mathbb{D}^\infty, V\in\mathbb{D}^\infty(H)$, 则

$$\mathbb{E}[(DF,V)_H]=\mathbb{E}[F\delta V], \tag{3.47}$$

$$\delta(FV)=F\delta V-(DF,V)_H, \tag{3.48}$$

$$\delta(FDG)=-F\mathcal{L}G-(DF,DG)_H, \tag{3.49}$$

$$\mathbb{E}[F\mathcal{L}G]=-\mathbb{E}[(DF,DG)_H]=\mathbb{E}[G\mathcal{L}F]. \tag{3.50}$$

读者试自行证明之. 此外, 分部积分公式 (3.47) 还可推广到广义泛函. 例如对 $s\in\mathbb{R}, F\in\mathbb{D}^p_{s+1}, V\in\mathbb{D}^q_{-s}(H)\,(p^{-1}+q^{-1}=1)$, 有

$$\langle DF,\ V\rangle=\langle F,\ \delta V\rangle, \tag{3.51}$$

等式两边分别为 $\mathbb{D}^p_s(H)\times\mathbb{D}^q_{-s}(H)$ 及 $\mathbb{D}^p_{s+1}\times\mathbb{D}^q_{-s-1}$ 上的典则双线性型. 值得注意的是, $\mathbb{D}^{-\infty}\times\mathbb{D}^\infty$ 上的典则双线性型 $\langle\cdot,\cdot\rangle$ 可以看作期望的自然推广, 因为当 $F,G\in\mathbb{D}^\infty$ 时 $\langle F,G\rangle=\mathbb{E}[FG]$. 特别若 $F\in\mathbb{D}^{-\infty}, G=1$, 则 $\langle F,1\rangle$ 是 F 的**广义期望**.

最后, 我们证明 OU 半群 $\{T_t,t>0\}$ 的一个引人注目的性质, 它类似于有限维空间中的"磨光算子".

命题 3.23 对 $t>0, p\in(1,\infty)$ 及任意实数 $r\le s$,

$$T_t:\mathbb{D}^p_r(E)\longrightarrow\mathbb{D}^p_s(E)$$

为连续. 对 $F\in\mathbb{D}^p_r(E), T_tF$ 当 $t\downarrow 0$ 时在 $\mathbb{D}^p_r(E)$ 中收敛于 F.

证明 不妨设 $E=\mathbb{R}$, 因 T_t 和 Q 可交换, 只需证 $s=0$ 的情形. 此时 $r\le 0$. 由 (3.16) 式可知, $\forall n_0\in\mathbb{N}, \exists c=c(p,n_0)$ 使 $\forall F\in L^p, n\ge n_0$ 有

$$\|T_tJ_nF\|_p\le ce^{-nt}\|F\|_p,$$

因此

$$\|T_tJ_n\|_{\mathcal{L}(L^p)}\le ce^{-nt},$$
$$\|Q^rT_tJ_n\|_{\mathcal{L}(L^p)}\le ce^{-nt}(1+n)^{-r/2}.$$

因为

$$\sum_{n=n_0}^{\infty} \|Q^r T_t J_n\|_{\mathcal{L}(L^p)} < \infty,$$

所以 $Q^r T_t \in \mathcal{L}(L^p)$. 但 $Q^r : L^p \to I\!D^p_r$ 为 Banach 空间同构，故 $T_t : I\!D^p_r \to L^p$ 连续. 后一结论可由 T_t 和 $Q^{\pm r}$ 的交换性得到. ∎

系 3.24 若 $F \in L^{\infty-}(\Omega; E), t > 0$. 则 $T_t F \in I\!D^{\infty}(E)$. 特别，若 F 为有界可测函数，则 $T_t F \in I\!D^{\infty}$.

§4. 非退化泛函的分布密度

Malliavin 随机变分学最重要的应用之一就是研究 Brown 运动轨道泛函分布密度的存在性、光滑性以及各种性质. 设 $(\Omega, \mathcal{F}, \mu; H)$ 为一 Gauss 概率空间，F 为其上 $I\!R^m$ 之一就是研究 Brown 运动轨道泛函分布密度的存在性、光滑性以及各种性质. 设 $(\Omega, \mathcal{F}, \mu; H)$ 为一 Gauss 概率空间，F 为其上 $I\!R^m$ 值泛函 (m 维随机向量)，其分布 $\mu \circ F^{-1}$ 是 $\mathcal{B}(I\!R^m)$ 上的概率测度. Malliavin 分析提供了一种极为有效的方法来研究在什么条件下 $\mu \circ F^{-1}$ 关于 $I\!R^m$ 中 Lebesgue 测度 λ^m 绝对连续以及其密度 (即 $\mu \circ F^{-1}$ 关于 λ^m 的 Radon-Nikodym 导数) 的各种性质.

4.1 Malliavin 协方差阵及若干引理

定义 4.1 设 $F = (F_1, \cdots, F_m) \in I\!D^1_1(I\!R^m)$, 令

$$\sigma_{ij} \equiv (DF_i, DF_j)_H, \quad 1 \le i, j \le m, \tag{4.1}$$

矩阵 $\Sigma(\omega) \equiv (\sigma_{ij}(\omega))_{1\le i,j\le m}$ 称为 **Malliavin 协方差阵**. 若 $\det\Sigma(\omega) > 0$ a.s. 且

$$[\det\Sigma(\omega)]^{-1} \in L^{\infty-}, \tag{4.2}$$

则称 Σ(或 F 本身)在 **Malliavin 意义下非退化**(简称非退化).

引理 4.2 若 $F \in \mathbb{D}_2^\infty(\mathbb{R}^m)$, 则其 Malliavin 协方差阵 $\Sigma \in \mathbb{D}_1^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$; 若 F 非退化, 则其逆阵 $\Sigma^{-1}(\omega) \equiv (\gamma_{ij}(\omega))_{1\le i,j\le m}$ a.s. 存在, 且 $\Sigma^{-1} \in \mathbb{D}_1^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$.

证明 对 $i,j=1,\cdots,m$, $|\sigma_{ij}| \le \|DF_i\|\|DF_j\|$, 从而 $\sigma_{ij} \in L^{\infty-}$. 由定义 (2.22) 式直接计算得

$$(D\sigma_{ij},h)_H = (D^2F_i, h\otimes DF_j)_{H\otimes H} + (D^2F_j, h\otimes DF_i)_{H\otimes H}, \quad \forall h \in H, \tag{4.3}$$

因 $F_i, F_j \in \mathbb{D}_2^\infty$, 故 $D\sigma_{ij} \in L^{\infty-}$, 从而 $\sigma_{ij} \in \mathbb{D}_1^\infty$. 由恒等式

$$\gamma_{ij} = \sum_{k,l=1}^m \gamma_{ik}\sigma_{kl}\gamma_{lj}, \quad 1 \le i,j \le m$$

微分可得

$$D\gamma_{ij} = -\sum_{k,l=1}^m \gamma_{ik}\gamma_{jl}D\sigma_{kl}, \quad 1 \le i,j \le m. \tag{4.4}$$

既然 γ_{ij} 等于 $(\det\Sigma)^{-1}$ 乘以 Σ 中元素的多项式, 由此可知 $\gamma_{ij} \in L^{\infty-}$, 由 (4.4) 式进一步推出 $D\gamma_{ij} \in L^{\infty-}$, 因此 $\gamma_{ij} \in \mathbb{D}_1^\infty$. ∎

注 根据同样的推理可知, 若 $F \in \mathbb{D}^\infty(\mathbb{R}^m)$, 则协方差阵 $\Sigma \in \mathbb{D}^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$; 若 F 非退化, 则 $\Sigma^{-1} \in \mathbb{D}^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$.

对于非退化泛函分布密度的存在性的证明, 关键是利用分部积分公式 (即 D 和 δ 的共轭性和 OU 算子 $\mathcal{L}$ 的自共轭性) 和以下的调和分析引理. 为此我们先证明有限维空间的一个不等式.

引理 4.3 (Gagliardo-Nirenberg 不等式) 设 $m > 1$, 记 $m^* = m/(m-1)$, 则 $\forall \varphi \in C_0^\infty(\mathbb{R}^m)$ 有

$$\|\varphi\|_{m^*} \le \frac{1}{m}\sum_{j=1}^m \|\partial_j\varphi\|_1. \tag{4.5}$$

证明 记 $x=(x_1,\cdots,x_m)$, 对 $j=1,\cdots,m$ 有

$$\varphi(x)=\int_{-\infty}^{x_j}\partial_j\varphi(x)dx_j,$$

故

$$|\varphi(x)|\le\int_{-\infty}^{\infty}|\partial_j\varphi(x)|dx_j,$$

$$|\varphi(x)|^{m^*}\le\prod_{j=1}^{m}\left(\int_{-\infty}^{\infty}|\partial_j\varphi(x)|dx_j\right)^{1/(m-1)}.$$

将上式依次对 $x_1,\cdots,x_m$ 积分，注意 $\int|\partial_j\varphi|dx_j$ 不依赖于 x_j, 在对 x_j 积分时，对其它 $m-1$ 个因子用推广的 Hölder 不等式

$$\left|\int(\prod_{j=1}^{n}f_j)\right|\le\prod_{j=1}^{n}\|f_j\|_n, \tag{4.6}$$

即得

$$\int_{\mathbb{R}^m}|\varphi(x)|^{m^*}dx\le\prod_{j=1}^{m}\|\partial_j\varphi\|_1^{1/(m-1)}.$$

于是

$$\|\varphi\|_{m^*}\le\prod_{j=1}^{m}\|\partial_j\varphi\|_1^{1/m}\le\frac{1}{m}\sum_{j=1}^{m}\|\partial_j\varphi\|_1. \qquad\blacksquare$$

下面是关键的调和分析引理：

引理 4.4 设 ν 为 $\mathcal{B}(\mathbb{R}^m)$ 上的有限测度，若对 $j=1,\cdots,m$, 存在常数 c_j 使 $\forall\varphi\in C_0^\infty(\mathbb{R}^m)$ 有

$$\left|\int_{\mathbb{R}^m}\partial_j\varphi(x)\nu(dx)\right|\le c_j\|\varphi\|_\infty, \tag{4.7}$$

则 ν 关于 Lebesgue 测度绝对连续. 当 $m>1$ 时具有密度 $\rho\in L^{m^*}(I\!R^m)$, 其中 $m^*=m/(m-1)$.

证明 当 $m=1$ 时, 任取区间 (a,b), 令 φ 为 (a,b) 上均匀分布随机变量之分布函数, 取 $\varphi_n\in C_0^\infty(I\!R)$ 使 $\varphi_n\longrightarrow\varphi$ 且 $\varphi_n'\longrightarrow\varphi'$. 由 (4.7) 即得 $\nu([a,b])\le c_1(b-a)$, 即 ν 关于 Lebesgue 测度绝对连续.

当 $m>1$ 时, 取磨光函数 $\psi_\epsilon\in C_0^\infty(I\!R^m)$, 作卷积

$$\rho_\epsilon(x)\equiv\int_{I\!R^m}\psi_\epsilon(x-y)\nu(dy),\quad \epsilon>0,$$

则 $\rho_\epsilon\in C_b^\infty(I\!R^m)$, 且 $\forall\varphi\in C_0^\infty(I\!R^m)$ 有

$$\lim_{\epsilon\downarrow0}\int_{I\!R^m}\rho_\epsilon(x)\varphi(x)dx=\int_{I\!R^m}\varphi(x)\nu(dx).\tag{4.8}$$

(作一些技术性处理, 不妨设 $\rho_\epsilon\in C_0^\infty(I\!R^m)$). 于是当 ϵ 足够小时有

$$\left|\int_{I\!R^m}\partial_j\rho_\epsilon(x)\varphi(x)dx\right|=\left|\int_{I\!R^m}\rho_\epsilon(x)\partial_j\varphi(x)dx\right|$$
$$\le c_j\|\varphi\|_\infty,\quad j=1,\cdots,m.$$

由此推出

$$\|\partial_j\rho_\epsilon\|_1\le c_j\quad j=1,\cdots,m.$$

由不等式 (4.5),$\exists c>0$ 使

$$\|\rho_\epsilon\|_{m^*}\le\frac{1}{m}\sum_{j=1}^m\|\partial_j\rho_\epsilon\|_1\le c.$$

因 L^{m^*} 中有界集弱紧, 故存在子序列 $\{\rho_{\epsilon_k}\}$ 在 L^{m^*} 中弱收敛于某函数 ρ, 由 (4.8) 即知

$$\int_{I\!R^m}\rho(x)\varphi(x)dx=\int_{I\!R^m}\varphi(x)\nu(dx),\quad\varphi\in C_0^\infty(I\!R^m),$$

从而 $\nu(dx) = \rho(x)dx$. ∎

4.2 分布密度的存在性

我们先在 $F \in \mathbb{D}_2^\infty(\mathbb{R}^m)$ 的假设下来研究其分布密度的存在性和连续性.

定理 4.5 设 $F = (F_1, \cdots, F_m) \in \mathbb{D}_2^\infty(\mathbb{R}^m)$ 且非退化，则对 $i = 1, \cdots, m, \exists \Phi_i \in L^{\infty-}$ 使 $\forall \varphi \in C_0^\infty(\mathbb{R}^m)$ 有

$$\mathbb{E}[\partial_i \varphi \circ F] = \mathbb{E}[\Phi_i \cdot (\varphi \circ F)], \tag{4.9}$$

从而 F 存在分布密度 ρ. 当 $m > 1$ 时， $\rho \in L^{m^*}(\mathbb{R}^m)$, 其中 $m^* = m/(m-1)$.

证明 由引理 4.2,F 的协方差阵 $\Sigma = (\sigma_{ij})$ 及其逆阵 $\Sigma^{-1} = (\gamma_{ij})$ 均属于 $\mathbb{D}_1^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$. 对 $i = 1, \cdots, m$, 令

$$Z_i \equiv \sum_{j=1}^m \gamma_{ij} DF_j, \quad \Phi_i \equiv \delta Z_i,$$

则 $Z_i \in \mathbb{D}_1^\infty(H)$, 由 (3.49) 式 (它显然可推广到 $\mathbb{D}_2^\infty$ 情形) 计算得

$$\Phi_i = -\sum_{j=1}^m (\gamma_{ij} \mathcal{L} F_j + (D\gamma_{ij}, DF_j)_H), \tag{4.10}$$

因此 $\Phi_i \in L^{\infty-}$. 由公式 (3.44)(显然可推广到 $\mathbb{D}_2^\infty$ 情形) 得

$$D(\varphi \circ F) = \sum_{i=1}^m (\partial_i \varphi \circ F) DF_i \in \mathbb{D}_1^\infty(H),$$

因此

$$(D(\varphi \circ F), DF_j)_H = \sum_{i=1}^m (\partial_i \varphi \circ F) \sigma_{ij}, \quad j = 1, \cdots, m. \tag{4.11}$$

由于 Σ 非退化，解此方程组得

$$
\begin{aligned}
\partial_i \varphi \circ F &= \sum_{j=1}^{m} \gamma_{ij} (D(\varphi \circ F), DF_j)_H \\
&= (D(\varphi \circ F), Z_i)_H, \quad i = 1, \cdots, m.
\end{aligned} \tag{4.12}
$$

由分部积分公式 (3.47) 得

$$
\begin{aligned}
\mathbb{E}[\partial_i \varphi \circ F] &= \mathbb{E}[(D(\varphi \circ F), Z_i)_H] \\
&= \mathbb{E}[(\varphi \circ F) \delta Z_i] \\
&= \mathbb{E}[(\varphi \circ F) \Phi_i], \quad i = 1, \cdots, m,
\end{aligned}
$$

(4.9) 式得证. 记 $\nu_F \equiv \mu \circ F^{-1}$ 为 F 之分布，则

$$
\begin{aligned}
\left| \int_{\mathbb{R}^m} \partial_i \varphi(x) \nu_F(dx) \right| &= |\mathbb{E}[\partial_i \varphi \circ F]| \\
&= |\mathbb{E}[(\varphi \circ F) \Phi_i]| \\
&\leq \mathbb{E}[|\Phi_i|] \cdot \|\varphi\|_\infty.
\end{aligned}
$$

由引理 4.4, 得证定理结论. ∎

注 若 $F \in \mathbb{D}_2^p(\mathbb{R}^m)$ 且 $(\det \Sigma)^{-1} \in L^p, p > 4m$, 则利用 Hölder 不等式进行估计，可以证明 $\Phi_i \in L^r, r = p/4m$.

设 $\beta < 1$. 令

$$
\mathcal{H}^\beta(\mathbb{R}^m) \equiv \left\{ \varphi \in C(\mathbb{R}^m) : \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|^\beta} < \infty \right\} \tag{4.13}
$$

为 β-Hölder 连续函数空间，我们有

定理 4.6 在定理 4.5 的假定下，对一切 $\beta < 1$ 及 $r > 0, F$ 的分布密度 ρ 满足

$$
\rho^r \in \mathcal{H}^\beta(\mathbb{R}^m). \tag{4.14}
$$

证明 对 $i=1,\cdots,m$ 令 $q_i(x)\equiv \mathbb{E}[\Phi_i|F=x]$, 其中 $\Phi_i\in L^{\infty-}$ 由定理 4.5 给出，于是 $\forall\varphi\in C_0^\infty(\mathbb{R}^m)$ 有

$$\begin{aligned}\int_{\mathbb{R}^m}\partial_i\varphi(x)\rho(x)dx&=\mathbb{E}[\partial_i\varphi\circ F]\\&=\mathbb{E}[(\varphi\circ F)\Phi_i]\\&=\mathbb{E}[(\varphi\circ F)\mathbb{E}[\Phi_i|F]]\\&=\int_{\mathbb{R}^m}\varphi(x)q_i(x)\rho(x)dx.\end{aligned}\tag{4.15}$$

上式表明，在广义导数意义下

$$\partial_i\rho=-q_i\rho,\quad i=1,\cdots,m.$$

记 $\|\nabla\rho\|\equiv(\sum_{i=1}^m(\partial_i\rho)^2)^{1/2}$, 则 $\forall p>1$,

$$\rho^{-p}\|\nabla\rho\|^p=\left(\sum_{i=1}^m q_i^2\right)^{p/2}\le m^p\sum_{i=1}^m|q_i|^p.$$

两边对 $\rho(x)dx$ 积分得

$$\begin{aligned}\int_{\mathbb{R}^m}\|\nabla\rho\|^p\rho^{1-p}dx&\le m^p\sum_{i=1}^m\mathbb{E}[|\mathbb{E}[\Phi_i|F]|^p]\\&\le m^p\sum_{i=1}^m\|\Phi_i\|_p^p<\infty.\end{aligned}$$

因 $\|\nabla\rho^{1/p}\|=\frac{1}{p}\rho^{1/p-1}\|\nabla\rho\|$, 故

$$\|\nabla\rho^{1/p}\|_p^p=p^{-p}\int_{\mathbb{R}^m}\|\nabla\rho\|^p\rho^{1-p}dx<\infty,$$

即 $\rho^{1/p}$ 属于 Sobolev 空间 $W^{1,p}(\mathbb{R}^m)$, 但由 Sobolev 嵌入定理

$$\bigcap_{p>1}W^{1,p}(\mathbb{R}^m)\subset\bigcap_{\beta<1}\mathcal{H}^\beta(\mathbb{R}^m),\tag{4.16}$$

故有定理结论. ∎

如果我们只要求密度存在，则条件可以大大减弱. Bouleau - Hirsch[1,2] 证明了如下结果:

定理 4.7 设 $p>1, F\in \mathbb{D}_1^p(\mathbb{R}^m)$, 其协方差阵 Σ a.s. 可逆，则 F 存在分布密度.

其证明较复杂，可参看 Bouleau - Hirsch[2] 或 Nualart[1]. 这里只证明 $m=1$ 的情形，此时只要假定 $p=1$ 即可.

证明 设 $F\in \mathbb{D}_1^1$, 且不妨设 F 有界, 例如 $|F|<1$. 为证 F 之分布绝对连续，只需证明对任意 Borel 可测函数 $g:(-1,1)\longrightarrow[0,1]$, 只要 $\int_{-1}^1 g(y)dy=0$, 则 $\mathbb{E}[g\circ F]=0$.

选取函数列 $\{g_n\}\subset C_b^1(-1,1)$, 使

$$\lim_{n\to\infty} g_n(y)=g(y)\quad \text{a.e. } [\mu\circ F^{-1}+\lambda^1].$$

令

$$\psi_n(y)=\int_{-1}^y g_n(x)dx,\quad \psi(y)=\int_{-1}^y g(x)dx.$$

由复合函数微分法则, $\psi_n\circ F\in \mathbb{D}_1^1$ 且 $D[\psi_n(F)]=g_n(F)DF$. 由 $g_n\longrightarrow g$ a.e.$[\lambda^1]$ 可知

$$\lim_{n\to\infty}\psi_n\circ F=\psi\circ F,\qquad \text{a.s. 及 } L^1,$$

又由 $g_n\longrightarrow g$ a.e.$[\mu\circ F^{-1}]$ 可知

$$\lim_{n\to\infty} D[\psi_n(F)]=g(F)DF,\qquad \text{a.s. 及 } L^1(H).$$

因 $\psi\circ F=0$ a.s., 由算子 D 的闭性可知 $g(F)DF=0$ a.s.. 但 $\|DF\|_H>0$ a.s., 故 $g\circ F=0$ a.s.. ∎

注 从证明中可以看出，如果除去 Σ a.s. 可逆条件，仍然可以得出在集合 $\{\det\Sigma>0\}$ 上 $\mu\circ F^{-1}$ 绝对连续，即对任意 Borel 集 B, 若其 Lebesgue 测度为 0, 则 $\mu\{F\in B, \det\Sigma>0\}=0$. 一维情形还可参看严加安 [3].

4.3 分布密度的光滑性

下面我们将在 $F \in \mathbb{D}^{\infty}(\mathbb{R}^m)$ 的假设下来证明分布密度的无穷次可微性. 这里我们采用 S.Watanabe[1] 的简捷而富有启发性的方法, 即研究广义函数和泛函的复合. 因为 F 的分布密度 $p(x)$ 可以形式上写成 $\mathbb{E}[\delta_x \circ F]$, 即 Dirac δ 函数和 F 的复合的期望值. 显然 $\delta_x \circ F$ 不可能是通常意义下的泛函. 我们已经知道, 当 $\varphi \in \mathcal{S}(\mathbb{R}^m)$ 时, 复合泛函 $\varphi \circ F \in \mathbb{D}^{\infty}$. 若将 F 固定, 则映射: $\varphi \mapsto \varphi \circ F$ 是 $\mathcal{S}(\mathbb{R}^m)$ 到 $\mathbb{D}^{\infty}$ 的线性映射. 我们的目的是将它延拓为 $\mathcal{S}^*(\mathbb{R}^m)$ 到 $\mathbb{D}^{-\infty}$ 的具有某种连续性质的线性映射, 从而 $\delta_x \circ F$ 可以解释为广义泛函.

考虑 Schwartz 的缓增广义函数空间 $\mathcal{S}^*(\mathbb{R}^m)$. 为证明方便, 我们引进与第一章 §3 的例中稍微不同但与它等价的一族范数:

$$\|f\|_{2k}^{\sim} \equiv \|(1+|x|^2-\Delta)^k f\|_{\infty}, \quad k \in \mathbb{Z}. \tag{4.17}$$

以 $\mathcal{T}_{2k}$ 表示 $\mathcal{S}(\mathbb{R}^m)$ 关于 $\|\cdot\|_{2k}^{\sim}$ 完备化而得的 Banach 空间, 于是仍然有 (参看 Reed-Simon[1])

$$\mathcal{S}(\mathbb{R}^m) = \varprojlim_k \mathcal{T}_{2k},$$

$$\mathcal{S}^*(\mathbb{R}^m) = \varinjlim_k \mathcal{T}_{-2k}.$$

引理 4.8 设 $\delta_y, y \in \mathbb{R}^m$ 为 Dirac δ 函数. $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m, |\alpha| \equiv \sum_{j=1}^m \alpha_j, \partial_\alpha \equiv \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$. 若 $n > m/2$, 则 $\delta_y \in \mathcal{T}_{-2n}$; 若 $|\alpha| \le 2k$, 则 $\partial_\alpha \delta_y \in \mathcal{T}_{-2n-2k}$, 且映射

$$\mathbb{R}^m \ni y \mapsto \delta_y \in \mathcal{T}_{-2n-2k}$$

为 $2k$ 次连续可微.

证明 δ_y 之 Fourier 变换为 $e^{-i(\xi,y)}$, 从而

$$\mathcal{F}[(1-\Delta)^{-n}\delta_y](\xi) = (1+|\xi|^2)^{-n} e^{-i(\xi,y)}.$$

进行逆变换得

$$[(1-\Delta)^{-n}\delta_y](x) = \left(\frac{1}{2\pi}\right)^m \int_{\mathbb{R}^m} \frac{e^{i(\xi,x-y)}}{(1+|\xi|^2)^n} d\xi.$$

当 $n > m/2$ 时

$$\|(1+|x|^2-\Delta)^{-n}\delta_y\|_\infty \le \|(1-\Delta)^{-n}\delta_y\|_\infty < \infty,$$

故 $\delta_y \in \mathcal{T}_{-2n}$，且映射 $y \mapsto (1+|x|^2-\Delta)^{-n}\delta_y \in \mathcal{T}_0$ 为连续．当 $|\alpha| \le 2k$ 及 $\varphi \in \mathcal{T}_{2n+2k}$ 时， $\partial_\alpha\varphi \in \mathcal{T}_{2n}$，且

$$\langle \partial_\alpha\delta_y, \varphi\rangle = (-1)^{|\alpha|}\partial_\alpha\varphi(y),$$

因此映射 $y \mapsto \delta_y \in \mathcal{T}_{-2n-2k}$ 为 $2k$ 次连续可微． ∎

定理 4.9 (S.Watanabe) 若 $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ 且非退化，则 $\forall p \in (1,\infty)$ 及 $n \in \mathbb{N}_0, \exists c = c(p,n) > 0, \forall \varphi \in \mathcal{S}(\mathbb{R}^m)$ 有

$$\|\varphi \circ F\|_{-2n,p} \le c\|\varphi\|^{\sim}_{-2n}, \tag{4.18}$$

从而映射 $\varphi \mapsto \varphi \circ F$ 可唯一延拓为 $\mathcal{S}^*(\mathbb{R}^m)$ 到 $D^{-\infty}$ 的线性映射，限制于 $\mathcal{T}_{-2n}$ 时为到 $\mathbb{D}^p_{-2n}$ 中的连续映射．

证明 类似于定理 4.5 的证明可得 (4.11) 及 (4.12) 式，所不同的是，此时 $\varphi \in \mathcal{S}(\mathbb{R}^m)$，式中泛函均属于 $\mathbb{D}^\infty$．

对 $G \in \mathbb{D}^\infty, i = 1, \cdots, m$，令

$$\Phi_i(G) \equiv \sum_{j=1}^m \delta(\gamma_{ij} G DF_j).$$

由 (3.49) 式得

$$\begin{aligned}
\Phi_i(G) &= -\sum_{j=1}^m \{\gamma_{ij} G \mathcal{L} F_j + (D(\gamma_{ij}G), DF_j)_H\} \\
&= -\sum_{j=1}^m \{\gamma_{ij}\mathcal{L}F_j + (D\gamma_{ij}, DF_j)_H\}G - \Big(\sum_{j=1}^m \gamma_{ij} DF_j, DG\Big)_H \\
&= \Psi_0 G + (\Psi_1, DG)_H,
\end{aligned}$$

其中 $\Psi_0 \in \mathbb{D}^\infty, \Psi_1 \in \mathbb{D}^\infty(H)$. 由 (4.12) 及分部积分公式 (3.47) 知

$$\begin{aligned}
\mathbb{E}[G(\partial_i\varphi \circ F)] &= \sum_{j=1}^m \mathbb{E}[(\gamma_{ij}GDF_j, D(\varphi\circ F))_H] \\
&= \sum_{j=1}^m \mathbb{E}[(\varphi\circ F)\delta(\gamma_{ij}GDF_j)] \\
&= \mathbb{E}[(\varphi\circ F)\Phi_i(G)], \qquad i=1,\cdots,m.
\end{aligned}$$

以 $\partial_i\varphi$ 代 $\varphi, \Phi_i(G)$ 代 G, 依此类推, 可将 i 代以任一多重指标 $\alpha=(\alpha_1,\cdots,\alpha_m)$, 得

$$\mathbb{E}[G(\partial_\alpha\varphi\circ F)] = \mathbb{E}[(\varphi\circ F)\Phi_\alpha(G)]. \tag{4.19}$$

当 $|\alpha|=2n$ 时, $\Phi_\alpha(G)$ 具有形式:

$$\Phi_\alpha(G) = \Psi_0 G + (\Psi_1, DG)_H + \cdots + (\Psi_{2n}, D^{2n}G)_{H^{\otimes 2n}}, \tag{4.20}$$

其中 $\Psi_k \in \mathbb{D}^\infty(H^{\otimes k})$ 为 $\gamma_{ij}, F_j (1\le i,j\le m)$ 及其导数的多项式. 特别, 对 $A = 1+|x|^2-\Delta$ 有

$$\mathbb{E}[G(A^n\varphi\circ F)] = \mathbb{E}[(\varphi\circ F)\Phi_A(G)],$$

其中 $\Phi_A(G)$ 仍具有 (4.20) 的形式, 于是

$$\begin{aligned}
|\mathbb{E}[G(\varphi\circ F)]| &= |\mathbb{E}[G(A^nA^{-n}\varphi\circ F)]| \\
&= |\mathbb{E}[(A^{-n}\varphi\circ F)\Phi_A(G)]| \\
&\le \|A^{-n}\varphi\|_\infty \|\Phi_A(G)\|_1 \\
&= \|\Phi_A(G)\|_1 \|\varphi\|^{\sim}_{-2n}.
\end{aligned}$$

若 $p^{-1}+q^{-1}=1$, 则

$$\begin{aligned}
\|\varphi\circ F\|_{-2n,p} &= \sup\{|\mathbb{E}[G(\varphi\circ F)]| : \|G\|_{2n,q}\le 1\} \\
&\le \|\varphi\|^{\sim}_{-2n} \sup\{\|\Phi_A(G)\|_1 : \|G\|_{2n,q}\le 1\}.
\end{aligned}$$

由 (4.20) 式可知

$$c(p,n) \equiv \sup\{\|\Phi_A(G)\|_1 : \|G\|_{2n,q} \leq 1\} < \infty. \qquad \blacksquare$$

上述定理确定了广义函数 $T \in \mathcal{S}^*(\mathbb{R}^m)$ 和非退化泛函 $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ 的 "广义复合": $T \circ F$, Watanabe[1] 称映射 $T \mapsto T \circ F$ 为 T 在映射 $F : \Omega \longrightarrow \mathbb{R}^m$ 下的 **拉回 (pullback)**, 而 Malliavin[5] 则称之为 T 借助于 F 的 **提升 (lifting up)**, 因为它将有限维空间上的广义函数提升为无穷维空间上的广义泛函. 下面我们将看到, 此映射的对偶映射恰好是关于 (随机向量)F 的条件期望 $\mathbb{E}^F$, 所以此映射可以记为

$$(\mathbb{E}^F)^* : \mathcal{S}^*(\mathbb{R}^m) \longrightarrow \mathbb{D}^{-\infty}. \tag{4.21}$$

$(\mathbb{E}^F)^*T \equiv T \circ F$, 特别取 $T = \delta_x$, 则

$$\rho_F(x) \equiv \langle \delta_x \circ F, 1 \rangle \tag{4.22}$$

是泛函 F 的分布密度, 于是我们得到以下重要结论:

定理 4.10 设 $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ 且非退化, 则 F 具有无穷次可微的分布密度 $\rho_F(x)$.

证明 由定理 4.9 及引理 4.8, 若 $n > m/2$, 则 $\forall p \in (1,\infty)$, $k \in \mathbb{N}_0$, 映射 $\mathbb{R}^m \ni x \mapsto \delta_x \circ F \in \mathbb{D}^p_{-2n-2k}$ 为 $2k$ 次连续可微, 但 k 是任意的, 故由 (4.22) 式定义的 ρ_F 属于 $C^\infty(\mathbb{R}^m)$.

考虑映射 (4.21) 的对偶映射 $\mathbb{E}^F$, 它是 $\mathbb{D}^q_{2n}$ 到 $\mathcal{T}_{2n}$ 的连续映射, 且 $\rho_F(x) = \langle \delta_x, \mathbb{E}^F 1\rangle = (\mathbb{E}^F 1)(x)$.$\forall \varphi \in \mathcal{S}(\mathbb{R}^m)$ 有

$$\begin{aligned}\mathbb{E}[\varphi \circ F] &= \langle \varphi \circ F, 1 \rangle = \langle \varphi, \mathbb{E}^F 1 \rangle \\ &= \int_{\mathbb{R}^m} \varphi(x)\rho_F(x)dx.\end{aligned}$$

这就证明了由 (4.22) 式定义的 ρ_F 是 F 的分布密度. $\blacksquare$

在定理 4.10 的证明中, 若以 $G \in \mathbb{D}^q_{2n+2k}$ 代替 1, 可得

$$\begin{aligned}\mathbb{E}[(\varphi \circ F)G] &= \langle \varphi \circ F, G \rangle = \langle \varphi, \mathbb{E}^F G \rangle \\ &= \int_{\mathbb{R}^m} \varphi(x)\langle \delta_x \circ F, G \rangle dx.\end{aligned}$$

由此可见，在集合 $\{x:\rho_F(x)>0\}$ 上

$$\langle\delta_x\circ F, G\rangle=\rho_F(x)\mathbb{E}[G|F=x], \quad \text{a.e..} \tag{4.23}$$

作为推论，我们得到以下关于条件期望正则性的重要结果：

系 4.11 若 $n>m/2, p\in(1,\infty), k\in\mathbb{N}_0, G\in\mathbb{D}^p_{2n+2k}$，则在集合 $\{x:\rho_F(x)>0\}$ 上条件期望 $\mathbb{E}[G|F=x]$ 具有 C^{2k} 修正．特别，若 $G\in\mathbb{D}^\infty$，则 $\mathbb{E}[G|F=x]$ 具有 C^∞ 修正．

4.4 例

典型的应用例子是关于偏微分算子亚椭圆性的 Hörmander 定理的概率证明．我们将在下一章阐述．这里只举两个简单的例子．

例 1 (Donsker δ 函数) 设 $H=L^2(\mathbb{R}_+;\mathbb{R}^d), \{W(t), t\in\mathbb{R}_+\}$ 为概率空间 $(\Omega,\mathcal{F},\mathbb{P})$ 上的 d 维 Brown 运动， $\mathcal{F}^W$ 为由它生成的完备 σ- 代数．对 $h\in H$，令

$$W_h(\omega)\equiv\sum_{j=1}^{d}\int_0^\infty h_j(t)dW_j(t),$$

则 $(\Omega,\mathcal{F}^W,\mathbb{P};H)$ 为不可约 Gauss 概率空间．对固定 $t\in\mathbb{R}_+, W(t)$ 为 $\mathbb{R}^d$ 值多项式泛函．若取 $\{e_1,\cdots,e_d\}$ 为 $\mathbb{R}^d$ 的基，则

$$DW_j(t)=1_{[0,t]}e_j, \quad j=1,\cdots,d.$$

于是 $W(t)\in\mathbb{D}^\infty(\mathbb{R}^d)$，且

$$\sigma_{ij}(t)\equiv(DW_i(t),DW_j(t))_H=\delta_{ij}t, \qquad i,j=1,\cdots,d.$$

当 $t>0$ 时， $W(t)$ 非退化，从而存在 C^∞ 分布密度

$$\begin{aligned}p_t(x)&=\langle\delta_x(W(t)), 1\rangle\\&=(2\pi t)^{-1/2}\exp\{-|x|^2/2t\},\end{aligned}$$

其中 $\delta_x(W(t))\in\mathbb{D}^{-\infty}$ 称为 Donsker δ 函数，对于 $F\in\mathbb{D}^\infty$，条件期望

$$\mathbb{E}[F|W(t)=x]=p_t(x)^{-1}\langle\delta_x(W(t)), F\rangle$$

具有 C^∞ 修正.

我们知道 (例如参看 Ikeda - Watanabe[3]), 抛物型方程

$$\partial_t u = \frac{1}{2}\Delta u + v \cdot u$$

的基本解可形式地表示为

$$p(t,x,y) = \mathbb{E}\left[\delta_y(x+W(t))\exp\left\{\int_0^t v(x+W(s))ds\right\}\right]. \quad (4.24)$$

若 $v \in C^{2n}(\mathbb{R}^d), n \geq [d/2]+1, v$ 的所有导数增长的阶不超过多项式且

$$\overline{\lim_{|x|\to\infty}} v(x)/|x|^2 = a < \infty,$$

则当 $a < 1/2t$ 时, 可以证明 $\exists p > 1$ 使

$$G \equiv \exp\left\{\int_0^t v(x+W(s))ds\right\} \in \mathbb{D}_{2n}^p.$$

因此, $p(t,x,\cdot) \in C^{2k}(\mathbb{R}^d)$, 其中 $k = n-1-[d/2]$.

在叙述例 2 之前, 我们先证明一个有用的命题.

命题 4.12 设 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间. 若 $\varphi_1, \cdots, \varphi_n \in \mathbb{D}_1^2, f$ 为 $\mathbb{R}^n$ 上的 Lipschitz 函数 (Lipschit 常数为 K), 则 $F \equiv f(\varphi_1,\cdots,\varphi_n) \in \mathbb{D}_1^2$, 且

$$DF = \sum_{j=1}^n G_j D\varphi_j, \quad (4.25)$$

其中随机向量 $G = (G_1,\cdots,G_n)$ 满足 $|G| \leq K$.

证明 先设 $f \in C_b^1(\mathbb{R}^n)$, 由微分法则 (2.26) 及光滑泛函逼近可知 $F \in \mathbb{D}_1^2$ 且当 $G_j = \partial_j f(\varphi_1,\cdots,\varphi_n)(1 \leq j \leq n)$ 时 (4.25) 成立.

现设 f 为 Lipschitz 函数 (Lipschit 常数为 K), 则存在 C^∞ 函数序列 $\{f_m\}$ 一致收敛于 f 且 $|\nabla f_m| \le K$. 于是

$$F_m \equiv f_m(\varphi_1, \cdots, \varphi_n) \xrightarrow{L^2(\Omega)} F,$$

且 $\{DF_m\}$ 在 $L^2(\Omega; H)$ 中一致有界, 从而存在子序列 $\{DF_{m_k}\}$ 在 $L^2(\Omega; H)$ 中弱收敛, 亦即 $\{F_{m_k}\}$ 在 $\mathbb{D}_1^2$ 中弱收敛. 因 $\mathbb{D}_1^2$ 是弱序列完备的, 既然 F_m 在 $L^2(\Omega)$ 中收敛于 F, 故 $\{F_{m_k}\}$ 在 $\mathbb{D}_1^2$ 中弱极限必为 F, 因此 $F \in \mathbb{D}_1^2$.

另一方面, $\{\nabla f_{m_k}(\varphi_1, \cdots, \varphi_n)\}$ 一致有界, 从而存在子序列在 $L^2(\Omega; \mathbb{R}^n)$ 中弱收敛于某 G, 于是 (4.25) 式成立. ∎

例 2 (连续过程极大值的分布密度) 设 $\{X(t), 0 \le t \le 1\}$ 为 Gauss 概率空间 $(\Omega, \mathcal{F}, \mu; H)$ 上的连续随机过程, 满足条件:

1° $\mathbb{E}[\sup_{0 \le t \le 1} X(t)^2] < \infty$;

2° $X(t) \in \mathbb{D}_1^2, 0 \le t \le 1$;

3° H 值过程 $\{DX(t), 0 \le t \le 1\}$ 有连续修正, 且

$$\mathbb{E}\Big[\sup_{0 \le t \le 1} \|DX(t)\|_H^2\Big] < \infty,$$

则随机变量 $G \equiv \sup_{0 \le t \le 1} X(t) \in \mathbb{D}_1^2$. 事实上, 设 $\{r_k\}$ 为 [0,1] 中全部有理数, 对任一 $n \in \mathbb{N}$, 将其前 n 个按次序重排, 记为: $t_1 < t_2 < \cdots < t_n$. 因为 $\max\{x_1, \cdots, x_n\}$ 为 Lipschitz 函数, 故

$$G_n \equiv \max\{X(t_1), \cdots, X(t_n)\} \in \mathbb{D}_1^2.$$

当 $n \to \infty$ 时 G_n 在 $L^2(\Omega)$ 中收敛于 G, 且 $\{DG_n\}$ 在 $L^2(\Omega; H)$ 中一致有界, 从而 $G \in \mathbb{D}_1^2$.

由定理 4.7 容易证明, 若在集合 $\{t \in [0,1] : X(t) = G\}$ 上有 $\|DX(t)\|_H \ne 0$, 则 G 的分布密度存在. 从证明过程还可以看出, 参数集 [0,1] 可以换成任一紧距离空间. 例如, Florit - Nualart[1] 证明了两参数 Brown 运动 (Brownian sheet) 的极大值具有 C^∞ 分布密度.

第三章　Wiener 泛函的随机变分

在应用中最重要的 Gauss 概率空间是经典 Wiener 空间，最重要的 Wiener 泛函是所谓 **Itô 泛函**，即 Itô 随机积分和 Itô 随机微分方程的解. 在本章中，我们将着重阐述 Wiener 泛函的随机变分理论及其在抛物型偏微分方程的基本解正则性研究等方面的应用，特别是给出关于偏微分算子亚椭圆性的 Hörmander 定理的概率方法证明. 此外我们还着重介绍新近发展起来的两个重要方向：拟必然分析 (Quasi sure analysis) 和非适应随机分析 (Anticipating stochastic calculus).

§1. Itô 泛函的微分分析与热核的正则性

1.1 Skorohod 积分

本节恒设 $H = L^2(\mathbb{R}_+;\mathbb{R}^d), \Omega = C_0(\mathbb{R}_+;\mathbb{R}^d)$，即 $\mathbb{R}_+$ 上在 0 点为 0 的 $\mathbb{R}^d$ 值连续函数全体按有界区间一致收敛拓扑构成的 Fréchet 空间，μ 为 $(\Omega,\mathcal{B}(\Omega))$ 上的 Wiener 测度. 对 $t\in\mathbb{R}_+,\omega\in\Omega$，令 $W_t(\omega)\equiv\omega(t)$，则 $\{W_t,t\in\mathbb{R}_+\}$ 为 $(\Omega,\mathcal{B}(\Omega),\mu)$ 上的 d 维 Brown 运动. 设 $\{\mathcal{F}_t,t\in\mathbb{R}_+\}$ 为由此 Brown 运动生成的自然 σ- 代数流，$\mathcal{F}=\mathcal{F}_\infty$ 为 $\mathcal{B}(\Omega)$ 关于 μ 的完备化 σ- 代数. 对 $h\in H$，令

$$W_h\equiv\int_0^\infty h(t)\cdot dW_t=\sum_{j=1}^d\int_0^\infty h_j(t)dW_t^j,$$

则 $(\Omega,\mathcal{F},\mu;H)$ 为不可约 Gauss 概率空间. 对 $h\in H$，令 $\widetilde{h}(t)\equiv\int_0^t h(s)ds$，则 $\widetilde{H}\equiv\{\widetilde{h}:h\in H\}\subset\Omega$ 为 Cameron-Martin 子空间.

设 E 为可分 Hilbert 空间，则 E 值泛函 $F\in L^2(\Omega;E)$ 有唯一

分解：

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{1.1}$$

其中 $f_n \in H^{\widehat{\otimes} n} \otimes E(n \geq 1)$. 注意 $H^{\otimes n} \cong L^2(\mathbb{R}_+^n; (\mathbb{R}^d)^{\otimes n})$, 故 $f_n \in \widehat{L^2}(\mathbb{R}_+^n; (\mathbb{R}^d)^{\otimes n} \otimes E)$, 即 $L^2(\mathbb{R}_+^n)$ 中对称函数构成的子空间. 记 $\Delta_n \equiv \{(t_1, \cdots, t_n) \in \mathbb{R}_+^n : t_1 \leq t_2 \leq \cdots \leq t_n\}$, 则

$$I_n(f_n) = n! \int_{\Delta_n} f_n(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} \tag{1.2}$$

为 n 次叠代 Itô 积分.

若 $E = L^2(\mathbb{R}_+; \mathbb{R}^m)$, 则 $L^2(\Omega; E) \cong L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^m)$, 因而 $X \in L^2(\Omega; E)$ 可以看作 $\mathbb{R}^m$ 值随机过程, 具有分解：

$$X_t = \mathbb{E}[X_t] + \sum_{n=1}^{\infty} I_n(f_{n+1}(\cdot, t)), \quad t \in \mathbb{R}_+, \tag{1.3}$$

其中 $f_{n+1} \in L^2(\mathbb{R}_+^{n+1}; (\mathbb{R}^d)^{\otimes n} \otimes \mathbb{R}^m)$, 对固定 $t \in \mathbb{R}_+$, $f_{n+1}(\cdot, t) \in \widehat{L^2}(\mathbb{R}_+^n; (\mathbb{R}^d)^{\otimes n} \otimes \mathbb{R}^m)$.

若 E 为任意可分 Hilbert 空间, $X \in L^2(\Omega; H \otimes E) \cong L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^d \otimes E)$ 且 $X \in \mathcal{D}(\delta)$, 则我们将 X 的散度 δX 记为

$$\delta X \equiv \int_0^{\infty} X_t dW_t, \tag{1.4}$$

并称为过程 X 的 **Skorohod 积分**.

定义 1.1 随机过程 $X : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}^m$, 若 $\forall t \in \mathbb{R}_+, X$ 在 $[0, t] \times \Omega$ 上的限制 $X|_{[0,t] \times \Omega}$ 为 $\mathcal{B}[0, t] \times \mathcal{F}_t$ 可测, 则称为 **循序过程**.

循序过程必为可测适应过程. 反之, 任一可测适应过程必存在循序修正 (例如参看 Dellacherie - Meyer[1]). 所以, 我们讨论可测适应过程时, 均采用其循序修正. 下面将要证明, 当 X 为循序过程时, 其 Skorohod 积分重合于 Itô 积分.

引理 1.2 设 $F \in L^2(\Omega; E)$ 具有分解式 (1.1), 则 $\forall t \in \mathbb{R}_+$

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n \mathbf{1}_{[0,t]}^{\otimes n}), \quad \text{a.s.}. \tag{1.5}$$

证明 不妨设 $F = I_n(f_n), n \geq 1$. 记 $\Delta_n(t) \equiv \{(t_1, \cdots, t_n) \in \Delta_n : t_n \leq t\}$, 则由 (1.2) 式得

$$\begin{aligned}\mathbb{E}[F|\mathcal{F}_t] &= n! \int_{\Delta_n(t)} f_n(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} \\ &= I_n(f_n 1_{[0,t]}^{\otimes n}), \quad \text{a.s.}.\end{aligned}$$ ∎

由此可见，若 $X \in L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^m)$ 具有分解式 (1.3), 则 X 为循序过程的充要条件是： $\forall t \in \mathbb{R}_+, n \in \mathbb{N}$,

$$f_{n+1}(\cdot, t) = f_{n+1}(\cdot, t)\mathbf{1}_{[0,t]}^{\otimes n}, \quad \text{a.e.}. \tag{1.6}$$

引理 1.3 设 $F \in \mathbb{D}_1^2(E)$, 则 $\forall t \in \mathbb{R}_+, \mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}_1^2(E)$ 且

$$D_t \mathbb{E}[F|\mathcal{F}_s] = \mathbb{E}[D_t F|\mathcal{F}_s]\mathbf{1}_{[0,s]}(t), \quad \text{a.s.}. \tag{1.7}$$

证明 由 (II.2.46) 及 (1.5) 式

$$\begin{aligned}D_t \mathbb{E}[F|\mathcal{F}_s] &= \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)\mathbf{1}_{[0,s]}^{\otimes(n-1)})\mathbf{1}_{[0,s]}(t) \\ &= \mathbb{E}[D_t F|\mathcal{F}_s]\mathbf{1}_{[0,s]}(t), \quad \text{a.s.}.\end{aligned}$$ ∎

注 若 F 为 $\mathcal{F}_s$ 可测，由 (1.7) 式，当 $t > s$ 时， $D_t F = 0$ a.s..

引理 1.4 若 $X \in \mathbb{D}_2^2(H \otimes E)$, 则 $\delta X \in \mathbb{D}_1^2(E)$ 且 $\forall t \in \mathbb{R}_+$

$$D_t(\delta X) = X_t + \int_0^{\infty} D_t X_s dW_s. \tag{1.8}$$

证明 设 X 具有分解式 (1.3), 则由 (II.2.46) 式得

$$D_t X_s = \sum_{n=1}^{\infty} n I_{n-1}(f_{n+1}(\cdot, t, s)).$$

由 (II.2.53) 式计算其 Skorohod 积分

$$\int_0^\infty D_t X_s dW_s = \sum_{n=1}^\infty n I_n(\widehat{f}_{n+1}(\cdot,t,\cdot)),$$

其中 $\widehat{f}_{n+1}(\cdot,t,\cdot)$ 表示 f_{n+1} 中固定变元 t 后关于其余 n 个变元的对称化. 另一方面, 由 (II.2.53) 式 $\delta X = \sum_{n=1}^\infty I_n(\widetilde{f}_n)$, 故

$$\begin{aligned}
D_t(\delta X) &= \sum_{n=1}^\infty n I_{n-1}(\widetilde{f}_n(\cdot,t)) \\
&= \sum_{n=0}^\infty I_n(\widetilde{f}_{n+1}(\cdot,t)) + \sum_{n=1}^\infty n I_n(\widetilde{f}_{n+1}(\cdot,t)) \\
&= X_t + \int_0^\infty D_t X_s dW_s . \qquad \blacksquare
\end{aligned}$$

引理 1.5 若 $X \in \mathbb{D}_3^2(H \otimes E)$, 则

$$\mathcal{L}\delta X = \delta\mathcal{L}X - \delta X. \tag{1.9}$$

证明 由 (1.8) 式

$$\begin{aligned}
\mathcal{L}\delta X &= -\delta D\delta X \\
&= -\int_0^\infty D_t(\delta X) dW_t \\
&= -\int_0^\infty X_t dW_t - \int_0^\infty \int_0^\infty D_t X_s dW_s dW_t \\
&= -\delta X + \delta\mathcal{L}X. \qquad \blacksquare
\end{aligned}$$

定理 1.6 设 $X \in L^2(\Omega; H \otimes \mathbb{R}^m) \cong L^2(\mathbb{R}_- \times \Omega; \mathbb{R}^m \otimes \mathbb{R}^d)$ 为循序过程, 则 $X \in \mathcal{D}(\delta)$ 且

$$\delta X = \int_0^\infty X_t \cdot dW_t$$

重合于 X 的 Itô 积分.

证明 设 X 具有分解式 (1.3) 且其中 f_{n+1} 满足 (1.6) 式 ($n \in \mathbb{N}$). 因

$$I_n(f_{n+1}(\cdot,t)) = n! \int_{\Delta_n(t)} f_{n+1}(t_1,\cdots,t_n,t)dW_{t_1}\cdots dW_{t_n},$$

计算其第 $n+1$ 次叠代 Itô 积分得

$$\begin{aligned}
&\int_0^\infty I_n(f_{n+1}(\cdot,t))\cdot dW_t\\
&= n!\int_{\Delta_{n+1}} f_{n+1}(t_1,\cdots,t_n,t)dW_{t_1}\cdots dW_{t_n}dW_t\\
&= (n+1)!\int_{\Delta_{n+1}} \widetilde{f}_{n+1}(t_1,\cdots,t_n,t)dW_{t_1}\cdots dW_{t_n}dW_t\\
&= I_{n+1}(\widetilde{f}_{n+1}).
\end{aligned}$$

由 (II.2.50) 及 (1.6) 式可知 $\|\widetilde{f}_{n+1}\|^2 \le \frac{1}{n+1}\|f_{n+1}\|^2$, 从而

$$\sum_{n=1}^\infty n!\|\widetilde{f}_n\|^2 \le \sum_{n=1}^\infty (n-1)!\|f_n\|^2 = \|X\|^2 < \infty.$$

由命题 II.2.14, $X \in \mathcal{D}(\delta)$ 且

$$\begin{aligned}
\delta X &= \sum_{n=0}^\infty I_{n+1}(\widetilde{f}_{n+1})\\
&= \sum_{n=0}^\infty \int_0^\infty I_n(f_{n+1}(\cdot,t))\cdot dW_t\\
&= \int_0^\infty X_t\cdot dW_t \quad (\text{Itô 积分}).
\end{aligned}$$ ∎

作为推论, 我们顺便得到泛函的随机积分表现的 Clark-Ocone 公式 (参看 Clark[1], Haussmann[1], Ocone[1]).

定理 1.7 设 $F \in \mathbb{D}_1^2(E)$, 则

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dW_t. \tag{1.10}$$

证明 设 F 具有分解式 (1.1), 由引理 1.3 有

$$\mathbb{E}[D_tF|\mathcal{F}_t] = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t)\mathbf{1}_{[0,t]}^{\otimes(n-1)}).$$

因为 $nf_n(\cdot,t)\mathbf{1}_{[0,t]}^{\otimes(n-1)}$ 关于 n 个变量对称化恰好等于 f_n, 由 (II.2.53) 式计算其 Skorohod 积分得

$$\int_0^{\infty} \mathbb{E}[D_tF|\mathcal{F}_t]dW_t = \sum_{n=1}^{\infty} I_n(f_n) = F - \mathbb{E}[F].$$ ∎

注意 $\{\mathbb{E}[D_tF|\mathcal{F}_t], t \in \mathbb{R}_+\}$ 为循序过程, 因此 (1.10) 式中的随机积分重合于 Itô 积分. 此外, 当 X 为循序过程时, (1.8) 式和 (1.9) 式中的 δX 均可理解为 Itô 积分, 因此引理 1.4 和引理 1.5 可以分别推广到 $X \in \mathbb{D}_1^2(H \otimes E)$ 及 $X \in \mathbb{D}_2^2(H \otimes E)$ 的情形.

下一定理是 Itô 积分等距性质的一个推广.

定理 1.8 设 $X, Y \in \mathbb{D}_1^2(H) \cong L^2(\mathbb{R}_+; \mathbb{D}_1^2)$, 则

$$\mathbb{E}[\delta X \delta Y] = \int_0^{\infty} \mathbb{E}[X_tY_t]dt + \int_0^{\infty}\int_0^{\infty} \mathbb{E}[D_sX_tD_tY_s]dsdt. \tag{1.11}$$

证明 由引理 1.4 及分部积分公式 (II.3.47)

$$\begin{aligned}
\mathbb{E}[\delta X \delta Y] &= \mathbb{E}\Big[\int_0^{\infty} X_tD_t(\delta Y)dt\Big] \\
&= \mathbb{E}\Big[\int_0^{\infty} X_tY_tdt\Big] + \mathbb{E}\Big[\int_0^{\infty} X_t\Big(\int_0^{\infty} D_tY_sdW_s\Big)dt\Big] \\
&= \int_0^{\infty} \mathbb{E}[X_tY_t]dt + \int_0^{\infty} \mathbb{E}\Big[X_t\int_0^{\infty} D_tY_sdW_s\Big]dt \\
&= \int_0^{\infty} \mathbb{E}[X_tY_t]dt + \int_0^{\infty} \mathbb{E}\Big[\int_0^{\infty} (D_sX_tD_tY_s)ds\Big]dt,
\end{aligned}$$

(1.11) 式得证. ∎

注 若 X 及 Y 为循序过程, 则当 $t > s$ 时 $D_tY_s = 0$ a.s., 而当 $s > t$ 时, $D_sX_t = 0$ a.s., 故 (1.11) 式最后一个积分为 0, 我们重新得到了 Itô 积分的等距公式.

1.2 随机微分方程解的光滑性

下面考察 Itô 随机微分方程

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s) \cdot dW_s, \quad t \geq 0, \tag{1.12}$$

其中 $x \in R^m, b : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ 及 $\sigma : \mathbb{R}^m \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^d$ 为 Borel 可测函数，满足 Lipschitz 条件： $\exists K > 0, \forall x, y \in \mathbb{R}^m$

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| \leq K|x - y|. \tag{1.13}$$

由随机微分方程理论 (例如参看 Ikeda - Watanabe[3] 或黄志远 [4]) 可知，此方程存在唯一强解 $X = X(x, t, \omega)$, 且

1° 对 $\forall x \in \mathbb{R}^m, X(x, \cdot, \cdot)$ 为扩散过程；

2° 对 a.a.$\omega[\mu], X(\cdot, \cdot, \omega)$ 关于 (x, t) 连续；

3° 对 a.a.$\omega[\mu], \forall t \in \mathbb{R}_+, X(\cdot, t, \omega)$ 为 $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ 的同胚；

4° 对 $p \geq 2, T > 0$ 及 $R > 0, \exists C = C(p, T, R)$ 使

$$\sup_{|x| \leq R} \mathbb{E}\Big[\sup_{0 \leq t \leq T} |X(x, t)|^p\Big] \leq C. \tag{1.14}$$

若函数 b 及 σ 无穷次可微且一切偏导数有界，则

5° 对 a.a.$\omega[\mu], \forall t \in \mathbb{R}_+, X(\cdot, t, \omega)$ 为 $\mathbb{R}^m \longrightarrow \mathbb{R}^m$ 的 C^∞ 微分同胚；

6° 对 $p \geq 2, T > 0, k \in \mathbb{N}$ 及 $R > 0, \exists C = C(p, T, k, R)$ 使

$$\sup_{|x| \leq R} \mathbb{E}\Big[\max_{|\alpha| \leq k} \sup_{0 \leq t \leq T} |\partial_\alpha X(x, t)|^p\Big] \leq C, \tag{1.15}$$

其中 $\alpha = (\alpha_1, \cdots, \alpha_m), |\alpha| = \sum_{j=1}^m \alpha_m, \partial_\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$ 为关于 x 的偏导数.

对 $t \in \mathbb{R}_+$, 令

$$J_t \equiv J(x, t, \omega) = (\partial_j X^i(x, t, \omega))_{1 \leq i, j \leq m} \tag{1.16}$$

为 X 关于初值 x 微分的 Jacobi 矩阵，则 J_t 及其逆矩阵 J_t^{-1} 分别满足如下 Itô 随机微分方程：

$$J_t = I + \int_0^t A_0^{(1)}(X_s)J_s ds + \sum_{k=1}^d \int_0^t A_k^{(1)}(X_s)J_s \cdot dW_s^k, \qquad (1.17)$$

$$\begin{aligned} J_t^{-1} = I - \int_0^t J_s^{-1}\Big\{A_0^{(1)}(X_s) - \sum_{k=1}^d [A_k^{(1)}(X_s)]^2\Big\}ds \\ - \sum_{k=1}^d \int_0^t J_s^{-1} A_k^{(1)}(X_s)\cdot dW_s^k, \end{aligned} \qquad (1.18)$$

其中 I 为 $m\times m$ 单位矩阵， $A_0^{(1)}(x) \equiv (\partial_j b^i(x))_{1\le i,j\le m}$, $A_k^{(1)}(x) \equiv (\partial_j \sigma_k^i(x))_{1\le i,j\le m}$, $(k=1,2,\cdots,d)$.

下面考虑 X 关于“样本点”ω 的微分，即 Malliavin 意义下的弱导数.

定理 1.9 设方程 (1.12) 中的系数 b 和 σ 为 C^∞ 函数且一切偏导数有界，则其唯一解 $X = X(x,t,\cdot) \in \mathbb{D}^\infty(\mathbb{R}^m)(\forall x\in\mathbb{R}^m, t>0)$, 其协方差阵 $\Sigma_t \equiv \Sigma(x,t,\omega)$ 可表示为

$$\Sigma_t = J_t\left[\int_0^t J_s^{-1} a(X_s)(J_s^{-1})^* ds\right] J_t^*, \qquad \text{a.s.}, \qquad (1.19)$$

其中 $a=\sigma\sigma^*$, J_t 为 X_t 关于初值 x 微分的 Jacobi 矩阵.

证明 由估计式 (1.14) 可知， $X_t \in L^{\infty-}(\Omega;\mathbb{R}^m)\,(t\ge 0)$. 因为 J_t 及 J_t^{-1} 分别为方程 (1.17) 及 (1.18) 的解，故仍有 $J_t, J_t^{-1} \in L^{\infty-}(\Omega;\mathbb{R}^m\otimes\mathbb{R}^m)$.

由 Picard 逼近

$$\begin{aligned} &X_t^{(0)} \equiv x; \\ &X_t^{(n+1)} = x + \int_0^t b(X_s^{(n)})ds + \int_0^t \sigma(X_s^{(n)})\cdot dW_s, \quad n\ge 0. \end{aligned}$$

应用命题 II.4.12 及引理 1.4 容易证明 $X \in \mathbb{D}_1^2(H \otimes \mathbb{R}^m)$.

既然 X 为连续适应过程，所以当 $s > t$ 时，$D_sX_t = 0$ a.s.. 当 $s \le t$ 时，由引理 1.4 及复合泛函微分法则可得

$$\begin{aligned} D_sX_t &= \int_s^t D_sb(X_r)dr + \sigma(X_s) + \sum_{k=1}^d \int_s^t D_s\sigma_k(X_r)dW_r^k \\ &= \int_s^t A_0^{(1)}(X_r)D_sX_rdr + \sigma(X_s) \\ &\quad + \sum_{k=1}^d \int_s^t A_k^{(1)}(X_r)D_sX_rdW_r^k, \end{aligned} \tag{1.20}$$

其中 σ_k 为矩阵 σ 之第 k 列 $(k = 1, \cdots, d)$.

另一方面，由 (1.17) 及 (1.18) 式可得

$$J_tJ_s^{-1} = I + \int_s^t A_0^{(1)}(X_r)J_rJ_s^{-1}dr + \sum_{k=1}^d \int_s^t A_k^{(1)}(X_r)J_rJ_s^{-1}dW_r^k,$$

从而

$$\begin{aligned} J_tJ_s^{-1}\sigma(X_s) &= \sigma(X_s) + \int_s^t A_0^{(1)}(X_r)J_rJ_s^{-1}\sigma(X_s)dr \\ &\quad + \sum_{k=1}^d \int_s^t A_k^{(1)}(X_r)J_rJ_s^{-1}\sigma(X_s)dW_r^k. \end{aligned} \tag{1.21}$$

比较 (1.20) 及 (1.21) 式，由随机微分方程解的唯一性即得

$$D_sX_t = J_tJ_s^{-1}\sigma(X_s)1_{[0,t]}(s), \quad \text{a.s..} \tag{1.22}$$

因此有

$$\begin{aligned} \Sigma_t &= \int_0^\infty (D_sX_t)(D_sX_t)^*ds \\ &= J_t\left[\int_0^t J_s^{-1}a(X_s)(J_s^{-1})^*ds\right]J_t^*, \quad \text{a.s.}, \end{aligned}$$

且

$$\|DX_t\|^2_{H\otimes \mathbb{R}^m} = \mathrm{Tr}\Sigma_t \in L^{\infty-}.$$

以方程 (1.20) 代替 (1.12), 类似地可以计算高阶导数, 并容易看出, $\forall k \in \mathbb{N}, t \in \mathbb{R}_+, \|D^k X_t\| \in L^{\infty-}$, 因此 $X_t \in \mathbb{D}^{\infty}(\mathbb{R}^m)$. ∎

利用光滑函数逼近系数 b 及 σ, 并应用命题 II.4.12, 及定理 II.4.7 可以证明如下结论 (详细证明可参看 Nualart[1] 或 Bouleau - Hirsch[1,2]):

定理 1.10 设方程 (1.12) 中的系数 b 和 σ 满足 Lipschitz 条件 (1.13), 则其唯一解 $X = X(x,t,\cdot) \in \mathbb{D}_1^{\infty}(\mathbb{R}^m)(\forall x \in \mathbb{R}^m, t > 0)$, 令

$$\tau \equiv \inf\left\{t > 0 : \int_0^t 1_{\{\det a(X_s)\neq 0\}} ds > 0\right\},$$

则在集合 $\{t > \tau\}$ 上, $\mu \circ X_t^{-1}$ 关于 Lebesgue 测度绝对连续.

1.3 亚椭圆性与 Hörmander 条件

考虑二阶偏微分算子

$$L = \frac{1}{2}\sum_{i,j=1}^m a^{ij}(\cdot)\partial_i\partial_j + \sum_{i=1}^m b^i(\cdot)\partial_i \tag{1.23}$$

及热方程的 Cauchy 问题:

$$\begin{cases} \partial_t u(t,x) = Lu(t,x), & t > 0, x \in \mathbb{R}^m; \\ u(0,x) = \varphi(x). \end{cases} \tag{1.24}$$

我们知道, 若 $\varphi \in C_b^2(\mathbb{R}^m)$, 则

$$u_{\varphi}(t,x) \equiv \mathbb{E}[\varphi(X(x,t,\cdot))]$$

为 (1.24) 的解. 由定理 II.4.10 可知, 若 X_t 非退化, 即

$$[\det \Sigma_t]^{-1} \in L^{\infty-},$$

则扩散过程 X 的转移概率 $P(t,x,\cdot)=\mu\circ X(x,t,\cdot)^{-1}$ 存在 C^∞ 密度

$$p(t,x,y)=I\!E[\delta_y(X(x,t,\cdot))],$$

亦即方程 (1.24) 的基本解 (所谓 **热核**).

由偏微分方程理论可知，当矩阵 $a(x)$ 一致正定，即 $\exists\eta>0$ 使 $a(\cdot)\geq\eta I$ 时，此结果是成立的. 1967 年，Hörmander[1] 得到了关于亚椭圆 (hypoelliptic) 算子的一个相当弱的充分条件，这就是著名的 Hörmander 定理. 为了叙述这个定理，需要先将算子 L 写成向量场的形式. 为了记号简化，以后我们均采用 Einstein 约定：当一个指标重复出现在上标及下标中时，均意味着对此指标求和. 令

$$A_k(\cdot)\equiv\sigma_k^i(\cdot)\partial_i,\quad k=1,\cdots,d,$$

$$A_0(\cdot)\equiv\left[b^i(\cdot)-\frac{1}{2}\sum_{k=1}^d\sigma_k^j(\cdot)\partial_j\sigma_k^i(\cdot)\right]\partial_i,$$

则 $A_0,A_1,\cdots,A_k$ 均为 $I\!R^m$ 上的 C^∞ 向量场. 注意

$$\sum_{k=1}^d A_k^2=a^{ij}\partial_i\partial_j+\sum_{k=1}^d\sigma_k^j[\partial_j\sigma_k^i]\partial_i,$$

故

$$L=\frac{1}{2}\sum_{k=1}^d A_k^2+A_0.\tag{1.25}$$

若令 $\widetilde{b}\equiv b-\frac{1}{2}\sum_{k=1}^d A_k^{(1)}\sigma_k$, 其中 $A_k^{(1)}\equiv(\partial_j\sigma_k^i)_{1\leq i,j\leq m}$, 则 $A_0(\cdot)=\widetilde{b}^i(\cdot)\partial_i$. 因为 Itô 方程 (1.12) 等价于 **Fisk-Stratonovich 方程**:

$$X_t=x+\int_0^t\widetilde{b}(X_s)ds+\int_0^t\sigma(X_s)\circ dW_s,\quad t\geq 0,\tag{1.26}$$

所以此方程也可写成向量场形式:

$$dX_t=A_0(X_t)dt+A_k(X_t)\circ dW_t^k.\tag{1.27}$$

此式应当理解为： $\forall f\in C_b^\infty(\mathbb{R}^m)$ 有

$$df(X_t)=(A_0f)(X_t)dt+(A_kf)(X_t)\circ dW_t^k.$$

在以下讨论中，若 $V\in C^\infty(\mathbb{R}^m,\mathbb{R}^m)$, 则同时以 V 表示 C^∞ 向量场： $V(\cdot)=V^i(\cdot)\partial_i$. 注意方程 (1.17) 及 (1.18) 分别等价于 Fisk-Stratonovich 方程：

$$dJ_t=\widetilde{A}_0^{(1)}(X_t)J_tdt+A_k^{(1)}(X_t)J_t\circ dW_t^k \tag{1.28}$$

及

$$dJ_t^{-1}=-J_t^{-1}\widetilde{A}_0^{(1)}(X_t)dt-J_t^{-1}A_k^{(1)}(X_t)\circ dW_t^k. \tag{1.29}$$

其中 $\widetilde{A}_0^{(1)}(x)=(\partial_j\widetilde{b}^i(x))_{1\le i,j\le m}$. 由 Stratonovich 微分规则得

$$\begin{aligned}
d[J_t^{-1}V(X_t)]&=(dJ_t^{-1})\circ V(X_t)+J_t^{-1}\circ dV(X_t)\\
&=-J_t^{-1}\widetilde{A}_0^{(1)}(X_t)V(X_t)dt-J_t^{-1}A_k^{(1)}(X_t)V(X_t)\circ dW_t^k\\
&\quad+J_t^{-1}(A_0V)(X_t)dt+J_t^{-1}(A_kV)(X_t)\circ dW_t^k.
\end{aligned}$$

由于 $[\widetilde{A}_0^{(1)}(x)V(x)]^i=V^j(x)\partial_j\widetilde{b}^i(x)=(V\widetilde{b}^i)(x)$, 故用向量场记号有

$$\widetilde{A}_0^{(1)}(x)V(x)=(VA_0)(x).$$

同样地，

$$A_k^{(1)}(x)V(x)=(VA_k)(x),\quad k=1,\cdots,d.$$

于是

$$\begin{aligned}
d[J_t^{-1}V(X_t)]&=J_t^{-1}(A_0V-VA_0)(X_t)dt\\
&\qquad+J_t^{-1}(A_kV-VA_k)(X_t)\circ dW_t^k\\
&=J_t^{-1}[A_0,V](X_t)dt\\
&\qquad+J_t^{-1}[A_k,V](X_t)\circ dW_t^k,
\end{aligned}\tag{1.30}$$

式中 $[\cdot,\cdot]$ 表示 Lie 括号. 将方程 (1.26) 和 (1.28) 联立起来，其解 $R_t\equiv(X_t,J_t)$ 为取值于 $\mathbb{R}^m\times(\mathbb{R}^m\otimes\mathbb{R}^m)$ 的随机过程，初值

为 $R_0=(x,I)$. 对任一向量场 V, 我们可以定义函数 $f_V: I\!R^m \times (I\!R^m \otimes I\!R^m) \longrightarrow I\!R^m$ 如下:

$$f_V(r) \equiv J^{-1}V(x), \quad 若 r=(x,J). \tag{1.31}$$

于是方程 (1.30) 可写成:

$$\begin{cases} df_V(R_t) = f_{[A_0,V]}(R_t)dt + f_{[A_k,V]}(R_t) \circ dW_t^k \\ f_V(R_0) = V(x). \end{cases} \tag{1.32}$$

为了将此方程转换成 Itô 方程, 引进以下记号:

$$\{A_k,V\} = [A_k,V], \quad k=1,\cdots,d,$$

$$\{A_0,V\} = [A_0,V] + \frac{1}{2}\sum_{k=1}^{d}[A_k,[A_k,V]].$$

在 (1.32) 中以 $[A_j,V]$ 代替 V, 得

$$df_{[A_j,V]}(R_t) = f_{[A_0,[A_j,V]]}(R_t)dt + f_{[A_k,[A_j,V]]}(R_t) \circ dW_t^k, \quad j=1,\cdots,d.$$

于是

$$f_{[A_k,V]}(R_t) \circ dW_t^k = f_{[A_k,V]}(R_t) \cdot dW_t^k + \frac{1}{2}\sum_{k=1}^{d} f_{[A_k,[A_k,V]]}(R_t)dt,$$

故方程 (1.32) 等价于 Itô 方程:

$$\begin{cases} df_V(R_t) = f_{\{A_0,V\}}(R_t)dt + f_{\{A_k,V\}}(R_t) \cdot dW_t^k, \\ f_V(R_0) = V(x). \end{cases} \tag{1.33}$$

我们定义如下向量场的集合 $\widehat{\mathcal{V}}_n$ 及 $\mathcal{V}_n$:

$$\widehat{\mathcal{V}}_0 \equiv \{A_1,\cdots,A_d\}(注意\ A_0\ 不在其中),$$

$$\widehat{\mathcal{V}}_n \equiv \{\{A_0,V\},\{A_k,V\}, V\in \widehat{\mathcal{V}}_{n-1}, k=1,\cdots,d\}, \quad n\geq 1,$$

$$\mathcal{V}_n \equiv \bigcup_{m=0}^{n} \widehat{\mathcal{V}}_m, \quad n\in I\!N_0.$$

所谓 Hörmander 条件，是指：

(H): 由向量场 $\{A_k, [A_0, A_k], k = 1, \cdots, d\}$ 生成的 Lie 代数在一切 $x \in \mathbb{R}^m$ 处具有维数 m(注意 A_0 只在 Lie 括号中出现).

显然，此条件等价于：

(H)$'$: $\forall x \in \mathbb{R}^m, \exists N \in \mathbb{N}_0$ 及 $V_1, \cdots, V_m \in \mathcal{V}_N$，使 $V_1(x), \cdots V_m(x)$ 线性独立.

它又等价于以下条件：

(H)$''$: $\forall x \in \mathbb{R}^m, \exists N \in \mathbb{N}_0$ 使

$$\inf_{l \in S} \max_{V \in \mathcal{V}_N} (l, V(x))^2 > 0. \tag{1.34}$$

其中 $S = S^{m-1} \equiv \{x \in \mathbb{R}^m : |x| = 1\}$，即 m 维单位球面. 实际上，在 $\mathcal{V}_n$ 中只含有限个向量场，将它们排成一个矩阵，若其秩小于 m，则各行线性相关，从而存在 $l \in S$ 使 (1.34) 左边为 0; 反之，若其秩为 m，则各行线性独立，从而对任意 $l \in S$,(1.34) 左边均大于 0.

Hörmander 定理断言，在条件 (H) 下，L 为**亚椭圆算子**，即对 $\mathbb{R}^m$ 中一切开集 U 及广义函数 $u \in \mathcal{D}^*(\mathbb{R}^m)$，若 $Lu|_U \in C^\infty(U)$，则 $u|_U \in C^\infty(U)$.

由经典的偏微分方程理论可知，当 L 为椭圆算子 (即 $a(x)$ 正定) 时，必为亚椭圆算子. 为了说明亚椭圆算子未必椭圆，我们先看一个例子：

例 (Kolmogorov 1934) 设 $m = 2, d = 1, A_0(x) = x_1\partial_2, A_1(x) = \partial_1$，则

$$L = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + x_1\frac{\partial}{\partial x_2}. \tag{1.35}$$

此时

$$a(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix},$$

从而 L 不是椭圆算子. 但

$$\begin{aligned}[A_1, A_0] &= \partial_1(x_1\partial_2) - x_1\partial_2\partial_1 \\ &= x_1\partial_1\partial_2 + \partial_2 - x_1\partial_2\partial_1 = \partial_2.\end{aligned}$$

因为 (∂_1, ∂_2) 为 $\mathbb{R}^2$ 上切空间的基底, 故条件 (H) 满足, 由 Hörmander 定理, L 为亚椭圆算子.

考虑 L 扩散过程 $X_t = (X_t^1, X_t^2)$, 它是如下 Itô 方程的解:

$$\begin{cases} X_t^1 = x_1 + W_t, \\ X_t^2 = x_2 + \int_0^t X_s^1 ds. \end{cases} \tag{1.36}$$

显然 X 是 Gauss 过程, 且

$$m_t \equiv \mathbb{E}[X_t] = \begin{pmatrix} x_1 \\ x_2 + x_1 t \end{pmatrix},$$

$$V_t \equiv \mathrm{cov}(X_t) = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix},$$

$$V_t^{-1} = \frac{2}{t^3}\begin{pmatrix} 2t^2 & -3t \\ -3t & 6 \end{pmatrix}, \det V_t = \frac{t^4}{12}.$$

其转移密度为

$$\begin{aligned} p(t,x,y) &= (2\pi)^{-1}(\det V_t)^{-1/2}\exp\{-\tfrac{1}{2}(y - m_t, V_t^{-1}(y - m_t))\} \\ &= \frac{\sqrt{3}}{\pi t^2}\exp\left\{ - \frac{(x_1 - y_1)^2}{2t} - \frac{6[y_2 - x_2 - (x_1 + y_1)t/2]^2}{t^3} \right\}. \end{aligned} \tag{1.37}$$

显然它是关于 y 的 C^∞ 函数, 此即方程 $\partial_t u = Lu$ 的基本解. 由 (1.36) 可知

$$J_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad J_t^{-1} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

由 (1.19) 式计算其 Malliavin 协方差阵得

$$\Sigma_t = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix} = V_t,$$

显然此协方差阵非退化.

证明 Hörmander 定理，关键的一步就是要证明在 (H) 条件下所对应的 Malliavin 协方差阵非退化.

1.4 Hörmander 定理的概率证明

前面已经提到， Hörmander 定理的概率证明首先由 Malliavin[1] 给出，此后 Bismut[1], Kusuoka - Stroock[3], Ikeda - Watanabe[1] 也给出了各不相同的证明. 我们这里采用 Norris[1] 根据 Stroock[3] 的思路所提出的简化方法. 先证明一个引理:

引理 1.11 设 X 为一维 Itô 过程:

$$X_t = x + \int_0^t Y_s^0 ds + \sum_{k=1}^d \int_0^t Y_s^k dW_s^k, \quad t \geq 0, \tag{1.38}$$

其中 Y^0 仍为一维 Itô 过程:

$$Y_t^0 = y + \int_0^t Z_s^0 ds + \sum_{k=1}^d \int_0^t Z_s^k dW_s^k, \quad t \geq 0, \tag{1.39}$$

式中 $x, y \in \mathbb{R}, Y = (Y^1, \cdots, Y^d)$ 及 $Z = (Z^1, \cdots, Z^d)$ 为 d 维循序过程，且 $\exists K > 0$ 及有界停时 $\tau > 0$ 使

$$\sup_{0 \leq t \leq \tau} \{|Y_t^0| + |Z_t^0| + |Y_t| + |Z_t|\} \leq K,$$

则 $\forall q > 8, \nu < (q-8)/9$. 当 ϵ 充分小时， $\exists c > 0$ 使

$$\mu\Big\{\int_0^\tau X_t^2 dt < \epsilon^q, \int_0^\tau (|Y_t^0|^2 + |Y_t|^2) dt \geq \epsilon\Big\} \leq c \exp\{-\epsilon^{-\nu}\}. \tag{1.40}$$

证明 记 $A_t \equiv \int_0^t Y_s^0 ds, M_t \equiv \int_0^t Y_s dW_s, Q_t \equiv \int_0^t A_s Z_s dW_s$, $N_t \equiv \int_0^t X_s Y_s dW_s$,

$$S_1 \equiv \{[N]_\tau < \epsilon_1, \quad \sup_{t \leq \tau} |N_t| \geq \delta_1\},$$
$$S_2 \equiv \{[M]_\tau < \epsilon_2, \quad \sup_{t \leq \tau} |M_t| \geq \delta_2\},$$
$$S_3 \equiv \{[Q]_\tau < \epsilon_3, \quad \sup_{t \leq \tau} |Q_t| \geq \delta_3\}.$$

式中 $[\cdot]$ 表示连续局部鞅的平方变差过程，由指数不等式（参看引理后的注）可知：

$$\mu(S_i) \le 2\exp\{-\delta_i^2/2\epsilon_i\}, \quad i=1,2,3.$$

选 $q_1=(q-\nu)/2, q_2=(q_1/2-\nu)/2, q_3=(2q_2-\nu)/2$，则 $q>q_1>q_2>q_3>1$. 选 $\delta_i=\epsilon^{q_i}(i=1,2,3), \epsilon_1=c_1\epsilon^q, \epsilon_2=c_2\epsilon^{q_1/2}, \epsilon_3=c_3\epsilon^{2q_2}$ (c_1,c_2,c_3 为待定常数)，则

$$\delta_i^2/\epsilon_i \sim \epsilon^{-\nu}, \quad i=1,2,3.$$

为证引理，只要证明当 ϵ 充分小时可选适当的常数 c_1,c_2,c_3 使 (1.40) 式左边的事件含于 $\cup_{i=1}^3 S_i$. 为此只要证明：当 ϵ 充分小时，若 $\omega \notin \cup_{i=1}^3 S_i$，且 $\int_0^\tau X_t^2 dt < \epsilon^q$，则必有 $\int_0^\tau (|Y_t^0|^2+|Y_t|^2)dt<\epsilon$. 设 $\tau \le T$，下面分三步来证明：

1° 设 $c_1=K^2$，则

$$[N]_\tau = \int_0^\tau X_t^2|Y_t|^2 dt < K^2\epsilon^q = \epsilon_1.$$

由 $\omega\notin S_1$，必有 $\sup_{t\le\tau}|N_t|<\delta_1=\epsilon^{q_1}$. 但

$$\sup_{t\le\tau}\left|\int_0^t X_s Y_s^0 ds\right| \le \left(T\int_0^\tau (X_s Y_s^0)^2 ds\right)^{1/2} < KT^{1/2}\epsilon^{q/2},$$

故

$$\sup_{t\le\tau}\left|\int_0^t X_s dX_s\right| < \epsilon^{q_1} + KT^{1/2}\epsilon^{q/2}.$$

由 Itô 公式

$$[M]_t = X_t^2 - x^2 - 2\int_0^t X_s dX_s$$

可知

$$\int_0^\tau [M]_t dt < \epsilon^q + 2T(\epsilon^{q_1}+KT^{1/2}\epsilon^{q/2}).$$

因 $q_1 < q/2$, 当 ϵ 充分小时， $\exists c_0 > 0$ 使上式小于 $c_0\epsilon^{q_1}$. 注意到 $[M]$ 为增过程，对 $\delta > 0$ 有

$$\delta[M]_{\tau-\delta} < \int_{\tau-\delta}^{\tau}[M]_t dt < c_0\epsilon^{q_1},$$

$$[M]_\tau < [M]_{\tau-\delta} + K^2\delta,$$

选 $\delta = \epsilon^{q_1/2}$, 则 $\exists c_2 > 0$ 使 $[M]_\tau < c_2\epsilon^{q_1/2}$.

2° 设 $\epsilon_2 = c_2\epsilon^{q_1/2}$, 由 $\omega \notin S_2$, 必有 $\sup_{t\le\tau}|M_t| < \delta_2 = \epsilon^{q_2}$. 因 $\int_0^\tau X_t^2 dt < \epsilon^q$, 故

$$\lambda^1\{0 \le t \le \tau : |X_t| \ge \epsilon^{q/3}\} \le \epsilon^{q/3},$$

式中 λ^1 为一维 Lebesgue 测度. 因 $X_t = x + A_t + M_t$, 故

$$\lambda^1\{0 \le t \le \tau : |x + A_t| \ge \epsilon^{q/3} + \epsilon^{q_2}\} \le \epsilon^{q/3},$$

从而 $\forall t \in [0,\tau], \exists s \in [0,\tau], |s-t| \le \epsilon^{q/3}$, 使 $|x + A_s| < \epsilon^{q/3} + \epsilon^{q_2}$, 于是

$$|x + A_t| \le |x + A_s| + \left|\int_s^t Y_r^0 dr\right| < (1+K)\epsilon^{q/3} + \epsilon^{q_2}.$$

特别有 $|x| < (1+K)\epsilon^{q/3} + \epsilon^{q_2}$, 所以 $\forall t \in [0,\tau], |A_t| < 2((1+K)\epsilon^{q/3} + \epsilon^{q_2})$, 因 $q_2 < q/3$, 当 ϵ 充分小时有 $|A_t| < 3\epsilon^{q_2}$.

3° 由 Itô 公式,

$$\begin{aligned}\int_0^\tau (Y_t^0)^2 dt &= \int_0^\tau Y_t^0 dA_t \\ &= Y_\tau^0 A_\tau - \int_0^\tau A_t(Z_t^0 dt + Z_t \cdot dW_t),\end{aligned} \tag{1.41}$$

注意到 $|Y_\tau^0 A_\tau| < 3K\epsilon^{q_2}$, $|\int_0^\tau A_t Z_t^0 dt| < 3KT\epsilon^{q_2}$ 以及 $[Q]_\tau = \int_0^\tau A_t^2|Z_t|^2 dt < 9K^2T\epsilon^{2q_2}$, 令 $c_3 = 9K^2T, \epsilon_3 = c_3\epsilon^{2q_2}$, 因 $\omega \notin S_3$, 必有 $\sup_{t\le\tau}|Q_t| < \delta_3 = \epsilon^{q_3}$. 特别， $|Q_\tau| < \epsilon^{q_3}$, 由 (1.41) 式即知

$$\int_0^\tau (Y_t^0)^2 dt < 3K(1+T)\epsilon^{q_2} + \epsilon^{q_3},$$

因 $q_3 < q_2$，当 ϵ 充分小时上式小于 $2\epsilon^{q_3}$，于是

$$\int_0^T (|Y_t^0|^2 + |Y_t|^2)dt < 2\epsilon^{q_3} + c_2\epsilon^{q_1/2} < \epsilon. \qquad \blacksquare$$

注 若 M 为零初值连续局部鞅，$a > 0, c > 0, t > 0$，令

$$Z_s \equiv \exp\left\{\frac{c}{a}M_s - \frac{c^2}{2a^2}[M]_s\right\}, \quad s \geq 0,$$

则 Z 为非负上鞅. 由上鞅极大值不等式得

$$\begin{aligned}&\mu\left\{[M]_t < a; \sup_{0\leq s\leq t}|M_s| \geq c\right\}\\ &\leq \mu\left\{\sup_{0\leq s\leq t} Z_s \geq \exp(c^2/2a)\right\}\\ &\leq 2\exp\{-c^2/2a\}.\end{aligned}$$

上式称为连续局部鞅的指数不等式.

定理 1.12 设方程 (1.12) 中系数 b 及 σ 为 C^∞ 函数且一切偏导数有界，且由 (1.25) 给出的算子 L 满足条件 (H)，则其唯一解 $X = X(x,t,\omega)$ 具有 C^∞ 转移概率密度 $p(t,x,\cdot)$，亦即热方程 $\partial_t u = Lu$ 具有 C^∞ 光滑基本解.

证明 只需证明在 (H'') 条件下，$\forall t > 0$，协方差阵 Σ_t 非退化，即 $(\det\Sigma_t)^{-1} \in L^{\infty-}$. 注意由定理 1.9 及 $(\det J_t)^{-1} \in L^{\infty-}$，只需证明下列矩阵非退化：

$$\tilde{\Sigma}_t \equiv \int_0^t J_s^{-1}a(X_s)(J_s^{-1})^* ds.$$

固定 $t > 0$ 及 $c > 0$，令

$$\tau_c \equiv \inf\{s \geq 0 : |X_s - x| \vee \|J_s^{-1} - I\| \geq c^{-1}\} \wedge t,$$

则 τ_c 为停时，且对 $\epsilon \in (0,t)$

$$\{\tau_c \leq \epsilon\} = \left\{\sup_{s\leq\epsilon}|X_s - x| \vee \|J_s^{-1} - I\| \geq c^{-1}\right\}.$$

由方程 (1.12) 及 (1.18) 解的估计可知，$\forall p>1$ 有

$$\mathbb{E}\left[\sup_{s\le\epsilon}|X_s-x|^p\vee\|J_s^{-1}-I\|^p\right]=O(\epsilon^{p/2}),$$

从而

$$\tau_c^{-1}\in L^{\infty-}.$$

由条件 (H)″ 及方程 (1.33) 解对初值的连续依赖性可知，$\forall l_0\in S\equiv S^{m-1},\exists N\in\mathbb{N}_0,V\in\mathcal{V}_N$，以及 l_0 之某邻域 S_0，对足够大的 c 及足够小的 $\delta>0$ 有

$$\inf_{l\in S_0}\inf_{s\le\tau_c}(l,f_V(R_s))^2\ge\delta,\tag{1.42}$$

从而 $\forall p>1$(简记 τ_c 为 τ)，

$$\sup_{l\in S_0}\mu\left\{\int_0^\tau(l,f_V(R_s))^2ds<\epsilon\right\}\le\mu\{\delta\tau<\epsilon\}=o(\epsilon^p).\tag{1.43}$$

设 $V=\{A_{k_j},\{A_{k_{j-1}},\cdots\{A_{k_1},A_{k_0}\}\cdots\}\}$，其中 $0\le j\le N,1\le k_0\le d,0\le k_1,\cdots,k_j\le d$. 令 $V_0\equiv A_{k_0},V_1\equiv\{A_{k_1},V_0\},\cdots,V_j\equiv\{A_{k_j},V_{j-1}\}=V$，我们用归纳法证明：对 $i=j,j-1,\cdots,0$ 有

$$\sup_{l\in S_0}\mu\left\{\int_0^\tau(l,f_{V_i}(R_s))^2ds<\epsilon\right\}=o(\epsilon^p),\quad 1<p<\infty.\tag{1.44}$$

当 $i=j$ 时此即 (1.43) 式. 设 (1.44) 式对 i 成立，要证明它对 $i-1$ 也成立. 注意对 $l\in S$ 及 C^∞ 向量场 V 我们有

$$\begin{cases}d(l,f_V(R_t))=(l,f_{\{A_0,V\}}(R_t))dt+(l,f_{\{A_k,V\}}(R_t))dW_t^k\\(l,f_V(R_0))=(l,V(x)).\end{cases}\tag{1.45}$$

由引理 1.11 对 $q>8$, 当 ϵ 充分小时

$$\mu\left\{\int_0^\tau(l,f_{V_{i-1}}(R_s))^2ds<\epsilon^q,\int_0^\tau\sum_{k=0}^d(l,f_{\{A_k,V_{i-1}\}}(R_s))^2ds\ge\epsilon\right\}$$
$$=o(\epsilon^p),\quad 1<p<\infty.$$

由归纳假设可知

$$\sup_{l\in S_0}\mu\left\{\int_0^\tau\sum_{k=0}^d(l,f_{\{A_k,V_{i-1}\}}(R_s))^2ds<\epsilon\right\}=o(\epsilon^p),$$

从而

$$\sup_{l\in S_0}\mu\left\{\int_0^\tau(l,f_{V_{i-1}}(R_s))^2ds<\epsilon^q\right\}=o(\epsilon^p),$$

(1.44) 式得证. 特别，当 $i=0$ 时即有 $k\in[1,d]$ 使

$$\sup_{l\in S_0}\mu\left\{\int_0^\tau(l,f_{A_k}(R_s))^2ds<\epsilon\right\}=o(\epsilon^p).$$

因 S 为紧集，可选有限个邻域覆盖，故

$$\mu\left\{\inf_{l\in S}\int_0^\tau\sum_{k=1}^d(l,f_{A_k}(R_s))^2ds<\epsilon\right\}=o(\epsilon^p),\quad 1<p<\infty. \tag{1.46}$$

因 $\tau\le t$, 上式中 τ 改为 t 时仍成立，但

$$\begin{aligned}&\inf_{l\in S}\int_0^t\sum_{k=1}^d(l,f_{A_k}(R_s))^2ds\\&=\inf_{l\in S}\int_0^t|J_s^{-1}\sigma(X_s)^*l|^2ds\\&=\inf_{l\in S}(l,\widetilde{\Sigma}_tl)\end{aligned}$$

恰好是 $\widetilde{\Sigma}_t$ 的最小特征根 $\lambda_{\min}$. 因此 $\lambda_{\min}^{-1}\in L^{\infty-}$, 由此推出

$$[\det\widetilde{\Sigma}_t]^{-1}\in L^{\infty-}. \qquad \blacksquare$$

§ 2. Wiener 空间中的位势理论与拟必然分析

Malliavin 分析提供了将有限维空间中的位势理论推广到无穷维空间中去的可能性. 本节设 (H, X, μ) 为一抽象 Wiener 空间, $\mathbb{D}_k^p (k \in \mathbb{N}_0, p \in (1, \infty))$ 为其上泛函的 Watanabe-Sobolev 空间. 在有限维 Sobolev 空间 $W^{k,p}(\mathbb{R}^m)$ 中, 我们知道, 若 $kp > m$, 则其中函数有连续修正. 但在无穷维空间中 $m = \infty$, 因此没有理由指望 $\mathbb{D}_k^p$ 中泛函也有连续修正. 为了讨论其正则性质, Malliavin[2] 引入了 (k,p)- 容度及 (k,p)- 拟连续的概念, 由此开创了所谓 **拟必然分析**(quasi-sure analysis) 的研究领域.

经典概率论允许忽略零概率事件, 即零测度集合, 因此可以称作几乎必然分析. 但在无穷维随机分析中, 某些零测度集合往往是不能忽略的. 例如设 $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ 为一非退化泛函, 根据第二章 §4 的讨论, $\nu_y \equiv \delta_y \circ F$ 可以解释为在 $F = y$ 时的条件分布. 显然 $\{\nu_y\}_{y \in \mathbb{R}^m}$ 不可能相互等价也未必关于 μ 绝对连续, 从而 μ 零集未必为 ν_y 零集, μ-a.s. 成立的命题未必 ν_y-a.s. 成立. 因此更精细的分析只容许忽略关于一族 (可能相互奇异) 的测度的 "普遍零集", 即所谓 **疏集**(slim sets), 这就是拟必然分析.

经典的随机过程理论研究轨道空间. 但许多物理问题要求人们研究 Riemann 流形上的 **回路空间**(loop space). 回路可以看作终点固定的轨道, 而拟必然分析则是研究回路空间的重要工具.

2.1 (k,p)- 容度

设 (H, X, μ) 为抽象 Wiener 空间, $\mathcal{F}$ 为 $\mathcal{B}(X)$ 关于 μ 的完备化 σ- 代数.

定义 2.1 对 $k \in \mathbb{N}_0, 1 < p < \infty$ 及 X 中开集 O, 令

$$\mathcal{V}_{k,p}^O \equiv \{\varphi \in \mathbb{D}_k^p : \varphi \geq 0, \varphi(x) \geq 1 \ \mu\text{-a.e.} x \in O\}, \tag{2.1}$$

$$C_{k,p}(O) \equiv \inf\{\|\varphi\|_{k,p} : \varphi \in \mathcal{V}_{k,p}^O\}, \tag{2.2}$$

对 X 中任意子集 A, 令

$$C_{k,p}(A) \equiv \inf\{C_{k,p}(O) : O\text{为开集且}O \supset A\}. \tag{2.3}$$

集函数 $C_{k,p}$ 称为 X 上的 **(k,p)- 容度**. 若 $C_{k,p}(A) = 0, A$ 称为 **(k,p)- 零容集**, 若此式对所有 $k \in \mathbb{N}_0$ 及 $1 < p < \infty$ 成立, 则 A 称为 **疏集**, 若使某命题 $\pi(x)$ 不成立的 $x \in X$ 构成 (k,p)- 零容集 (相应地, 疏集), 则称 $\pi(x)$ 为 **(k,p)- 拟必然** 或 **拟处处**(相应地, **拟必然** 或 **拟处处**) 成立. 简记为 (k,p)-q.s. 或 (k,p)-q.e. (相应地, q.s. 或 q.e.).

容易证明容度具有以下性质 (以下我们均设 $1 < p, q < \infty$, $k, l \in \mathbb{N}_0, A, B, A_j, B_j$ 等为 X 的子集):

引理 2.2 设 O 为 X 中开集, 则存在唯一元素 $e_O \in \mathcal{V}^O_{k,p} \subset \mathbb{D}^p_k$, 使

$$C_{k,p}(O) = \|e_O\|_{k,p}. \tag{2.4}$$

证明 注意由 (II.3.23) 式定义的算子 $Q^k = (I - \mathcal{L})^{-k/2}$ 为 $L^p \longrightarrow \mathbb{D}^p_k$ 的 Banach 空间同构, 从而 $\mathbb{D}^p_k$ 为均匀凸空间 (参看附录 B). 既然 $\mathcal{V}^O_{k,p}$ 为 $\mathbb{D}^p_k$ 中闭凸子集, 故存在唯一元素 $e_O \in V^O_{k,p}$ 使其范数 $\|e_O\|_{k,p}$ 达到下确界 $C_{k,p}(O)$. ∎

注 因为

$$Q^k = (I - \mathcal{L})^{-k/2} = \frac{1}{\Gamma(k/2)} \int_0^\infty t^{k/2-1} e^{-t} T_t dt,$$

故 Q^k 保持正性. 反之, 若 $e_O = Q^k\varphi, \varphi \in L^p$, 则 $\varphi \geq 0$ a.s.. 这一事实可以证明如下:

令 $B = \{x : \varphi(x) < 0\}, \psi = Q^k \mathbf{1}_B$, 则 $\psi \geq 0$ a.s., 故 $\forall \lambda \geq 0$, $e_O + \lambda\psi \in \mathcal{V}^O_{k,p}$. 令 $f(\lambda) \equiv \|e_O + \lambda\psi\|^p_{k,p} = \|\varphi + \lambda \mathbf{1}_B\|^p_p$. 因 $f(\lambda)$ 在 $\lambda = 0$ 处达到极小值, 故右导数

$$f'_+(0) = p \int_B |\varphi|^{p-2}\varphi d\mu \geq 0.$$

由此可见 $\mu(B)=0$.

命题 2.3 我们有

1° 若 $A\in\mathcal{F}$, 则 $\mu(A)\le C_{k,p}^{p}(A)$.

2° 若 $k\le l, p\le q$, 则 $C_{k,p}(A)\le C_{l,q}(A)$.

3° 若 $A\subset B$, 则 $C_{k,p}(A)\le C_{k,p}(B)$.

4° 若 $A=\bigcup_{j=1}^{\infty}A_j$, 则

$$C_{k,p}(A)\le\sum_{j=1}^{\infty}C_{k,p}(A_j), \tag{2.5}$$

$$C_{k,p}^{p}(A)\le\sum_{j=1}^{\infty}C_{k,p}^{p}(A_j). \tag{2.6}$$

5° (Borel-Cantelli 引理)

$$\sum_{j=1}^{\infty}C_{k,p}(A_j)<\infty\Longrightarrow C_{k,p}(\overline{\lim}_j A_j)=0. \tag{2.7}$$

证明 设 O 为开集, $0\le k\le l, 1<p\le q$, 则由定义

$$\mu(O)^{1/p}=C_{0,p}(O)\le C_{k,p}(O)\le C_{l,q}(O),$$

分别取下确界得证 1°, 2° 及 3°. 在证明 4° 时不妨设 A_j 为开集且 (2.5) 及 (2.6) 式右边级数收敛. 由定义, $\forall j\in I\!\!N, \epsilon>0, \exists\varphi_j\in\mathcal{V}_{k,p}^{A_j}$ 使 $\|\varphi_j\|_{k,p}<C_{k,p}(A_j)+\epsilon 2^{-j}$. 令 $\varphi\equiv\sum_j\varphi_j$, 则 $\varphi\in\mathcal{V}_{k,p}^{A}$ 且

$$\|\varphi\|_{k,p}<\sum_j\|\varphi_j\|_{k,p}<\sum_j C_{k,p}(A_j)+\epsilon,$$

由 ϵ 任意性推得 (2.5) 式. 为证 (2.6) 式, 注意由引理 2.2, $\forall j\in I\!\!N$, $\exists e_{A_j}\in\mathcal{V}_{k,p}^{A_j}\subset I\!\!D_k^p$, 使 $C_{k,p}(A_j)=\|e_{A_j}\|_{k,p}$. 由 $Q^k: L^p\longrightarrow I\!\!D_k^p$ 为

Banach 空间同构，设 $e_{A_j} = Q^k\varphi_j, \varphi_j \in L^p$ 且 $\varphi_j \geq 0$ $(j \in \mathbb{N})$. 令 $\varphi \equiv \sup_j \varphi_j, \psi = Q^k\varphi$, 则 $\psi \geq \sup_j e_{A_j}$, 可见 $\psi \in \mathcal{V}_{k,p}^A$, 因此

$$\begin{aligned} C_{k,p}^p(A) &\leq \|\psi\|_{k,p}^p = \|\varphi\|_p^p \\ &\leq \sum_j \|\varphi_j\|_p^p = \sum_j \|e_{A_j}\|_{k,p}^p \\ &= \sum_j C_{k,p}^p(A_j). \end{aligned}$$

(2.7) 式是可列次可加性 (2.5) 式的简单推论. ∎

由性质 1° 可知，(k,p)- 零容集必为 μ- 零集，从而 (k,p)-q.s. 可推出 μ-a.s., 拟必然成立的命题均几乎必然成立. 由性质 4° 可知，可列个疏集的并集仍然是疏集.

下面的引理是 Tchebycheff 不等式的一种较弱的形式.

引理 2.4 设 ψ 连续，$\varphi = Q^k\psi (k \in \mathbb{N}_0)$, 则 $\forall\epsilon > 0$ 及 $p \in (1,\infty)$ 有

$$C_{k,p}(x : |\varphi(x)| > \epsilon) \leq \epsilon^{-1}\|\varphi\|_{k,p}. \tag{2.8}$$

证明 不妨设 ψ 有界. 注意

$$Q^k = (I - \mathcal{L})^{-k/2} = \frac{1}{\Gamma(k/2)} \int_0^\infty t^{k/2-1}e^{-t}T_t dt,$$

故 φ 连续，$\{|\varphi| > \epsilon\}$ 为开集. 因 $\psi = \psi_+ - \psi_-, \varphi = Q^k\psi_+ - Q^k\psi_-$, 故 $\{|\varphi| > \epsilon\} \subset \{Q^k\psi_+ > \epsilon\} \cup \{Q^k\psi_- > \epsilon\}$. 由 (2.6) 式

$$\begin{aligned} C_{k,p}^p(|\varphi| > \epsilon) &\leq C_{k,p}^p(Q^k\psi_+ > \epsilon) + C_{k,p}^p(Q^k\psi_- > \epsilon) \\ &\leq \epsilon^{-p}(\|Q^k\psi_+\|_{k,p}^p + \|Q^k\psi_-\|_{k,p}^p) \\ &= \epsilon^{-p}(\|\psi_+\|_p^p + \|\psi_-\|_p^p) \\ &= \epsilon^{-p}\|\varphi\|_{k,p}^p. \end{aligned}$$

得证 (2.8) 式. ∎

2.2 拟连续修正

定义 2.5 X 上的泛函 φ, 若 $\forall\epsilon>0$, 存在开集 O_ϵ, 使 $C_{k,p}(O_\epsilon) < \epsilon$ 且 φ 在 O_ϵ^c 的限制为连续函数，则 φ 称为 **(k,p)- 拟连续**; 若 $\forall k\in \mathbb{N}_0$ 及 $p\in(1,\infty)$, φ 为 (k,p)- 拟连续，则 φ 称为 **拟连续**.

注 设 K 为一闭集， O 为使 $\mu(O\cap K)=0$ 的最大开集，则 $\mathrm{ess}(K)=K\setminus O$ 称为 K 的 **μ- 本质部分**. 易见， $\mu(\mathrm{ess}(K))=\mu(K)$, 若函数 φ 在 K 上连续且 $\varphi=0$ μ-a.e. 于 K, 则 φ 在 $\mathrm{ess}(K)$ 上处处为 0.

根椐容度性质可知， φ 为 (k,p)- 拟连续的充要条件是：存在递降的开集序列 $\{O_n\}$, $\lim_{n\to\infty}C_{k,p}(O_n)=0$, 使 $\forall n$, $\varphi|_{O_n^c}$ 连续, $\mathrm{ess}(O_n^c)=O_n^c$, 且由容度的胎紧性 (定理 2.12), 可选 O_n^c 为紧集. 这样的序列称为 **套**. 通过选取对角线序列，容易看出， φ 为拟连续的充要条件是：存在上述开集套 $\{O_n\}$, 使 $\lim_{n\to\infty}C_{n,n}(O_n)=0$.

拟连续泛函有如下重要性质：

定理 2.6 若 φ (k,p)- 拟连续，则

1° $\varphi=0$ μ-a.s. $\Longrightarrow \varphi=0$ (k,p)-q.s.;

2° $\varphi\geq 0$ μ-a.s. $\Longrightarrow \varphi\geq 0$ (k,p)-q.s..

证明 设 $\{O_n\}$ 为联系于 φ 的开集套，因 $\varphi=0\,\mu$-a.e. 于 O_n^c 且 $O_n^c=\mathrm{ess}(O_n^c)$, 故 φ 在 O_n^c 上处处为 $0\,(\forall n\in\mathbb{N})$, 于是

$$C_{k,p}(x:\varphi(x)\neq 0)\leq C_{k,p}(\cap_n O_n)=0,$$

1° 得证. 应用 1° 于泛函 $\psi\circ\varphi$, 其中 $\psi\in C_b^\infty(\mathbb{R})$, 当 $t<0, \psi(t)>0$, 当 $t\geq 0, \psi(t)=0$, 于是 $\{x:\varphi(x)<0\}=\{x:\psi(\varphi(x))\neq 0\}$, 2° 得证. ∎

系 2.7 设 φ 拟连续， O 为 X 中开集，则

1° $\varphi=0$ μ-a.e. 于 $O\Longrightarrow\varphi=0$ q.e. 于 O;

2° $\varphi\geq 0$ μ-a.e. 于 $O\Longrightarrow\varphi\geq 0$ q.e. 于 O.

定义 2.8 设 φ 为一泛函，若泛函 φ^* 为 (k,p)- 拟连续 (相应地，拟连续), 且 $\varphi^*=\varphi$ μ-a.s., 则 φ^* 称为 φ 之 **(k,p)- 拟连续**(相应地， **拟连续)修正**, 或 **再定义**(redefinition).

由定理 2.6 可知，拟连续修正在 q.s. 相等意义下是唯一的.

下面是有关容度的 Lusin 定理：

定理 2.9 若 $\varphi \in \mathbb{D}_k^p$, 则存在 (k,p)- 拟连续修正 φ^*, 它除开 (k,p)- 零容集外是唯一确定的；若 $\varphi \in \mathbb{D}^\infty$, 则存在拟连续修正 φ^*, 它除开疏集外是唯一确定的.

证明 只需证明前一结论. 令 $Q = (I - \mathcal{L})^{-1/2}$,

$$C \equiv \{\varphi \in \mathbb{D}_k^p : \varphi = Q^k\psi, \psi\text{有界连续}\},$$

则 C 在 $\mathbb{D}_k^p$ 中稠密，从而对 $\varphi \in \mathbb{D}_k^p$, 存在 $\{\varphi_j\} \subset C, \varphi_j = Q^k\psi_j, \psi_j$ 有界连续，且

$$\|\varphi_{j+1} - \varphi_j\|_{k,p} + \|\varphi_j - \varphi\|_{k,p} < 4^{-j}, \quad j \geq 1.$$

令

$$O_n \equiv \bigcup_{j=n+1}^{\infty} \{x : |\varphi_{j+1}(x) - \varphi_j(x)| > 2^{-j}\}, \quad n \geq 0.$$

由引理 2.4

$$\begin{aligned} C_{k,p}(O_n) &\leq \sum_{j=n+1}^{\infty} C_{k,p}(|\varphi_{j+1} - \varphi_j| > 2^{-j}) \\ &\leq \sum_{j=n-1}^{\infty} 2^j \|\varphi_{j+1} - \varphi_j\|_{k,p} < 2^{-n}. \end{aligned}$$

在 O_n^c 上，因

$$\sum_{j=1}^{\infty} \sup_x |\varphi_{j+1}(x) - \varphi_j(x)| < \infty,$$

故 $\{\varphi_j\}$ 一致收敛，令

$$\varphi^*(x) = \begin{cases} \lim_{j\to\infty} \varphi_j(x), & x \in \bigcup_n O_n^c; \\ 0, & x \in \bigcap_n O_n. \end{cases}$$

易见 φ^* 为 φ 之 (k,p)- 拟连续修正. ∎

命题 2.10 (Tchebycheff 不等式) 若 $\varphi\in \mathbb{D}_k^p, \epsilon>0$, 则

$$C_{k,p}(x:|\varphi^*(x)|>\epsilon)\le \epsilon^{-1}\|\varphi\|_{k,p}, \tag{2.9}$$

其中 φ^* 为 φ 之 (k,p)- 拟连续修正.

证明 令 $\varphi=Q^k\psi=Q^k\psi_+-Q^k\psi_-$, $\psi\in L^p$, 则 $\varphi^*=(Q^k\psi_+)^*-(Q^k\psi_-)^*$ (k,p)-q.s., 因此, 除开一个 (k,p)- 零容集外,

$$\{|\varphi^*|>\epsilon\}\subset\{(Q^k\psi_+)^*>\epsilon\}\cup\{(Q^k\psi_-)^*>\epsilon\}.$$

选取开集套 $\{O_m\}$ 使 $\lim_{m\to\infty}C_{k,p}(O_m)=0$, 且在每个 O_m^c 上 $(Q^k\psi_+)^*$ 及 $(Q^k\psi_-)^*$ 连续. 易见, $\forall m\in\mathbb{N}$, $\exists$ 开集 G_m 使 $\{(Q^k\psi_\pm)^*>\epsilon\}\cap O_m^c=G_m\cap O_m^c$, 于是由定义 2.1 知

$$\begin{aligned}C_{k,p}^p(|\varphi^*|>\epsilon)&\le C_{k,p}^p((Q^k\psi_+)^*>\epsilon)+C_{k,p}^p((Q^k\psi_-)^*>\epsilon)\\&\le \epsilon^{-p}(\|Q^k\psi_+\|_{k,p}^p+\|Q^k\psi_-\|_{k,p}^p)+2C_{k,p}^p(O_m),\end{aligned}$$

令 $m\to\infty$, 得

$$\begin{aligned}C_{k,p}^p(|\varphi^*|>\epsilon)&\le \epsilon^{-p}(\|\psi_+\|_p^p+\|\psi_-\|_p^p)\\&=\epsilon^{-p}\|\psi\|_p^p=\epsilon^{-p}\|\varphi\|_{k,p}^p.\end{aligned}$$ ∎

定理 2.11 若 $\{\varphi_n\}$ 在 $\mathbb{D}_k^p$ 中收敛于 φ, 则存在子序列 $\{\varphi_{n_j}\}$ 及开集套 $\{O_m\}$ 使 $\lim_{m\to\infty}C_{k,p}(O_m)=0$, 且在每个 O_m^c 上, $\{\varphi_{n_j}^*\}$ 一致收敛于 φ^*, 特别, $\varphi_{n_j}^*\longrightarrow\varphi^*$, (k,p)-q.s..

证明 不妨设 $\varphi=0$. 由 (2.9) 式, $\forall\epsilon>0$

$$\lim_{n\to\infty}C_{k,p}(|\varphi_n^*|>\epsilon)=0.$$

选子序列 $\{n_j\}$, 使 $\forall j$

$$C_{k,p}(|\varphi_{n_j}^*|>j^{-1})<j^{-2},$$

于是存在开集 $\widetilde{O}_j \supset \{|\varphi^*_{n_j}| > j^{-1}\}$ 使 $C_{k,p}(\widetilde{O}_j) < j^{-2}$. 取 $O_m \equiv \bigcup_{j=m}^{\infty} \widetilde{O}_j$, 即满足要求. ∎

2.3 容度的胎紧性、连续性与不变性

下一定理表明，容度具有胎紧性 (tightness).

定理 2.12 X 中存在紧集序列:

$$K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$$

使

$$\lim_{n\to\infty} C_{k,p}(K_n^c) = 0.$$

证明 由定理 I.4.18, 存在另一 Banach 空间 Y 使 $\mu(Y) = 1$, $H \subset Y \subset X$, 且嵌入映射 $i : Y \to X$ 为紧. 令 $\varphi(y) = \|y\|_Y$, 它在 X 上 μ-a.s. 有定义. 由 Fernique 定理 (I.4.20), $\exists\lambda > 0$ 使 $\mathbb{E}[\exp(\lambda\varphi^2)] < \infty$. 从而 $\varphi \in L^{\infty-}$. 由系 II.3.24, $\forall t > 0, T_t\varphi \in \mathbb{D}^\infty$. 因为

$$(T_t\varphi)(y) = \int \|e^{-t}y + \sqrt{1-e^{-2t}}z\|_Y \mu(dz),$$

令 $\delta_1 = e^{-t}, \delta_2 = \sqrt{1-e^{-2t}}\int \|z\|_Y \mu(dz)$, 则

$$\delta_1\|y\|_Y - \delta_2 \le (T_t\varphi)(y) \le \delta_1\|y\|_Y + \delta_2.$$

因 $(T_t\varphi)^{-1}([0,n]) \subset \{y : \|y\|_Y \le (n+\delta_2)/\delta_1\}$ 为 Y 中有界集，令 K_n 为其在 Y 中闭包，则 K_n 为 X 中紧集. 因 $T_t\varphi > n$ μ-a.e. 于 K_n^c, 由定理 2.6, 其 (k,p)– 拟连续修正 $(T_t\varphi)^* > n$ (k,p)-q.e. 于 K_n^c, 由命题 2.10

$$\begin{aligned} C_{k,p}(K_n^c) &\le C_{k,p}(y : (T_t\varphi)^*(y) > n) \\ &\le n^{-1}\|T_t\varphi\|_{k,p} \longrightarrow 0, \end{aligned}$$

定理获证. ∎

命题 2.13 对 X 中子集 A, 令

$$\widetilde{\mathcal{V}}^A_{k,p} \equiv \{\varphi \in \mathbb{D}^p_k : \varphi^* \ge 1 \ (k,p)\text{-q.e. 于} A\}, \tag{2.10}$$

则 $\widetilde{\mathcal{V}}_{k,p}^A$ 是 $I\!D_k^p$ 中闭凸子集，且

$$C_{k,p}(A)=\inf\{\|\varphi\|_{k,p}:\quad \varphi\in\widetilde{\mathcal{V}}_{k,p}^A\}. \tag{2.11}$$

从而存在唯一元素 $e_A\in\widetilde{\mathcal{V}}_{k,p}^A\subset I\!D_k^p$ 使

$$C_{k,p}(A)=\|e_A\|_{k,p}. \tag{2.12}$$

e_A 称为 A 的 **(k,p)-平衡位势**.

证明 $\widetilde{\mathcal{V}}_{k,p}^A$ 的闭性由定理 2.11 可以看出. 因 $I\!D_k^p$ 与 L^p 同构，为均匀凸空间，故存在唯一 e_A 使 (2.11) 和 (2.12) 两式的右边相等. 先设 A 为开集. 由定理 2.6 可知，若 $\varphi\geq 1$ a.e. 于 A, 必有 $\varphi^*\geq 1$ (k,p)-q.e. 于 A, 从而

$$\|e_A\|_{k,p}=C_{k,p}(A)\ (A\text{为开集}). \tag{2.13}$$

次设 A 为任意集且开集 $O\supset A$. 于是

$$\widetilde{\mathcal{V}}_{k,p}^O\subset\widetilde{\mathcal{V}}_{k,p}^A,$$

取 $\|\varphi\|_{k,p}$ 的下确界，得 $C_{k,p}(O)\geq\|e_A\|_{k,p}$, 再对一切包含 A 的开集 O 取下确界，得

$$C_{k,p}(A)\geq\|e_A\|_{k,p}. \tag{2.14}$$

为证相反不等式，设 e_A^* 为 e_A 之 (k,p)- 拟连续修正，则 $\forall\epsilon>0$, $\exists$ 开集 $O_\epsilon, C_{k,p}(O_\epsilon)<\epsilon$ 且 e_A^* 于 O_ϵ^c 连续， $e_A^*\geq 1$ 于 $O_\epsilon^c\cap A$. 令

$$\Omega_\epsilon\equiv\{x\in O_\epsilon^c:e_A^*(x)>1-\epsilon\}\cup O_\epsilon,$$

则 Ω_ϵ 为开集且 $\Omega_\epsilon\supset A$. 设 O_ϵ 之平衡位势为 e_ϵ, 由 $e_A+e_\epsilon>1-\epsilon$ a.e. 于 Ω_ϵ 可知

$$\begin{aligned}C_{k,p}(A)&\leq C_{k,p}(\Omega_\epsilon)\\&\leq(1-\epsilon)^{-1}\|e_A+e_\epsilon\|_{k,p}\\&\leq(1-\epsilon)^{-1}(\|e_A\|_{k,p}+\epsilon),\end{aligned}$$

令 $\epsilon \downarrow 0$ 即得 $C_{k,p}(A) \le \|e_A\|_{k,p}$. ∎

利用平衡位势，可以证明容度的连续性.

定理 2.14 容度 $C_{k,p}$ 具有以下连续性质:

1° 若集合序列 $\{A_n\}$ 单调非降，则

$$C_{k,p}(A_n) \uparrow C_{k,p}(\cup_n A_n); \tag{2.15}$$

2° 若紧集序列 $\{K_n\}$ 单调非升，则

$$C_{k,p}(K_n) \downarrow C_{k,p}(\cap_n K_n). \tag{2.16}$$

证明 1° 记 $A = \bigcup_n A_n, A_n$ 之平衡位势为 e_n, 则 $\|e_n\|_{k,p} = C_{k,p}(A_n), \{e_n\}$ 为 $I\!D_k^p$ 中有界序列. 由 Banach-Saks-Kakutani 定理 (参看附录 B), 存在子序列 $\{e_{n_j}\}$ 使 $S_m \equiv \frac{1}{m}\sum_{j=1}^m e_{n_j}$ 在 $I\!D_k^p$ 中收敛. 设其极限为 e_A, 由定理 2.11,$\{S_m\}$ 存在子序列 $\{S_{m_j}\}$ 使

$$\lim_{j\to\infty} S_{m_j}^* = e_A^*, \quad (k,p)\text{-q.s.}.$$

由此推出 $e_A^* \in \widetilde{\mathcal{V}}_{k,p}^A$. 由 (2.11) 式可知

$$\begin{aligned} C_{k,p}(A) \le \|e_A\|_{k,p} &\le \sup_n \|e_n\|_{k,p} \\ &= \sup_n C_{k,p}(A_n), \end{aligned}$$

相反的不等式是显然的.

2° 记 $K = \bigcap_n K_n$, 则 $\forall \epsilon > 0, \exists$ 开集 $O_\epsilon \supset K$, 使 $C_{k,p}(O_\epsilon) < C_{k,p}(K) + \epsilon$. 因 $\{O_\epsilon^c \cap K_n\}$ 为单调非升紧集列，其交 $O_\epsilon^c \cap K = \emptyset$, 从而 $\exists n_0$, 当 $n \ge n_0$ 时 $O_\epsilon^c \cap K_n = \emptyset$, 亦即 $O_\epsilon \supset K_n$, 故

$$C_{k,p}(K_n) < C_{k,p}(K) + \epsilon.$$

令 $\epsilon \downarrow 0$, 得

$$\lim_{n\to\infty} C_{k,p}(K_n) \le C_{k,p}(K),$$

相反的不等式也是显然的. ∎

注 定理 2.14 表明, $C_{k,p}$ 为 **Choquet 容度**(参看 Dellacherie - Meyer[1]), 根据 Choquet 容度定理, 对任意 Borel 集 B 及 $\epsilon > 0$, 存在紧集 $K_\epsilon \subset B$, 使

$$C_{k,p}(K_\epsilon) \geq C_{k,p}(B) - \epsilon. \tag{2.17}$$

即 Borel 集为 $C_{k,p}$**可容集**.

容度的定义表面上依赖于 X 的拓扑结构. 在一般的 Gauss 概率空间 $(\Omega, \mathcal{F}, \mu; H)$ 中, Ω 中没有拓扑结构, 当取定 H 的一组基时, 得到一个数值模型 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$, 从而可以定义 $\mathbb{R}^\infty$ 上的 (k,p)- 容度. 人们自然要问: 拟必然分析是否内蕴性质? 或者从抽象 Wiener 空间角度来说, 若 (H, X_1, μ_1) 和 (H, X_2, μ_2) 为两个抽象 Wiener 空间, 那么它们上面的容度是否等价? 这个问题首先由 Itô 提出, Albeverio 、 Fukushima 等 [1] 给出了肯定的答案, 其中关键是利用容度的胎紧性. 因为连续单射限制于紧集上实际上是拓扑同胚. 下面的定理就是这个结果的一种表述. 进一步的讨论可参看巩馥洲与马志明 [1].

定理 2.15 设 (H, X, μ) 为抽象 Wiener 空间, 任选 H 的一组属于 X^* 的基 $\{e_j\}$, 由映射 $x \longmapsto \{< x, e_j >\}_{j \in \mathbb{N}}$ 定义的连续单射 $\iota : X \longrightarrow \mathbb{R}^\infty$ 建立一个数值模型 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$. 设 $C_{k,p}$ 及 $\widetilde{C}_{k,p}$ 分别为 X 及 $\mathbb{R}^\infty$ 上定义的 (k,p)- 容度, 则

1° $(\iota X)^c$ 为 $\mathbb{R}^\infty$ 中疏集;

2° $\forall k \in \mathbb{N}_0, p \in (1, \infty)$ 及 X 之子集 A,

$$C_{k,p}(A) = \widetilde{C}_{k,p}(\iota A). \tag{2.18}$$

证明 显然 $\gamma^\infty = \mu \circ \iota^{-1}$ 且 ι 在 X 上诱导的拓扑弱于 X 的范数拓扑, 故

$$C_{k,p}(A) \leq \widetilde{C}_{k,p}(\iota A). \tag{2.19}$$

由胎紧性, X 中存在紧集序列 $\{K_n\}$ 使 $C_{k,p}(K_n^c) < \frac{1}{n}$, 令 $\widetilde{K}_n = \iota K_n$, 则 $\widetilde{K}_n$ 为 $\mathbb{R}^\infty$ 中紧集, 且

$$\gamma^\infty(\widetilde{K}_n^c \setminus \iota(K_n^c)) = 0.$$

故

$$\varphi \geq 1\ \gamma^\infty\text{-a.e. 于}\widetilde{K}_n^c \Longleftrightarrow \varphi \circ \iota \geq 1\ \mu\text{-a.e. 于}K_n^c.$$

由此推出

$$\widetilde{C}_{k,p}((\iota X)^c) \leq \widetilde{C}_{k,p}(\widetilde{K}_n^c) = C_{k,p}(K_n^c) \longrightarrow 0,$$

1° 得证.

设开集 $O \supset A$, 使

$$C_{k,p}(O) < C_{k,p}(A) + \epsilon,$$

因 $\iota: K_n \longrightarrow \widetilde{K}_n$ 为同胚，故 $\exists \widetilde{O} \supset \iota A$ 使 $\iota(O \cap K_n) = \widetilde{O} \cap \widetilde{K}_n$, 于是由 $\iota^{-1}\widetilde{O} \subset (O \cap K_n) \cup K_n^c$ 有

$$\begin{aligned}\widetilde{C}_{k,p}(\iota A) &\leq \widetilde{C}_{k,p}(\widetilde{O}) = C_{k,p}(\iota^{-1}\widetilde{O}) \\ &\leq C_{k,p}(O \cap K_n) + C_{k,p}(K_n^c) \\ &\leq C_{k,p}(A) + \epsilon + 1/n,\end{aligned}$$

令 $n \to \infty, \epsilon \downarrow 0$, 即得与 (2.19) 相反的不等式. ∎

2.4 正广义泛函与有限能量测度

在有限维空间中，正广义函数都是测度. 在抽象 Wiener 空间 (H, X, μ) 中 Sugita[3] 证明了正广义泛函 (在 S.Watanabe 意义下) 都是 X 上的 Borel 测度；在核空间情形， Kondratiev - Samoylenko[1] 和 Yokoi[1] 也证明了正广义泛函 (在 Hida 意义下) 都是测度，因为 Hida 广义泛函比 Meyer-Watanabe 广义泛函更广，且在核空间情形证明更简单，所以我们在这里只叙述 Sugita 的定理，详细证明可参看 Sugita[3], 经典 Wiener 空间情形的简单证明参看 Huang[5].

我们假定 $k \in \mathbb{N}_0, 1 < p < \infty, p^{-1} + q^{-1} = 1$.

定义 2.16 设 $G \in \mathbb{D}^{-\infty}$, 若对一切 $\varphi \in \mathbb{D}^\infty, \varphi \geq 0$ a.s., 有 $\langle G, \varphi \rangle \geq 0$, 则 G 称为 **正广义泛函**, 记为 $G \in \mathbb{D}_+^{-\infty}$.

容易看出，若 $G \in \mathbb{D}_{+}^{-\infty} \cap L^p$, 则 $G \geq 0$ a.s., 且 Q^k, T_t 均保持正性不变.

定理 2.17 (Sugita[3]) 一切 $G \in \mathbb{D}_{+}^{-\infty}$, 存在 $(X, \mathcal{B}(X))$ 上唯一有限测度 ν_G 使

$$\langle G, \varphi \rangle = \int_X \varphi^*(x) \nu_G(dx), \quad \forall \varphi \in \mathbb{D}^{\infty}, \tag{2.20}$$

其中 φ^* 为 φ 之拟连续修正. 若 $G \in \mathbb{D}_{-k}^q$, 则 (2.20) 对一切 $\varphi \in \mathbb{D}_k^p$ 成立，其中 φ^* 为 φ 之 (k,p)- 拟连续修正.

注 由于拟连续修正 q.s. 唯一确定，而 ν_G 在疏集上无负荷 (参看系 2.20), 故表示式 (2.20) 是确定的.

特别, 若 $G \in L_{+}^q$, 则 (2.20) 对一切 $\varphi \in L^p$ 成立, 于是 $\nu_G << \mu$, 且 $d\nu_G = G d\mu$.

注意若 $G \in \mathbb{D}_{-k}^q$, 则 $Q^k G \in L^q$, 因 $q-1 = q/p$, $|Q^k G|^{q-1} \in L^p$, 令

$$\varphi \equiv Q^k(|Q^k G|^{q-2} Q^k G), \tag{2.21}$$

则 $\varphi \in \mathbb{D}_k^p$, 且

$$G = Q^{-k}(|Q^{-k}\varphi|^{p-2} Q^{-k}\varphi). \tag{2.22}$$

(2.21) 和 (2.22) 式建立了 $\mathbb{D}_k^p$ 和 $\mathbb{D}_{-k}^q$ 中元素之间 $1-1$ 对应.

定理 2.18 设 e_A 为 X 的子集 A 的 (k,p)- 平衡位势，令

$$G_A \equiv Q^{-k}(|Q^{-k} e_A|^{p-2} Q^{-k} e_A), \tag{2.23}$$

则 $G_A \in \mathbb{D}_{-k}^q \cap \mathbb{D}_{+}^{-\infty}$, 它所对应的测度 ν_A(称为 A 的 **(k,p)-平衡测度**) 的支集含于 $\overline{A}$ 中，且

$$\begin{aligned} \int_X e_A^*(x) \nu_A(dx) &= \|e_A\|_{k,p}^p = \|G_A\|_{-k,q}^q \\ &= C_{k,p}^p(A), \end{aligned} \tag{2.24}$$

称为 e_A 或 ν_A 的 **(k,p)-能量**.

证明 由 (2.21) 式, $e_A = Q^k(|Q^kG_A|^{q-2}Q^kG_A)$, 若 $\psi \in \mathbb{D}_k^p$ 且 $\psi \geq 0$ a.s., 由定理 2.6, 其 (k,p)- 拟连续修正 $\psi^* \geq 0$ (k,p)-q.s.. 对 $\lambda \geq 0, e_A + \lambda\psi^* \in \widetilde{\mathcal{V}}_{k,p}^A$. 令

$$f(\lambda) \equiv \|e_A + \lambda\psi\|_{k,p}^p = \|Q^{-k}(e_A + \lambda\psi)\|_p^p.$$

因 $f(\lambda)$ 在 $\lambda = 0$ 处达到极小值, 故右导数 $f'_+(0) \geq 0$ 但

$$\begin{aligned}
f'_+(0) &= p\int |Q^{-k}e_A|^{p-2}(Q^{-k}e_A)(Q^{-k}\psi)d\mu \\
&= p\int (Q^kG_A)(Q^{-k}\psi)d\mu \\
&= p\langle G_A, \psi\rangle.
\end{aligned}$$

故 $G_A \in \mathbb{D}_{-k}^q \cap \mathbb{D}_+^{\infty}$. 注意到上式中只要 $\psi \geq 0$ (k,p)-q.e. 于 A 而不管 ψ 在 A 外取什么值, 都有 $\langle G_A, \psi\rangle = \int \psi^* d\nu_A \geq 0$. 由此推出 (证明细节可参看 Sugita[3]) $\mathrm{supp}(\nu_A) \subset \overline{A}$. 而

$$\begin{aligned}
\int_X e_A^* d\nu_A &= \langle G_A, e_A\rangle \\
&= \langle G_A, Q^k(|Q^kG_A|^{q-2}Q^kG_A)\rangle \\
&= \langle Q^kG_A, |Q^kG_A|^{q-2}Q^kG_A\rangle \\
&= \|Q^kG_A\|_q^q = \|G_A\|_{-k,q}^q,
\end{aligned}$$

另一方面,

$$\begin{aligned}
\langle G_A, e_A\rangle &= \langle Q^{-k}(|Q^{-k}e_A|^{p-2}Q^{-k}e_A), e_A\rangle \\
&= \langle |Q^{-k}e_A|^{p-2}Q^{-k}e_A, Q^{-k}e_A\rangle \\
&= \|Q^{-k}e_A\|_p^p = \|e_A\|_{k,p}^p \\
&= C_{k,p}^p(A).
\end{aligned}$$

定理证毕. ∎

定理 2.19 设 $G \in \mathbb{D}_{-k}^q \cap \mathbb{D}_+^{-\infty}, \widetilde{\nu}_G$ 为对应 ν_G 的外测度, 则对 X 的一切子集 A 有

$$\widetilde{\nu}_G(A) \leq \|G\|_{-k,q}C_{k,p}(A). \tag{2.25}$$

证明 令 $G_n = T_{1/n}G$, 则 G_n 在 $\mathbb{D}^q_{-k}$ 中收敛于 G, 由此推出(参看后面的注)ν_{G_n} 弱收敛于 ν_G. 设 O 为开集, e_O 为其 (k,p)-平衡位势, 则

$$\langle G_n, e_O\rangle \geq \int_O G_n d\mu = \nu_{G_n}(O). \tag{2.26}$$

当 $n \to \infty$ 时, $\underline{\lim}_{n\to\infty}\nu_{G_n}(O) \geq \nu_G(O)$, 上式左边收敛于 $\langle G, e_O\rangle$, 且有

$$\begin{aligned}\langle G, e_O\rangle &\leq \|G\|_{-k,q}\|e_O\|_{k,p}\\ &= \|G\|_{-k,q}C_{k,p}(O),\end{aligned}$$

因此 (2.25) 式对开集 O 成立. 对一切含 A 的开集 O 取下确界, 即得 (2.25) 式. ∎

注 由容度的胎紧性, $\forall\epsilon > 0, \exists$ 紧集 K 使 $C_{k,p}(K^c) < \epsilon$, 设 e_{K^c} 为其 (k,p)- 平衡位势, 于是 $\forall n$

$$\begin{aligned}\nu_{G_n}(K^c) = \int_{K^c} T_{1/n}G d\mu &\leq \int_{K^c} e_{K^c}T_{1/n}G d\mu\\ &= \langle T_{1/n}e_{K^c}, G\rangle \leq \|T_{1/n}e_{K^c}\|_{k,p}\|G\|_{-k,q}\\ &\leq \epsilon\|G\|_{-k,q},\end{aligned}$$

由此可见 $\{\nu_{G_n}\}$ 胎紧. 因为在有限维空间上有界连续函数可由光滑函数逼近, 故 ν_{G_n} 的任意有限维分布弱收敛于 ν_G 的相应有限维分布. 根据 $\{\nu_{G_n}\}$ 的胎紧性即可推出 ν_{G_n} 弱收敛于 ν_G.

由定理 2.19 可以得到如下重要推论.

系 2.20 若 $G \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$, 则 ν_G 在 (k,p)- 零容集上无负荷. 特别, 若 $G \in \mathbb{D}^{-\infty}_+$, 则 ν_G 在疏集上无负荷.

系 2.21 设 B 为 Borel 集, 则 B 为 (k,p)- 零容集之充要条件是: $\nu_G(B) = 0, \forall G \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$. 特别, B 为疏集之充要条件是: $\nu_G(B) = 0, \forall G \in \mathbb{D}^{-\infty}_+$.

证明 必要性已证. 为证充分性, 设 $C_{k,p}(B)>0$, 则存在紧集 $K\subset B$ 使 $C_{k,p}(K)>0$. 取 K 之 (k,p)- 平衡测度 ν_K, 则 $\mathrm{supp}(\nu_K)\subset K$ 且 $\nu_K(K)>0$(否则由 (2.24) 式, $C_{k,p}(K)=0$). ∎

我们称由正广义泛函所表示的测度为 **有限能量测度**. 因此, 疏集可以刻画为所有有限能量测度的 "普遍零集".

作为应用, 我们得到如下积分分解定理.

定理 2.22 设 $F\in \mathbb{D}^{\infty}(\mathbb{R}^m)$ 且非退化, ρ_F 为其分布密度, 对 $y\in\mathcal{R}(F)$(F 的值域), 令

$$\nu_y=\mu(\cdot\,|F=y), \tag{2.27}$$

则 $\exists k\in \mathbb{N}_0, p\in(1,\infty)$, 使映射 $y\longmapsto \nu_y$ 在 $\mathbb{D}^p_{-k}$ 中连续, 且 $\forall\varphi\in C^{\infty}(\mathbb{R}^m), G\in\mathbb{D}^{\infty}$, 有

$$\int_X G(\varphi\circ F)d\mu=\int_{\mathbb{R}^m}\varphi(y)\rho_F(y)dy\int_X G^*d\nu_y. \tag{2.28}$$

证明 由定理 II.4.9 及引理 II.4.8, 若 $k>m$, 则 $y\longmapsto\delta_y\circ F$ 在 $\mathbb{D}^p_{-k}$ 中连续. 因 $\delta_y\circ F\in\mathbb{D}_+^{-\infty}$, 由 (II.4.23) 式可知 $\rho_F(y)^{-1}\delta_y\circ F$ 恰好是条件概率 ν_y, 且

$$\begin{aligned}\mathbb{E}[G|F=y]&=\rho_F(y)^{-1}\langle\delta_y\circ F,G\rangle\\&=\int_X G^*d\nu_y,\end{aligned}$$

因此 (2.28) 式成立. ∎

系 2.23 (降维原理) 在定理 2.22 条件下, 凡拟必然成立的命题在一切条件 $F=y, y\in\mathcal{R}(F)$ 下都几乎必然成立.

证明 因为疏集是 "普遍零集". ∎

2.5 随机过程的拟必然轨道性质

在经典 Wiener 空间中, Brown 运动轨道的许多性质, 如 Hölder 连续性, 不可微性, 重对数律, 单点集的不可达性 (维数 $d>3$ 时)

无重点性 ($d>5$) 等等，不仅是几乎必然性质，而且被证明是拟必然性质．例如参看 Fukushima[1] 和 Takeda[1]．某些经典的极限定理，如平方变差的收敛性，大偏差原理，Doob 不等式和鞅极限定理等，其中测度也可代之以容度使之更精密化．例如参看 Yoshida[1]，任佳刚 [3] 和 Denis[1]．这里我们只简略介绍有关随机过程轨道连续和扩散过程轨道光滑逼近的拟必然性质的一些结果．详细证明可参看任佳刚 [1,5]．

回到上节的经典 Wiener 空间．设 $\{X(t),t\in[0,1]^d\}$ 为其上随机场．我们称 $\{X^*(t),t\in[0,1]^d\}$ 为其 **(k,p)-连续修正**(或相应地，**∞- 连续修正**)，若：

1° $\forall t\in[0,1]^d, X^*(t)$ 为 $X(t)$ 的 (k,p)- 拟连续 (相应地，拟连续) 修正；

2° 对 (k,p)-q.e.ω(相应地，对 q.e.ω)，$X^*(\cdot,\omega)$ 连续．

我们有以下推广的 Kolmogorov 准则：

定理 2.24 (任佳刚 [1]) 设 $k\in\mathbb{N}_0, 1<p<\infty$．若 $\exists\alpha>0,c>0$ 及正整数 β，使

1° $\forall t\in[0,1]^d,\quad X(t)\in\mathbb{D}_k^p$；

2° $\forall s,t\in[0,1]^d,(X(t)-X(s))^\beta\in\mathbb{D}_k^p$，且

$$\|(X(t)-X(s))^\beta\|_{k,p}\le c|t-s|^{d+\alpha},\tag{2.29}$$

其中 $|t-s|\equiv\sum_{j=1}^d|t_j-s_j|$，则 X 存在 (k,p)- 连续修正．

证明 不妨设 $\forall t, X(t)$ 本身 (k,p)- 拟连续 (否则取其 (k,p)- 拟连续修正)．选 ν 及 δ 使 $0<\nu<\alpha/\beta,(1-\delta)(\alpha+d-\beta\nu)>(1+\delta)d$．记

$$T_n\equiv\{(i_1 2^{-n},\cdots,i_d 2^{-n}):0\le i_1,\cdots,i_d\le 2^n\},$$
$$T_n(\delta)\equiv\{(s,t)\in T_n\times T_n:|t-s|<2^{-n(1-\delta)}\},$$

$T\equiv\cup_n T_n, l\equiv(1-\delta)(\alpha+d-\beta\nu)-(1+\delta)d>0$，由命题 2.10 得

$$C_{k,p}\Big(\bigcup_{(s,t)\in T_n(\delta)}\{|X(t)-X(s)|>|t-s|^\nu\}\Big)$$
$$\le c\sum_{(s,t)\in T_n(\delta)}|t-s|^{\alpha+d-\beta\nu}\le c2^{-nl}.$$

因为 $\sum_n 2^{-nl} < \infty$, 由 (2.7) 式, 对 (k,p)-q.e.$\omega, \exists n_0 = n_0(\omega)$ 使 $n \geq n_0, (s,t) \in T_n(\delta)$ 时有

$$|X(t) - X(s)| \leq |t-s|^\nu.$$

任取 $s,t \in T, |t-s| < 2^{-n_0(1-\delta)}$, 可选 $n \geq n_0$ 使 $n/(n+1) > 1-\delta$ 且

$$2^{-(n+1)(1-\delta)} \leq |t-s| < 2^{-n(1-\delta)},$$

对每个坐标逐一讨论, 可证

$$|X(t) - X(s)| \leq c|t-s|^\nu.$$

此式说明 $X(\cdot,\omega)$ 除开一个 ω 的 (k,p)- 零容集外在 T 上均匀连续, 从而可连续延拓到 $[0,1]^d$ 上, 记此延拓为 $X^*(\cdot,\omega)$. 对一切 $t \in [0,1]^d, \{t_n\} \subset T, t_n \to t$, 由 (2.29) 式知, $\{X(t_n)\}$ 在 $\mathbb{D}_k^p$ 中收敛于 $X(t)$, 从而存在子序列 $\{t_{n_j}\}$ 使 $X^*(t_{n_j}) = X(t_{n_j}) \to X(t)$ (k,p)-q.s., 故 $X^*(t) = X(t)$ (k,p)-q.s., 显然 X^* 为 X 的 (k,p)- 连续修正. ∎

系 2.25 若 $\forall k \in \mathbb{N}_0, p \in (1,\infty)$ $\exists \alpha = \alpha(k,p), c = c(k,p)$ 及 $\beta = \beta(k,p)$ 满足定理 2.24 的条件, 则 X 存在 ∞- 连续修正.

考虑 Fisk-Stratonovich 方程 (1.27):

$$\begin{cases} dX_t = A_0(X_t)dt + A_i(X_t) \circ dW_t^i, \\ X_0 = x, \qquad\qquad (0 \leq t \leq 1), \end{cases} \tag{2.30}$$

其中 $A_0, A_1, \cdots, A_d$ 为 $\mathbb{R}^m$ 上的 C^∞ 向量场且一切偏导数有界. $\forall n \in \mathbb{N}$, 令 $t_n \equiv 2^{-n}[2^n t], t_n^+ \equiv 2^{-n}([2^n t]+1), 0 \leq t \leq 1, \dot{W}_n^i(t) \equiv 2^n(W^i(t_n^+) - W^i(t_n)), i = 1, \cdots, d$. 考虑逼近方程 (2.30) 的常微分方程序列:

$$\begin{cases} dX_n(t) = (A_0(X_n(t)) + A_i(X_n(t))\dot{W}_n^i(t))dt, \\ X_n(0) = x, \qquad\qquad (0 \leq t \leq 1). \end{cases} \tag{2.31}$$

已知方程 (2.31) 的解 $X_n(t)$ a.s. 收敛于方程 (2.30) 的解 $X(t)$(例如参看 Bismut[2]), 任佳刚 [1] 证明了如下拟必然收敛定理:

定理 2.26 在上述条件下, 存在疏集 S, 使 $\forall\omega\notin S$ 有

$$\lim_{n\to\infty}\sup_{0\le t\le 1}|X_n(t,\omega)-X(t,\omega)|=0. \tag{2.32}$$

证明 我们只给出证明的思路, 详细证明可参看任佳刚 [1,5]. 令

$$Z(s,t)=\begin{cases} X(t) & (s=0);\\ X_n(t)+(s-\frac{1}{n})(\frac{1}{n+1}-\frac{1}{n})^{-1}(X_{n+1}(t)-X_n(t)) & \\ & (\frac{1}{n+1}<s\le\frac{1}{n},n\in\mathbb{N}).\end{cases} \tag{2.33}$$

由于 $Z(0,t)=X(t),Z(1/n,t)=X_n(t)$, 若能证明随机场 (2.33): $\{Z(s,t),(s,t)\in[0,1]^2\}$ 具有 ∞- 连续修正, 则定理得证.

由随机积分的估计, $\forall p\ge 1$, 存在不依赖于 n 的常数 c, 使 $\forall s,t\in[0,1]$ 有

$$\mathbb{E}[|X_n(t)-X_n(s)|^{2p}]\le c|t-s|^p; \tag{2.34}$$

$$\mathbb{E}[|X_n(t)-X(t)|^{2p}]\le c2^{-np}. \tag{2.35}$$

为证 $Z(s,t)$ 满足定理 2.24 的条件, 只需证明: $\forall k\in\mathbb{N}_0,p\in(1,\infty)$,

$$\sup_{0\le t\le 1}\sup_{n\in\mathbb{N}}\|X_n(t)\|_{k,p}<\infty. \tag{2.36}$$

而这可以利用 $X_n(t)$ 各阶导数所满足的随机微分方程对其进行估计, 通过类似于定理 1.9 的证明方法获得. ∎

§ 3. 非适应随机分析

Itô 泛函的一个显著特点是: 它作为一个随机过程是适应于 Brown 运动所生成的 $\sigma-$ 代数流的. 这在应用中受到一定限制. 例

如我们考虑随机微分方程的终值或边值问题，一般来说，不能指望它的解仍然是一个适应随机过程 (除非扩大原来的 σ- 代数流). 前面我们注意到， Skorohod 积分 (散度算子 δ) 是 Itô 积分的推广，它对非适应过程仍有意义. 因此， Malliavin 分析提供了非适应随机分析的一个有效途径.

本节我们考虑有限时间区间上的 Brown 运动. 因此我们假定 $H = L^2([0,1]; I\!R^d), \Omega = C_0([0,1]; I\!R^d), \mu$ 为其上 Wiener 测度，其余记号都和 §1 相同.

3.1 Skorohod 积分的 Riemann 和逼近

我们先用 Riemann 和来逼近 Skorohod 积分. 设

$$\pi_n : 0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = 1, \quad n \in I\!N$$

为 $[0,1]$ 的分割序列. 为记号简单起见，我们将略去上标 n, 记

$$\begin{aligned} \Delta_j &\equiv (t_{j-1}, t_j],\ |\Delta_j| \equiv t_j - t_{j-1}, \\ W(\Delta_j) &\equiv W(t_j) - W(t_{j-1}),\ (1 \le j \le k_n), \\ |\pi_n| &\equiv \max_{1 \le j \le k_n} |\Delta_j|, \end{aligned}$$

$$\mathcal{F}_{\Delta_j^c} \equiv \sigma\{W_t - W_s : (s,t] \cap \Delta_j = \emptyset\}, \quad 1 \le j \le k_n. \tag{3.1}$$

对 $X \in L^2(\Omega; H) \cong L^2([0,1] \times \Omega; I\!R^d)$, 令

$$\pi_n(X) \equiv \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|} \left(\int_{\Delta_j} X_s ds \right) \mathbf{1}_{\Delta_j}; \tag{3.2}$$

$$\widehat{\pi_n}(X) \equiv \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|} \left(\int_{\Delta_j} I\!E[X_s | \mathcal{F}_{\Delta_j^c}] ds \right) \mathbf{1}_{\Delta_j}. \tag{3.3}$$

则有以下简单命题:

命题 3.1 若 $X \in L^2(\Omega; H)$, 则当 $|\pi_n| \to 0$ 时 $\pi_n(X)$ 及 $\widehat{\pi_n}(X)$ 在 $L^2(\Omega; H)$ 中收敛于 X; 若 $X \in I\!D_1^2(H)$, 则上述收敛为 $I\!D_1^2(H)$ 中收敛.

证明 易见映射 $X \longmapsto \pi_n(X)$ 及 $X \longmapsto \widehat{\pi_n}(X)$ 为 $L^2(\Omega; H)$ 中的线性算子. 令 $\mathcal{B}_n$ 为 [0,1] 中由分割 π_n 生成的有限 σ- 代数, λ 为 $[0,1]$ 上的 Lebesgue 测度, 则 $\lambda \times \mu$ 为 $[0,1] \times \Omega$ 上的概率测度, 且

$$\pi_n(X) = \mathbb{E}_{\lambda\times\mu}[X|\mathcal{B}_n \times \mathcal{F}], \quad n \in \mathbb{N} \tag{3.4}$$

为 L^2 有界鞅, 由鞅收敛定理, 它在 $L^2(\Omega; H)$ 中收敛于 X. 类似地, 若令 $\mathcal{G}_n$ 为 $[0,1]\times\Omega$ 的子集族 $\{\Delta_j \times A_j : A_j \in \mathcal{F}_{\Delta_j^c}; 1 \le j \le k_n\}$ 生成的 σ- 代数, 则

$$\widehat{\pi_n}(X) = \mathbb{E}_{\lambda\times\mu}[X|\mathcal{G}_n], \quad n \in \mathbb{N} \tag{3.5}$$

仍为 L^2 有界鞅, 同样在 $L^2(\Omega; H)$ 中收敛于 X.

若 $X \in \mathbb{D}_1^2(H)$, 对 $r \in [0,1]$, 因

$$D_r\pi_n(X)_t = \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} D_r X_s ds\right)\mathbf{1}_{\Delta_j}(t), \tag{3.6}$$

$$D_r\widehat{\pi}_n(X)_t = \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} \mathbb{E}[D_r X_s|\mathcal{F}_{\Delta_j^c}]ds\right)\mathbf{1}_{\Delta_j^c}(r)\mathbf{1}_{\Delta_j}(t), \tag{3.7}$$

(后一式的证明参看引理 1.3) 类似讨论可知 $D\pi_n(X)$ 及 $D\widehat{\pi_n}(X)$ 均在 $L^2(\Omega; H\otimes H)$ 中收敛于 DX, 从而 $\pi_n(X)$ 及 $\widehat{\pi_n}(X)$ 均在 $\mathbb{D}_1^2(H)$ 中收敛于 X. ∎

以下的讨论中, 为记号简单起见, 设 $d=1$.

命题 3.2 若 $X \in L^2(\Omega; H)$, 则 $\widehat{\pi_n}(X) \in \mathcal{D}(\delta)$ 且

$$\delta\widehat{\pi_n}(X) = \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]ds\right)\cdot W(\Delta_j). \tag{3.8}$$

若当 $|\pi_n| \to 0$ 时, $\delta\widehat{\pi_n}(X)$ 在 $L^2(\Omega)$ 中收敛, 则 $X \in \mathcal{D}(\delta)$ 且 δX 等于此极限.

证明 先设 $X \in \mathbb{D}_1^2(H)$, 由 (II.3.48) 式并注意到 (3.7) 式，我们有

$$\begin{aligned}\delta(\mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]\mathbf{1}_{\Delta_j}) &= \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]\cdot W(\Delta_j)\\&\quad-\int_0^1 \mathbb{E}[D_rX_s|\mathcal{F}_{\Delta_j^c}]\mathbf{1}_{\Delta_j^c}(r)\mathbf{1}_{\Delta_j}(r)dr\\&= \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]\cdot W(\Delta_j).\end{aligned}$$

因为 $\mathbb{D}_1^2(H)$ 在 $L^2(\Omega;H)$ 中稠密，上式对 $X \in L^2(\Omega;H)$ 也成立，故有 (3.8) 式. 由命题 3.1 及 δ 的闭性容易证明后一结论. ∎

命题 3.3 若 $X \in \mathbb{D}_1^2(H)$, 则 $\pi_n(X) \in \mathbb{D}_1^2(H)$ 且

$$\begin{aligned}\delta\pi_n(X) &= \sum_{j=1}^{k_n}\frac{1}{|\Delta_j|}\left(\int_{\Delta_j}X_sds\right)\cdot W(\Delta_j)\\&\quad-\sum_{j=1}^{k_n}\frac{1}{|\Delta_j|}\int_{\Delta_j}\int_{\Delta_j}D_rX_sdrds,\end{aligned}\tag{3.9}$$

当 $|\pi_n| \to 0$ 时，$\delta\pi_n(X)$ 及 $\delta\widehat{\pi_n}(X)$ 均在 $L^2(\Omega)$ 中收敛于 δX.

证明 由 (II.3.48) 式，对 $1 \le j \le k_n$ 有

$$\begin{aligned}\delta(X_s\mathbf{1}_{\Delta_j}) &= X_s\cdot W(\Delta_j)-\int_0^1 D_rX_s\mathbf{1}_{\Delta_j}(r)dr\\&= X_s\cdot W(\Delta_j)-\int_{\Delta_j}D_rX_sdr,\end{aligned}$$

由此推得 (3.9) 式. 由命题 3.1 及 δ 从 $\mathbb{D}_1^2(H)$ 到 $L^2(\Omega)$ 中的连续性即得后一结论. ∎

下面给出 Stratonovich 积分的定义.

定义 3.4 可测过程 $X = \{X_t, 0 \le t \le 1\}$, 若 $\int_0^1 |X_t|dt < \infty$ a.s., 且

$$S_n \equiv \sum_{j=1}^{k_n}\frac{1}{|\Delta_j|}\left(\int_{\Delta_j}X_sds\right)\cdot W(\Delta_j)\tag{3.10}$$

当 $|\pi_n| \to 0$ 时依概率收敛，则 X 称为 **Stratonovich** 可积，此极限称为 **Stratonovich** 积分，记为 $\int_0^1 X_t \circ dW_t$.

由 (3.9) 式可知，为使 S_n 收敛，必须 (3.9) 式中后一积分和收敛. 此积分和的收敛性显然与 $D_r X_s$ 在对角线 $r = s$ 两侧的正则性有关. 为此，我们给出一个充分条件：

若 $X \in \mathbb{D}_1^2(H)$, 且 DX 具有修正，使 $t \longmapsto D_{s\wedge t}X_{s\vee t}$ 及 $t \longmapsto D_{s\vee t}X_{s\wedge t}$ 为 $[0,1]$ 到 $L^2(\Omega)$ 的关于 $s \in [0,1]$ 一致连续映射，且

$$\operatorname{ess}\sup_{0\le t,s\le 1} \mathbb{E}[|D_t X_s|^2] < \infty, \tag{3.11}$$

则记为 $X \in \widehat{\mathbb{D}}_1^2(H)$. 特别，若 $D_t X_s$ 有连续修正，则 $X \in \widehat{\mathbb{D}}_1^2(H)$.

若 $X \in \widehat{\mathbb{D}}_1^2(H)$, 则下列极限存在：

$$D_t^+ X_t \equiv \lim_{\epsilon\downarrow 0} D_t X_{t+\epsilon},$$

$$D_t^- X_t \equiv \lim_{\epsilon\downarrow 0} D_t X_{t-\epsilon},$$

上述收敛为 $L^2(\Omega)$ 中收敛，且关于 t 一致. 记 $\nabla \equiv D^+ + D^-$, 则我们有以下定理.

定理 3.5 若 $X \in \widehat{\mathbb{D}}_1^2(H)$, 则 X 为 Stratonovich 可积，且

$$\int_0^1 X_t \circ dW_t = \int_0^1 X_t dW_t + \frac{1}{2}\int_0^1 (\nabla X)_t dt. \tag{3.12}$$

证明 由命题 3.3, 只需证明当 $|\pi_n| \to 0$ 时

$$\sum_{j=1}^{k_n} \frac{1}{|\Delta_j|} \int_{\Delta_j}\int_{\Delta_j} D_t X_s dt ds \xrightarrow{\ \mu\ } \frac{1}{2}\int_0^1 (\nabla X)_t dt. \tag{3.13}$$

注意

$$
\begin{aligned}
&\mathbb{E}\left|\sum_j \frac{1}{|\Delta_j|}\int_{\Delta_j} dt\int_t^{t_j} D_tX_sds-\frac{1}{2}\int_0^1 D_t^+X_tdt\right| \\
&\leq \mathbb{E}\left|\sum_j \frac{1}{|\Delta_j|}\int_{\Delta_j} dt\int_t^{t_j}(D_tX_s-D_t^+X_t)ds\right| \\
&\quad +\mathbb{E}\left|\sum_j\int_{\Delta_j}\frac{t_j-t}{|\Delta_j|}D_t^+X_tdt-\frac{1}{2}\int_0^1 D_t^+X_tdt\right| \\
&\leq \sup_{\substack{0\leq t<s\leq 1\\ s-t\leq|\pi_n|}} \mathbb{E}[|D_tX_s-D_t^+X_t|] \\
&\quad +\mathbb{E}\left|\int_0^1 D_t^+X_t\left(\sum_j\frac{t_j-t}{|\Delta_j|}\mathbf{1}_{\Delta_j}(t)-\frac{1}{2}\right)dt\right|,
\end{aligned}
$$

当 $|\pi_n|\to 0$ 时， $\sum_j|\Delta_j|^{-1}(t_j-t)\mathbf{1}_{\Delta_j}(t)$ 在 $L^2([0,1])$ 中弱收敛于 1/2, 由 $\widehat{\mathbb{D}}_1^2(H)$ 的定义容易看出，上述两项均收敛于 0, 类似地可证

$$
\mathbb{E}\left|\sum_j \frac{1}{|\Delta_j|}\int_{\Delta_j} dt\int_{t_{j-1}}^{t} D_tX_sds-\frac{1}{2}\int_0^1 D_t^-X_tdt\right|\longrightarrow 0,
$$

于是 (3.13) 得证. ∎

注 当 X 为循序过程时， $D_t^-X_t=0, (\nabla X)_t=D_t^+X_t$. 若 X 为连续半鞅，则由 (3.12) 式可知， X 与 W 的交互变差为

$$
[X,W]_1=\int_0^1 D_t^+X_tdt. \tag{3.14}
$$

3.2 非适应过程的 Itô 公式

设 $V\in\mathbb{D}_1^2(H)$, 对 $t\in[0,1]$, 令

$$
X_t=\delta(V\mathbf{1}_{[0,t]})=\int_0^t V_sdW_s. \tag{3.15}
$$

称其为 Skorohod 不定积分. 由于 V 未必适应, 故一般来说, X 未必是鞅, 也未必有连续修正. 下面给出它具有连续修正的一个充分条件.

命题 3.6 若 $p>2, V\in L^{2p}([0,1];\mathbb{D}_1^{2p})$, 则 $X_t=\int_0^t V_s dW_s$ 具有连续修正.

证明 因 $\delta:\mathbb{D}_1^{2p}(H)\longrightarrow L^{2p}$ 为连续, 故 $\exists C_p>0$ 使

$$\|\delta V\|_{2p}^{2p}\le C_p(\|V\|_{2p}^{2p}+\|DV\|_{2p}^{2p}).$$

在上式中以 $V\mathbf{1}_{(s,t]}$ 代替 V, 应用 Hölder 不等式得

$$\begin{aligned}\mathbb{E}\left[\left|\int_s^t V_r dW_r\right|^{2p}\right]&\le C_p(t-s)^{p-1}\left\{\mathbb{E}\left[\int_0^1 V_r^{2p}dr\right]\right.\\&\qquad\left.+\mathbb{E}\left[\int_0^1\left(\int_0^1(D_uV_r)^2du\right)^p dr\right]\right\}\\&=C_p(t-s)^{p-1}\int_0^1\|V_r\|_{1,2p}^{2p}dr.\end{aligned}$$

根据 Kolmogorov 连续性准则, X 存在连续修正. ∎

设 $\{\pi_n\}$ 为 $[0,1]$ 的分割序列, 令

$$Q^{\pi_n}(V)\equiv\sum_{j=1}^{k_n}\left(\int_{\Delta_j}V_s dW_s\right)^2 \tag{3.16}$$

为 X 的平方变差. 我们有如下收敛定理:

定理 3.7 设 $V\in\mathbb{D}_1^2(H)$, 则当 $|\pi_n|\longrightarrow 0$ 时 $Q^{\pi_n}(V)$ 在 $L^1(\Omega)$ 中收敛于 $\int_0^1 V_s^2 ds$.

证明 若 $U,V\in\mathbb{D}_1^2(H)$, 则 $\exists C>0$ 使

$$\begin{aligned}&\mathbb{E}[|Q^{\pi_n}(U)-Q^{\pi_n}(V)|]\\&\le\left(\mathbb{E}\left[\sum_j\left(\int_{\Delta_j}(U_s-V_s)dW_s\right)^2\right]\right)^{1/2}\\&\quad\times\left(\mathbb{E}\left[\sum_j\left(\int_{\Delta_j}(U_s+V_s)dW_s\right)^2\right]\right)^{1/2}\\&\le C\|U-V\|_{1,2}\|U+V\|_{1,2}.\end{aligned}$$

根据稠密性，不妨设 V 具有形式：

$$V = \sum_{i=1}^{m} \varphi_i \mathbf{1}_{(s_{i-1}, s_i]},$$

其中 $0 = s_0 < s_1 < \cdots < s_m = 1, \varphi_i \in \mathbb{D}_1^2$ 且 $\{s_i\}$ 为 π_n 之分点. 于是

$$\delta V = \sum_{i=1}^{m} \left\{ \varphi_i(W(s_i) - W(s_{i-1})) - \int_{s_{i-1}}^{s_i} D_s\varphi_i ds \right\},$$

$$\begin{aligned} Q^{\pi_n}(V) &= \sum_{i=1}^{m} \sum_{j: s_{i-1} < t_j \le s_i} \left(\varphi_i W(\Delta_j) - \int_{\Delta_j} D_s\varphi_i ds \right)^2 \\ &= \sum_{i=1}^{m} \sum_{j} \left[\varphi_i^2 W(\Delta_j)^2 - 2W(\Delta_j) \int_{\Delta_j} D_s\varphi_i ds \right. \\ &\quad \left. + \left(\int_{\Delta_j} D_s\varphi_i ds \right)^2 \right]. \end{aligned}$$

由 Brown 运动平方变差性质可知，上式当 $|\pi_n| \to 0$ 时在 $L^1(\Omega)$ 中收敛于 $\sum_{i=1}^{m} \varphi_i^2(s_i - s_{i-1}) = \int_0^1 V_s^2 ds$. ∎

由上述定理可知，若 $X_t = \int_0^t V_s dW_s$ 有连续有界变差修正，则 $V = 0$.

Malliavin 导数和 Skorohod 积分具有如下的局部性质.

命题 3.8 设 $A \in \mathcal{F}, F \in \mathbb{D}_1^1$, 则

$$F = 0 \text{ a.s. 于} A \Longrightarrow D_t F = 0 \text{ a.e. 于}[0,1] \times A.$$

证明 设 $\varphi \in C_0^\infty(\mathbb{R}), \varphi \ge 0, \varphi(0) = 1$ 且 $\text{supp}(\varphi) \subset [-1,1]$. 令 $\varphi_\epsilon(t) = \varphi(t/\epsilon), \epsilon > 0$, 则 $\text{supp}(\varphi_\epsilon) \subset [-\epsilon, \epsilon]$. 令 $\psi_\epsilon(t) = \int_{-\infty}^t \varphi_\epsilon(s) ds$, 则 $D\psi_\epsilon(F) = \varphi_\epsilon(F) DF$. $\forall h \in H$,

$$\begin{aligned} |\mathbb{E}[\varphi_\epsilon(F) D_h F]| &= |\mathbb{E}[D_h(\psi_\epsilon(F))]| \\ &= |\mathbb{E}[\psi_\epsilon(F) \delta h]| \le \epsilon \|\varphi\|_\infty \mathbb{E}[|\delta h|]. \end{aligned}$$

令 $\epsilon \downarrow 0$, 得

$$\mathbb{E}\left[\mathbf{1}_{\{F=0\}}\int_0^1 (D_tF)h(t)dt\right] = 0. \qquad \blacksquare$$

命题 3.9 设 $A \in \mathcal{F}, V \in \mathbb{D}_1^2(H)$, 则

$$V_t = 0 \text{ a.e. 于}[0,1] \times A \Longrightarrow \delta V = 0 \text{ a.s. 于} A.$$

证明 由命题 3.3, 在 (3.9) 式中选取 a.s. 收敛子序列并考虑到命题 3.8 即可证明该命题. ▮

利用此局部性质, 我们可以扩大算子 D 和 δ 的定义域. 设 $k \in \mathbb{N}_0, p \geq 1, F$ 为 E 值泛函. 若存在集合序列 $\{A_n\} \subset \mathcal{F}$ 及泛函序列 $\{F_n\} \subset \mathbb{D}_k^p(E)$, 满足以下性质:

1° $A_n \uparrow \Omega$ a.s.;

2° $\forall n \in \mathbb{N}, F = F_n$ a.s. 于 A_n,

则称泛函 F 为**局部 $\mathbb{D}_k^p(E)$ 泛函**, 记作 $F \in loc\mathbb{D}_k^p(E)$. 易见, 当 $F \in loc\mathbb{D}_1^p$ 时, 可以在每一集合 A_n 上定义 $DF = DF_n$; 当 $V \in loc\mathbb{D}_1^2(H)$ 时, 可以在每一 A_n 上定义 $\delta V = \delta V_n$ 而不会有含混. 利用局部化手续, 以前的许多命题都可以推广到局部 $\mathbb{D}_k^p(E)$ 空间上去.

下面是本节主要定理:

定理 3.10 (Itô 公式) 设 $f \in C^2(\mathbb{R}), V \in locL^4([0,1];\mathbb{D}_2^4)$, $U \in locL^4([0,1];\mathbb{D}_1^4)$, 若

$$X_t = \int_0^t V_s dW_s + \int_0^t U_s ds \tag{3.17}$$

具有连续修正, 则

$$f(X_t) = f(0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)(\nabla X)_s V_s ds, \tag{3.18}$$

其中 $\nabla = D^+ + D^-$.

注 由引理 1.4 可知

$$D_sX_t = V_s\mathbf{1}_{[0,t]}(s) + \int_0^t D_sV_rdW_r + \int_0^t D_sU_rdr, \tag{3.19}$$

从而

$$(\nabla X)_s = V_s + 2\int_0^s D_sV_rdW_r + 2\int_0^s D_sU_rdr. \tag{3.20}$$

特别，当 U 及 V 适应时，后二项均为 0,(3.18) 式成了通常的 Itô 公式.

证明 利用局部化手续，不妨设 f, f' 及 f'' 有界且 $V \in L^4([0,1];\mathbb{D}_2^4), U \in L^4([0,1];\mathbb{D}_1^4)$. 设 $\{\pi_n\}$ 为 $[0,t]$ 的分割序列，由 Taylor 展式得

$$\begin{aligned} f(X_t) = f(0) &+ \sum_{j=1}^{k_n} f'(X(t_{j-1}))(X(t_j) - X(t_{j-1})) \\ &+ \frac{1}{2}\sum_{j=1}^{k_n} f''(\widehat{X}_j)(X(t_j) - X(t_{j-1}))^2, \end{aligned} \tag{3.21}$$

其中 $\widehat{X}_j$ 为介于 $X(t_j)$ 和 $X(t_{j-1})$ 之间的随机变量. 因为对 $j = 1, \cdots, k_n$,

$$X(t_j) - X(t_{j-1}) = \int_{\Delta_j} V_sdW_s + \int_{\Delta_j} U_sds,$$

故由 (II.3.48) 式得

$$\begin{aligned} f'(X(t_{j-1}))\int_{\Delta_j} V_sdW_s = &\int_{\Delta_j} f'(X(t_{j-1}))V_sdW_s \\ &+ \int_{\Delta_j} V_sD_sf'(X(t_{j-1}))ds, \end{aligned}$$

再由 (3.19) 式得

$$
\begin{aligned}
D_s f'(X_{t_{j-1}}) &= f''(X_{t_{j-1}}) D_s X_{t_{j-1}} \\
&= f''(X_{t_{j-1}}) \Big[V_s \mathbf{1}_{[0,t_{j-1}]}(s) + \int_0^{t_{j-1}} D_s V_r dW_r + \int_0^{t_{j-1}} D_s U_r dr \Big].
\end{aligned}
$$

当 $s \in \Delta_j$ 时, 第一项为 0. 于是定理归结为证明: 当 $|\pi_n| \to 0$ 时下列和依概率收敛:

$$
\sum_j f'(X(t_{j-1})) \int_{\Delta_j} U_s ds \longrightarrow \int_0^t f'(X_s) U_s ds, \tag{3.22}
$$

$$
\sum_j \int_{\Delta_j} f'(X(t_{j-1})) V_s dW_s \longrightarrow \int_0^t f'(X_s) V_s dW_s, \tag{3.23}
$$

$$
\begin{aligned}
&\sum_j \int_{\Delta_j} f''(X(t_{j-1})) V_s \Big(\int_0^{t_{j-1}} D_s U_r dr \Big) ds \\
&\qquad \longrightarrow \int_0^t f''(X_s) V_s \Big(\int_0^s D_s U_r dr \Big) ds,
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
&\sum_j \int_{\Delta_j} f''(X(t_{j-1})) V_s \Big(\int_0^{t_{j-1}} D_s V_r dW_r \Big) ds \\
&\qquad \longrightarrow \int_0^t f''(X_s) V_s \Big(\int_0^s D_s V_r dW_r \Big) ds,
\end{aligned} \tag{3.25}
$$

$$
\sum_j f''(\widehat{X}_j)(X(t_j) - X(t_{j-1}))^2 \longrightarrow \int_0^t f''(X_s) V_s^2 ds. \tag{3.26}
$$

(3.22) 中的 Stieltjes 和显然 a.s. 收敛. 注意

$$
\begin{aligned}
(X(t_j) - X(t_{j-1}))^2 &= \Big(\int_{\Delta_j} V_s dW_s \Big)^2 + \Big(\int_{\Delta_j} U_s ds \Big)^2 \\
&\quad + 2 \int_{\Delta_j} V_s dW_s \int_{\Delta_j} U_s ds,
\end{aligned}
$$

由于 f'' 有界连续及 $\int_0^t U_s ds$ 变差有界, 后两项之和趋于 0. 另一

方面，由定理 3.7 容易证明:

$$\sum_j f''(\widehat{X}_j)\left(\int_{\Delta_j} V_s dW_s\right)^2 \longrightarrow \int_0^t f''(X_s)V_s^2 ds,$$

从而 (3.26) 式得证. (3.24) 及 (3.25) 式的证明类似，例如对 (3.25) 式我们有如下估计:

$$\left|\sum_j \int_{\Delta_j}\left[f''(X_{t_{j-1}})\int_0^{t_{j-1}} D_s V_r dW_r - f''(X_s)\int_0^s D_s V_r dW_r\right] V_s ds\right|$$

$$\leq \left|\sum_j \int_{\Delta_j} f''(X_{t_{j-1}})\left(\int_{t_{j-1}}^s D_s V_r dW_r\right) V_s ds\right|$$

$$+\left|\sum_j \int_{\Delta_j} [f''(X_{t_{j-1}}) - f''(X_s)]\left(\int_0^s D_s V_r dW_r\right) V_s ds\right|$$

$$\leq \|f''\|_\infty \sum_j \int_{\Delta_j}\left|\int_{t_{j-1}}^s D_s V_r dW_r\right| |V_s| ds$$

$$+\sup_j \sup_{s\in\Delta_j} |f''(X_{t_{j-1}}) - f''(X_s)| \int_0^t \left|V_s \int_0^s D_s V_r dW_s\right| ds.$$

因 $V \in L^4([0,1]; \mathbb{D}_2^4) \subset L^2([0,1]; \mathbb{D}_2^2), f''$ 连续，上式右边第二项 a.s. 收敛于 0, 由定理 1.8 及 Cauchy-Schwartz 不等式，第一项之期望不超过

$$\|f''\|_\infty \left(\mathbb{E}\left[\int_0^1 V_s^2 ds\right]\right)^{1/2} \left(\mathbb{E}\left[\sum_j \int_{\Delta_j}\int_{t_{j-1}}^s |D_s V_r|^2 dr ds\right]\right.$$

$$\left.+\mathbb{E}\left[\sum_j \int_{\Delta_j}\int_{t_{j-1}}^s \int_0^1 |D_u D_s V_r|^2 du dr ds\right]\right)^{1/2},$$

从而也收敛于 0. 最后只剩下 (3.23) 式. 我们证明它在 $L^2(\Omega)$ 中收敛. 因 Skorohod 积分算子 δ 为 $\mathbb{D}_1^2(H) \longrightarrow L^2(\Omega)$ 的连续线性算子，只需证明

$$V_s^{(n)} \equiv V_s \sum_{j=1}^{k_n} f'(X(t_{j-1}))\mathbf{1}_{\Delta_j}(s), \quad 0 \leq s \leq 1$$

在 $\mathbb{D}_1^2(H)$ 中收敛于 $V_sf'(X_s)$. 显然 $V^{(n)}$ 在 $L^2([0,1]\times\Omega)$ 中收敛于 $Vf'(X)$, 剩下要证明 $DV^{(n)}$ 在 $L^2([0,1]^2\times\Omega)$ 中收敛于 $D[Vf'(X)]$, 而

$$\begin{aligned}D_rV_s^{(n)} = (D_rV_s)\sum_j f'(X(t_{j-1}))\mathbf{1}_{\Delta_j}(s)\\ + V_s\sum_j f''(X(t_{j-1}))(D_rX(t_{j-1}))\mathbf{1}_{\Delta_j}(s),\end{aligned}$$

因 $V\in\mathbb{D}_1^2(H)$, 显然第一项和收敛于 $D_rV_sf'(X_s)$. 为证第二项和收敛于 $V_sf''(X_s)D_rX_s$, 利用 (3.19) 式将 $D_rX(t_{j-1})$ 分成三项之和, 显然有

$$V_s\sum_j f''(X(t_{j-1}))V_r\mathbf{1}_{[0,t_{j-1}]}(r)\mathbf{1}_{\Delta_j}(s)\longrightarrow V_sf''(X_s)V_r\mathbf{1}_{[0,s]}(r)$$

以及

$$V_s\sum_j f''(X(t_{j-1}))\left(\int_0^s D_rU_udu\right)\mathbf{1}_{\Delta_j}(s)\longrightarrow V_sf''(X_s)\int_0^s D_rU_udu.$$

而

$$\begin{aligned}&\mathbb{E}\left[\sum_j\int_{\Delta_j}\int_0^1 V_s^2f''(X(t_{j-1}))^2\left(\int_{t_{j-1}}^s D_rU_udu\right)^2drds\right]\\ &\le\|f''\|_\infty^2\left(\mathbb{E}\int_0^1 V_s^4ds\right)^{1/2}\left(\mathbb{E}\int_0^1\left[\int_0^1(D_rU_u)^2dr\right]^2du\right)^{1/2}|\pi_n|\\ &\longrightarrow 0,\end{aligned}$$

因此在 $L^2([0,1]^2\times\Omega)$ 中

$$\begin{aligned}V_s\sum_j f''(X(t_{j-1}))\left(\int_0^{t_{j-1}} D_rU_udu\right)\mathbf{1}_{\Delta_j}(s)\\ \longrightarrow V_sf''(X_s)\int_0^s D_rU_udu.\end{aligned}$$

类似地有

$$V_s\sum_j f''(X(t_{j-1}))\left(\int_0^{t_{j-1}} D_rV_udW_u\right)\mathbf{1}_{\Delta_j}(s)$$
$$\longrightarrow V_sf''(X_s)\int_0^s D_rV_udW_u,$$

定理得证. ∎

关于 Stratonovich 积分，也有类似公式. 我们约定，若 $V\in L^4([0,1];\mathbb{D}_2^4), DV$ 具有修正，使 $t\longmapsto D_{s\wedge t}V_{s\vee t}$ 及 $t\longmapsto D_{s\vee t}V_{s\wedge t}$ 为 $[0,1]$ 到 $L^4(\Omega)$ 的关于 $s\in[0,1]$ 一致连续映射，且

$$\text{ess.sup}_{0\le t,s\le 1}\mathbb{E}[|D_tV_s|^4]<\infty,$$

则记为 $V\in\widehat{L}^4([0,1];\mathbb{D}_2^4)$.

定理 3.11 设 $V\in loc\widehat{L}^4([0,1];\mathbb{D}_2^4), DV\in locL^4([0,1]^2;\mathbb{D}_1^4)$, $U\in locL^4([0,1];\mathbb{D}_1^4), f\in C^2(\mathbb{R})$. 若

$$X_t=\int_0^t V_s\circ dW_s+\int_0^t U_sds \tag{3.27}$$

具有连续修正，则

$$f(X_t)=f(0)+\int_0^t f'(X_s)U_sds+\int_0^t f'(X_s)V_s\circ dW_s. \tag{3.28}$$

证明 由 (3.12) 式，

$$X_t=\int_0^t V_sdW_s+\int_0^t U_sds+\frac{1}{2}\int_0^t(\nabla V)_sds.$$

根据 Itô 公式 (3.18)，

$$\begin{aligned}f(X_t)=&f(0)+\int_0^t f'(X_s)U_sds+\frac{1}{2}\int_0^t f'(X_s)(\nabla V)_sds\\&+\int_0^t f'(X_s)V_sdW_s+\frac{1}{2}\int_0^t f''(X_s)(\nabla X)_sV_sds.\end{aligned} \tag{3.29}$$

再由 (3.12) 式得

$$\int_0^t f'(X_s)V_s \circ dW_s = \int_0^t f'(X_s)V_s dW_s + \frac{1}{2}\int_0^t \nabla(f'(X_s)V_s)ds.$$

而

$$\nabla(f'(X_s)V_s) = f'(X_s)(\nabla V)_s + f''(X_s)V_s(\nabla X)_s,$$

代入 (3.29) 式即得 (3.28) 式. ∎

3.3 非适应随机微分方程

考虑随机微分方程

$$X_t = \eta + \int_0^1 b(t,s,X_s)ds + \int_0^1 \sigma(t,s,X_s)dW_s \quad 0 \le t \le 1,$$

其中 η 为随机变量, $b(t,\cdot,x)$ 及 $\sigma(t,\cdot,x)$ 为依赖于参数 t,x 的随机过程 (未必适应). 其中随机积分为 Skorohod 积分. 由定理 1.8 可知, 解的 L^2 估计涉及它的导数的估计, 而导数的估计涉及其二阶导数的估计. 因此, 通常的 Picard 迭代过程是不封闭的, 这正是问题的困难所在. 对于这一类方程, 已经有许多特别的方法, 在某些特殊情形, 给出了解的存在性和唯一性, 甚至解的明显表达式. 但整个说来, 理论的发展仍然是不成熟和不完善的. 关于它的发展现状读者可参看 Pardoux 的综合性论文 [1]. 这里我们着重介绍由 Buckdahn[1,2] 发展的 Girsanov 变换方法.

在经典 Wiener 空间中, Girsanov[1] 将 Cameron-Martin 关于 Wiener 测度在平移下的拟不变性结果 (定理 II.2.5) 推广到了 "随机平移" 的情形, 但所涉及的随机过程是适应的. Ramer[1] 与 Kusuoka[1] 在抽象 Wiener 空间中, 将 Girsanov 定理推广到了非适应情形. 关于非适应 Girsanov 变换, 近年来还有许多研究, 例如参看 Buckdahn[1],Enchev - Stroock[1], Ustunel - Zakai[3,5], 张荫南 [1]. 这里我们简要介绍 Kusuoka 的定理.

定义 3.12 设 (H,X,μ) 为抽象 Wiener 空间， $J:H\longrightarrow X$ 为嵌入映射． X 上的 H 值泛函 V，若对 μ-a.e. $x\in X$，映射 $h\longmapsto V(x+Jh)$ 在 H 中连续可微，即存在 Hilbert-Schmidt 算子 $DV(x):H\longrightarrow H$，使

1° $\|V(x+Jh)-V(x)-DV(x)h\|=o(\|h\|)$;

2° $DV(x+J\cdot):H\longrightarrow\mathcal{L}_{(2)}(H)$ 连续，

则称 V 为 **H 连续可微**，记为 $V\in C_H^1(H)$.

可以证明， $C_H^1(H)\subset loc\mathbb{D}_1^2(H)$，且 DV 重合于 Malliavin 导数 (注意 $\mathcal{L}_{(2)}(H)\cong H\otimes H$). 例如参看 Kusuoka[1].

我们先给出有限维情形的一个引理：

引理 3.13 设 $(\mathbb{R}^n,\mathcal{B}^n,\gamma^n;\mathbb{R}^n)$ 为有限维 Gauss 概率空间，$\psi\in\mathcal{S}_M(\mathbb{R}^n)$, $D\psi=\{\partial_i\psi_j\}_{1\le i,j\le n}$ 为 Jacobi 矩阵，令

$$\eta(x)\equiv|\det(I+D\psi(x))|\exp\{-(\psi(x),x)-\tfrac{1}{2}|\psi(x)|^2\}. \tag{3.30}$$

若由 $Tx=x+\psi(x)$ 定义的映射 $T:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ 为单射，则 $\gamma^n\circ T^{-1}\ll\gamma^n$; 若 T 为双射，则 $\mathcal{B}^n$ 上存在概率 ν 使 $\nu\sim\gamma^n$, $\nu\circ T^{-1}=\gamma^n$ 且 $d\nu/d\gamma^n=\eta$.

证明 对 $\mathbb{R}^n$ 上一切非负有界可测函数 f 有

$$\begin{aligned}&\int_{\mathbb{R}^n}f(Tx)\eta(x)\gamma^n(dx)\\&=(2\pi)^{-n/2}\int_{\mathbb{R}^n}f(Tx)\exp\{-\tfrac{1}{2}|Tx|^2\}|\det(I+D\psi(x))|dx\\&=\int_{T(\mathbb{R}^n)}f(y)\gamma^n(dy)\le\int_{\mathbb{R}^n}f(y)\gamma^n(dy).\end{aligned} \tag{3.31}$$

故 $\gamma^n\circ T^{-1}\ll\gamma^n$. 若 T 为双射， (3.31) 式中最后一个不等号成为等号，令 $\nu(dx)=\eta(x)\gamma^n(dx)$ 即得 $\nu\circ T^{-1}=\gamma^n$. ∎

为将此结果推广到无穷维空间，需要引进所谓 **Carleman-Fredholm 行列式**. 设 $S\in\mathcal{L}_{(2)}(H)$，具有特征值 $\{\lambda_j\}$. 我们知道 $\det(I+S)=\prod_j(1+\lambda_j)$ 仅对 $S\in\mathcal{L}_{(1)}(H)$ 有意义. 我们定义

$$\det{}_2(I+S)\equiv\prod_j(1+\lambda_j)e^{-\lambda_j}. \tag{3.32}$$

称为 Carleman-Fredholm 行列式. 易见，当 $S \in \mathcal{L}_{(2)}(H)$ 时，(3.32) 式有意义，而且可以证明： $S \longmapsto \det_2(I+S)$ 在 $\mathcal{L}_{(2)}(H)$ 的有界集上一致连续 (例如参看 B. Simon[2]). 显然，当 $S \in \mathcal{L}_{(1)}(H)$ 时，

$$\det_2(I+S) = \det(I+S)e^{-TrS}. \tag{3.33}$$

利用公式 (II.2.4),$\delta\psi(x) = (\psi(x), x) - \mathrm{Tr}(D\psi(x))$, 我们可以将 (3.30) 式改写为

$$\eta(x) = |\det_2(I + D\psi(x))| \exp\{-\delta\psi(x) - \tfrac{1}{2}|\psi(x)|^2\}. \tag{3.34}$$

利用有限维逼近，可以证明如下定理.

定理 3.14 (Kusuoka[1]) 设 (H, X, μ) 为抽象 Wiener 空间，$V \in C_H^1(H), Tx = x + JV(x)$. 若 $T: X \longrightarrow X$ 为双射，且对 μ-a.e. $x \in X$, 映射 $I+DV(x):\ H \to H$ 可逆 (即 $\det_2(I+DV(x)) \neq 0$), 则 $\mathcal{B}(X)$ 上存在与 μ 等价的概率测度 ν 使 $\nu \circ T^{-1} = \mu$ 且

$$\frac{d\nu}{d\mu} = |\det_2(I + DV)| \exp\{-\delta V - \tfrac{1}{2}\|V\|^2\}. \tag{3.35}$$

特别，在经典 Wiener 空间 $\Omega = C_0([0,1])$ 中，变换 $T: \Omega \longrightarrow \Omega$ 为如下形式：

$$T\omega = \omega + \int_0^{\cdot} V_s(\omega)ds. \tag{3.36}$$

若 V 为适应过程，则当 $t > s$ 时， $D_tV_s = 0$ a.s., 由此可推出 $|\det_2(I + DV)| = 1$ a.s., 于是 (3.35) 式化为通常的 Girsanov 公式：

$$\frac{d\nu}{d\mu} = \exp\left\{ -\int_0^1 V_s dW_s - \frac{1}{2}\int_0^1 V_s^2 ds \right\}. \tag{3.37}$$

下面考察非适应线性随机微分方程：

$$X_t = X_0 + \int_0^t b_s X_s ds + \int_0^t \sigma_s X_s dW_s \quad 0 \leq t \leq 1, \tag{3.38}$$

其中 X_0 为有界随机变量 (不必 $\mathcal{F}_0$ 可测),$b=\{b_s(\omega),0\le s\le 1\}$ 及 $\sigma=\{\sigma_s(\omega),0\le s\le 1\}$ 为有界随机过程 (不必适应).Buckdahn[2] 定义了一族变换:

$$T_t\omega=\omega+\int_0^{t\wedge\cdot}\sigma_s(T_s\omega)ds,\quad 0\le t\le 1, \tag{3.39}$$

证明了如下定理:

定理 3.15 设 $\sigma\in L^\infty([0,1]\times\Omega),D\sigma\in L^\infty([0,1]^2\times\Omega)$, 则由方程 (3.39) 确定唯一一族可逆变换 $\{T_t,0\le t\le 1\}$, 且 $\forall t\in[0,1],\mu\circ T_t^{-1}\ll\mu$, 其 Radon-Nikodym 导数 $L_t\equiv d\mu\circ T_t^{-1}/d\mu$ 满足方程:

$$L_t=1+\int_0^t\sigma_sL_sdW_s,\quad 0\le t\le 1. \tag{3.40}$$

若 $b\in L^\infty([0,1]\times\Omega),X_0\in L^\infty(\Omega)$, 则

$$X_t=X_0(T_t^{-1})\exp\left\{\int_0^t b_s(T_sT_t^{-1})ds\right\}L_t \tag{3.41}$$

为方程 (3.38) 在 $L^1([0,t]\times\Omega)$ 中的唯一解.

证明 我们只给出证明的思路, 详细证明可参看 Buckdahn[2].

先对如下形式的 σ 证明定理:

$$\sigma_t=f_t(W_{t_1},\cdots,W_{t_n}),\quad n\ge 1,t_1,\cdots,t_n\in[0,1], \tag{3.42}$$

其中 $f\in L^\infty([0,1]\times\mathbb{R}^n)$, 且 $\forall t\in[0,1],f_t\in C_b^\infty(\mathbb{R}^n)$. 注意 σ_t 满足条件:

$$|\sigma_t(\omega)-\sigma_t(\omega')|\le C\sup_{0\le s\le 1}|\omega(s)-\omega'(s)|,\quad \forall\omega,\omega'\in\Omega,t\in[0,1].$$

利用 Picard 迭代可证方程 (3.39) 存在唯一 Ω 值解 $\{T_t\omega,0\le t\le 1\}$, 其逆 $\{A_t\omega,0\le t\le 1\}$ 满足方程

$$A_t\omega=\omega-\int_0^{t\wedge\cdot}\sigma_s(T_sA_t\omega)ds. \tag{3.43}$$

记 $\sigma(T)\equiv\{\sigma_t(T_t\omega),0\le t\le 1\}$, 直接计算其 Malliavin 导数及 Carleman-Fredholm 行列式，可知 $\sigma(T)\in C_H^1(H), I+D\sigma(T)$ 可逆. 由定理 3.14 可得

$$\frac{d\mu\circ A_t^{-1}}{d\mu}=\exp\left\{-\int_0^t\sigma_s(T_s)dW_s-\frac{1}{2}\int_0^t\sigma_s(T_s)^2ds\right.\\\left.-\int_0^t\int_0^s(D_r\sigma_s)(T_s)D_s[\sigma_r(T_r)]drds\right\}. \tag{3.44}$$

记 (3.44) 式右端为 η_t, 则

$$L_t\equiv\frac{d\mu\circ T_t^{-1}}{d\mu}=\eta_t(A_t)^{-1} \tag{3.45}$$

满足方程 (3.40). 因为形如 (3.42) 的过程类在 $\mathbb{D}_1^2(H)$ 中稠密，对满足定理条件的 σ, 存在形如 (3.42) 的过程序列 $\{\sigma^n\}$ 在 $\mathbb{D}_1^2(H)$ 中收敛于 σ 并使 $\{\sigma^n\}$ 及 $\{D\sigma^n\}$ 一致有界. 由刚才所证，存在变换 T_t^n 及逆变换 A_t^n, R-N 导数 L_t^n 及 η_t^n 满足 (3.44),(3.45) 及方程 (3.40). 不难证明，当 $n\to\infty$ 时， $\{\sigma^n(T^n)\}$ 在 $\mathbb{D}_1^2(H)$ 中收敛且 $\{L_t^n:0\le t\le 1,n\ge 1\}$ 一致可积，从而定理前半部分得证.

显然由 (3.41) 式定义的 X 属于 $L^1([0,t]\times\Omega)$, 为验证其满足方程 (3.38), 我们对任意 $G\in\mathcal{S}_M$, 计算 $\mathbb{E}[\int_0^t\sigma_sX_sD_sGds]$. 作测度变换 $\mu\longmapsto\mu\circ T_s^{-1}=L_s\cdot\mu$ 得

$$\begin{aligned}&\mathbb{E}\left[\int_0^t\sigma_sX_sD_sGds\right]\\&=\mathbb{E}\left[\int_0^t\sigma_sX_0(A_s)L_s\exp\left\{\int_0^s b_r(T_rA_s)dr\right\}D_sGds\right]\\&=\mathbb{E}\left[\int_0^t\sigma_s(T_s)X_0\exp\left\{\int_0^s b_r(T_r)dr\right\}(D_sG)(T_s)ds\right].\end{aligned}$$

注意到由 (3.39) 可推知 $\frac{d}{ds}G(T_s)=\sigma_s(T_s)(D_sG)(T_s)$, 由分部积分

和再一次测度变换可得

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \sigma_s X_s D_s G ds\right] \\
&= \mathbb{E}\left[\int_0^t X_0 \exp\left\{\int_0^s b_r(T_r)dr\right\}\frac{d}{ds}G(T_s)ds\right] \\
&= \mathbb{E}\left[X_0 \exp\left\{\int_0^t b_s(T_s)ds\right\}G(T_t) - X_0 G \right. \\
&\quad \left. - \int_0^t X_0 b_s(T_s)\exp\left\{\int_0^s b_r(T_r)dr\right\}G(T_s)ds\right] \\
&= \mathbb{E}\left[X_0(A_t)\exp\left\{\int_0^t b_s(T_s A_t)ds\right\}L_t G\right] - \mathbb{E}[X_0 G] \\
&\quad - \mathbb{E}\left[\int_0^t X_0(A_s)b_s \exp\left\{\int_0^s b_r(T_r A_s)dr\right\}L_s G ds\right] \\
&= \mathbb{E}\left[\left(X_t - X_0 - \int_0^t b_s X_s ds\right)G\right],
\end{aligned}
$$

从而由 (II.3.47) 式知

$$
\int_0^t \sigma_s X_s dW_s = X_t - X_0 - \int_0^t b_s X_s ds, \quad \text{a.s.}.
$$

唯一性也可通过逼近和分部积分而得到，定理获证. ∎

注 若 σ 为适应过程，则 (3.43) 式化为

$$
A_t\omega = \omega - \int_0^{t\wedge\cdot} \sigma_s(\omega)ds,
$$

(3.44) 式化为

$$
\eta_t = \exp\left\{-\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right\},
$$

而

$$
L_t = \exp\left\{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right\}.
$$

若 b 亦为适应过程，则 (3.41) 式化为

$$X_t = X_0(A_t)\exp\left\{\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t b_s ds\right\}.$$

当 X_0 为 $\mathcal{F}_0$ 可测时，它就是经典 Itô 随机微分方程的解.

第四章　白噪声分析的一般理论

白噪声分析 (white noise analysis) 是 1975 年由飞田武幸 (T. Hida) 首先提出的一种无穷维随机分析，其基本思想是把 Wiener 泛函视为白噪声的泛函. 具体来说，令 Ω 表示 $\mathbb{R}$ 上在 0 点为 0 的连续函数全体，它按有界区间上一致收敛拓扑为一 Fréchet 空间，$\mathcal{B}(\Omega)$ 表示 Ω 上的 Borel σ- 代数，$\mathbb{P}$ 为 $(\Omega, \mathcal{B}(\Omega))$ 上的标准 Wiener 测度. 令

$$W_t(\omega) = \omega(t)\ , \quad t \in \mathbb{R}\ , \ \omega \in \Omega\ ,$$

则 $\{W_t, t \in \mathbb{R}\}$ 为一 Brown 运动，且 $W_0 = 0, \mathrm{a.s.}$. 于是 Wiener 泛函 $f(\omega)$ 可以看作 Brown 运动 W 的轨道的泛函. 进一步，令

$$\Omega_1 = \left\{\omega \in \Omega : \lim_{|t| \to \infty} (1+t^2)^{-1/2} |\omega(t)| = 0\right\}\ ,$$

则由 Brown 运动轨道的性质知 $\mathbb{P}(\Omega_1) = 1$. 令 $\mathcal{F} = \Omega_1 \cap \mathcal{B}(\Omega)$, 则 $\{W_t, t \in \mathbb{R}\}$ 限于 $(\Omega_1, \mathcal{F}, \mathbb{P})$ 仍为 Brown 运动. 由于 $\Omega_1 \subset \mathcal{S}^*(\mathbb{R})$ ($\mathcal{S}^*(\mathbb{R})$ 为 Schwartz 缓增广义函数空间), 可将 Brown 运动 W 及其按轨道的广义微分 $\dot{W}$ 看作为 $\mathcal{S}^*(\mathbb{R})$- 值随机元. $\dot{W}$ 即是所谓的 **"白噪声"**, 它在样本函数空间 $(\mathcal{S}^*(\mathbb{R}), \mathcal{B}(\mathcal{S}^*(\mathbb{R}))$ 上的分布 μ 称为 **白噪声测度**. $(\mathcal{S}^*(\mathbb{R}), \mathcal{B}(\mathcal{S}^*(\mathbb{R})), \mu)$ 上的可测函数称为 **白噪声泛函**. 这样一来，可以把 Wiener 泛函 $f(\omega)$ 看成为白噪声泛函 $F(x)$, 其中 $F = f \circ J^{-1}$, $J(\omega) = \dot{\omega}$ 为 Ω_1 到 $\mathcal{S}^*(\mathbb{R})$ 中的可测映射. 这一看法的优点是可以利用 $\mathcal{S}^*(\mathbb{R})$ 作为核空间对偶的线性拓扑结构，用"二次量子化"方法从 $\mathbb{R}$ 上的 Sobolev 空间 $\mathcal{S}_p(\mathbb{R})$ 出发，建立白噪声泛函的 Sobolev 空间 $(\mathcal{S})_p$. 然后令 $(\mathcal{S})$ 为 $\{(\mathcal{S})_p, p \in \mathbb{N}_0\}$ 的投影极限，$(\mathcal{S})^*$ 为 $\{(\mathcal{S})_{-p}, p \in \mathbb{N}_0\}$ 的归纳极限，分别称 $(\mathcal{S})$ 与 $(\mathcal{S})^*$ 为检验泛函与广义泛函空间. 这样我们从 Gel'fand 三元组

$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}, dx) \hookrightarrow \mathcal{S}^*(\mathbb{R})$ 出发得到一个新的 Gel'fand 三元组 $(\mathcal{S}) \hookrightarrow L^2(\mathcal{S}^*(\mathbb{R}), \mu) \hookrightarrow (\mathcal{S})^*$. 后者称为经典的 Hida 白噪声分析框架. 下面将看到, Hida 检验泛函空间 $(\mathcal{S})$ 比 Meyer-Watanabe 检验泛函空间 $\mathbb{D}^\infty$ 小, 并且 $(\mathcal{S})$ 可连续、稠密地嵌入到 $\mathbb{D}^\infty$ 中, 从而 Hida 广义泛函空间 $(\mathcal{S})^*$ 比 Meyer-Watanabe 广义泛函空间 $\mathbb{D}^{-\infty}$ 大, 它能包含物理中许多概念, 使量子物理中许多形式演算得到严格的数学解释. 因此, 白噪声分析在量子物理中得到了成功的应用.

上述"二次量子化"方法容易推广到一般的 Gel'fand 三元组 (见 Kondratiev-Leukert-Potthoff-Streit-Westerkamp[1]). 下面我们将要介绍的构造白噪声分析框架方法是"二次量子化"方法的一种自然推广, 它是首先由 Kondratiev-Streit[1] 提出的. 这一方法的基本出发点是在定义泛函的 Sobolev 范数时, 对混沌分解的第 n 个因子乘以 $(n!)^\beta$ 这一权系数以得到较小的检验泛函空间 (相应地, 较大的广义泛函空间). 这给白噪声分析的应用提供了灵活选择框架的余地.

§1. 白噪声分析的一般框架

在第二章中, 我们研究了不可约 Gauss 概率空间 $(\Omega, \mathcal{F}, \mu; H)$ 上的泛函. 在那里, 对基本概率空间 $(\Omega, \mathcal{F}, \mu)$ 中的 Ω 未作特别假定, 甚至不要求 Ω 有拓扑结构. 只是为了证明的方便, 我们才使用了所谓的 Gauss 概率空间的数值模型 $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$. 即使如此, 我们也未利用 $\mathbb{R}^\infty$ 的拓扑结构. 在那一章里, 我们引进了 Meyer-Watanabe 广义泛函, 但在这类广义泛函 φ 的混沌分解中, 序列 $\{f_n\}$ 仍是 $\{H^{\widehat{\otimes} n}\}$ 中的元素. 这对某些应用 (如对量子物理) 是一个限制. 在本节我们将要介绍的白噪声分析的一般框架在一定程度上拓广了 Meyer-Watanabe 广义泛函的范围. 这一拓广强烈依赖于基本概率空间 $(\Omega, \mathcal{F}, \mu)$ 中 Ω 的线性拓扑结构. 具体来说, 我们要求 Ω 是某个可列 Hilbert 核空间的对偶空间.

1.1 Wick 张量积与 Wiener-Itô-Segal同构

设 $E \hookrightarrow H \hookrightarrow E^*$ 为一 Gel'fand 三元组，即 H 为一实可分 Hilbert 空间，E 为一可列 Hiblert 核空间，它连续稠密地嵌入到 H 中，E^* 为 E 的对偶 (将 H 与它的对偶等同).H 或 $H^{\otimes n}$ 中的内积及范数统一用 $(\cdot,\cdot)$ 及 $|\cdot|$ 表示. 今后，我们恒将 $H^{\widehat{\otimes} n}$ 看成为 $H^{\otimes n}$ 的子空间，$H^{\widehat{\otimes} n}$ 中的原内积一律换算成 $H^{\otimes n}$ 的内积，即 $(\cdot,\cdot)_{H^{\widehat{\otimes} n}} = n!(\cdot,\cdot)$. 由第一章中的 Minlos 定理知，存在 $(E^*,\mathcal{B}(E^*))$ 上唯一的 Gauss 测度 μ 使得

$$\int_{E^*} e^{i\langle x,f\rangle}\mu(dx) = \exp\{-\tfrac{1}{2}|f|^2\}\ , f \in E. \tag{1.1}$$

这里及今后 $\langle\cdot,\cdot\rangle$ 均表示 $E^* \times E$(或 $E \times E^*$) 上的典则双线性型.

设 $f \in E$. 令 $W_f(x) = \langle x, f\rangle, x \in E^*$. 则 W_f 为 (E^*,μ) 上的 Gauss 随机变量，且由 (1.1) 知

$$\mathbb{E}[W_f] = 0\ ,\quad \mathbb{E}[W_f^2] = |f|^2\ .$$

于是 E 到 $L^2(E^*,\mu)$ 中的线性映射 $f \mapsto W_f$ 可以延拓成为 H 到 $L^2(E^*,\mu)$ 中的线性等距映射. 这样一来，与任一 Gel'fand 三元组 $E \hookrightarrow H \hookrightarrow E^*$ 可以唯一联系一 Gauss 概率空间 $(E^*,\mathcal{B}(E^*),\mu;H)$. 我们称这类 Gauss 概率空间为 **典则 Gauss 概率空间**. 经典的白噪声空间 $(\mathcal{S}^*(\mathbb{R}^d),\mathcal{B}(\mathcal{S}^*(\mathbb{R}^d)),\mu)$ 连同 $L^2(\mathbb{R}^d,dx)$ 构成一个典则 Gauss 概率空间.

在第二章中，我们曾对 $H = L^2(T,\mathcal{B},\lambda)$ 这一特殊情形，借助于多重 Wiener-Itô 积分建立了 $L^2(\Omega,\mathcal{F},\mu)$ 与 H 上的对称 Fock 空间的同构. 下面我们将对典则 Gauss 概率空间情形建立这一同构关系. 为此先引进 E^* 中元素的 Wick 张量积概念，借助于这一概念，我们可以定义多重 Wiener-Itô 积分的类似物.

定义 1.1 令 τ 为 $E^*\widehat{\otimes}E^*$ 中如下定义的元素：

$$\langle\tau, f\otimes g\rangle = (f,g)\ ,\quad f,g \in E\ . \tag{1.2}$$

对 $x \in E^*$, 令 $:x^{\otimes 0}:\equiv 1$ $:x^{\otimes 1}:\equiv x$, 并归纳定义

$$:x^{\otimes n}:\equiv x\widehat{\otimes}:x^{\otimes n-1}:-(n-1)\tau\widehat{\otimes}:x^{\otimes n-2}:,\quad n\geq 2.$$

易知

$$:x^{\otimes n}:=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!2^k}x^{\otimes(n-2k)}\widehat{\otimes}\tau^{\widehat{\otimes}k}\ ,\ n\geq 1\ . \tag{1.3}$$

显然有 $:x^{\otimes n}:\in E^{*\widehat{\otimes}n}$. 我们称 $:x^{\otimes n}:$ 为 x 的 n- **重 Wick 张量积**.

注 由 Hermite 多项式的表达式 (A.11) 及 (A.3) 容易证明

$$:x^{\otimes n}:=\int_{E^*}(x\pm iy)^{\otimes n}\mu(dy)\ , \tag{1.4}$$

等式右端为 Bochner 意义下的积分. 此外有

$$x^{\otimes n}=\int_{E^*}:(x+y)^{\otimes n}:\mu(dy)\ , \tag{1.5}$$

$$x^{\otimes n}=\sum_{k=0}^{[n/2]}\frac{n!}{k!(n-2k)!2^k}:x^{\otimes(n-2k)}:\widehat{\otimes}\tau^{\widehat{\otimes}k}\ . \tag{1.6}$$

引理 1.2 我们有

$$\langle f^{\otimes n},:x^{\otimes n}:\rangle=|f|^n H_n(|f|^{-1}\langle f,x\rangle)\ ,\ f\in E, \tag{1.7}$$

$$\int_{E^*}\langle f_n,:x^{\otimes n}:\rangle\langle g_m,:x^{\otimes m}:\rangle\mu(dx)=\delta_{nm}n!(f_n,g_m)\ , \tag{1.8}$$

其中 $f_n\in E^{\widehat{\otimes}n}$, $g_m\in E^{\widehat{\otimes}m}$, H_n 为 n- 阶 Hermite 多项式.

证明 由 (1.3) 及 (A.3) 立得 (1.7). 令 $E^{\circ n}$ 表示由 $\{f^{\otimes n},f\in E\}$ 生成的线性空间. 由极化公式的一个推论 (第一章 (2.12)) 知 $E^{\circ n}$ 在 $E^{\widehat{\otimes}n}$ 中稠. 由 (1.7) 及 (A.2) 得

$$\exp\{\langle f,x\rangle-\tfrac{1}{2}|f|^2\}=\textstyle\sum_{n=0}^{\infty}\frac{1}{n!}\langle f^{\otimes n},:x^{\otimes n}:\rangle.$$

于是有

$$
\begin{aligned}
&\int \sum_{n=0}^{\infty} \frac{u^n}{n!}\langle f^{\otimes n}, : x^{\otimes n} :\rangle \sum_{m=0}^{\infty} \frac{v^m}{m!}\langle g^{\otimes m}, : x^{\otimes m} :\rangle \mu(dx) \\
&= \int \exp\left[\langle uf + vg, x\rangle\right]\mu(dx) \exp\{-\tfrac{1}{2}(u^2|f|^2 + v^2|g|^2)\} \\
&= \exp\{uv\langle f, g\rangle\} \\
&= \sum_{n=0}^{\infty} \frac{u^n v^n}{n!}\langle f, g\rangle^n.
\end{aligned}
$$

由此推知，对 $f_n = f^{\otimes n}$ 及 $g_m = g^{\otimes m}$ 这一特殊情形，(1.8) 成立．于是对 $f_n \in E^{\circ n}$ 及 $g_m \in E^{\circ m}$, (1.8) 也成立．这表明：作为 $H^{\widehat{\otimes} n}$ 的稠子空间 $E^{\circ n}$ 到 $L^2(E^*, \mu)$ 中的线性等距映射，$f_n \mapsto (n!)^{-1/2}\langle f_n, : x^{\otimes n} :\rangle$ 可以唯一地扩张成为 $H^{\widehat{\otimes} n}$ 到 $L^2(E^*, \mu)$ 中的线性等距映射．特别 (1.8) 对 $f_n \in E^{\widehat{\otimes} n}$ 及 $g_m \in E^{\widehat{\otimes} m}$ 成立． ∎

注 今后我们用 $I_n(f_n)$ 表示映射 $f_n \mapsto \langle f_n, : x^{\otimes n} :\rangle$ 在 $H^{\widehat{\otimes} n}$ 上的连续延拓，它是多重 Wiener-Itô 积分的类似物．由引理 1.2 的证明知：

$$
I_n(f^{\otimes n}) = |f|^n H_n(|f|^{-1} W_f) \ , \ f \in H \ , \tag{1.9}
$$

$$
\int_{E^*} I_n(f_n) I_m(g_m) d\mu = \delta_{nm} n! (f_n, g_m) \ , \ f_n \in H^{\widehat{\otimes} n}, g_m \in H^{\widehat{\otimes} m} . \tag{1.10}
$$

作为几乎处处定义的随机变量，我们有时也用 $\langle f_n, : x^{\otimes n} :\rangle$ 形式地表示 $I_n(f_n)(x)$. 但这时 $\langle \cdot, \cdot \rangle$ 不再是 $E^{\otimes n} \times E^{*\otimes n}$ 上的典则双线性型的记号了．

下一定理建立了 $L^2(E^*, \mu)$ 与 H 上的对称 Fock 空间 $\Gamma(H)$ 之间的同构 (**Wiener-Itô-Segal 同构**).

定理 1.3 设 $\varphi \in L^2(E^*, \mu)$, 则存在唯一的 $\{f_n\}_{n \in \boldsymbol{N}_0} \in \Gamma(H)$ 使得

$$
\varphi = \sum_{n=0}^{\infty} I_n(f_n) \ , \tag{1.11}
$$

这里级数是在 L^2 意义下收敛. 此外有

$$\|\varphi\|^2 = \sum_{n=1}^{\infty} n!|f_n|^2 . \tag{1.12}$$

反之, 设 $\{f_n\}_{n\in \mathbb{N}_0} \in \Gamma(H)$, 则 (1.11) 定义了 $L^2(E^*,\mu)$ 中一元素.

今后, 我们常用 $\varphi \sim \{f_n\}$ 表示由 (1.11) 确定的 $L^2(E^*,\mu)$ 与 $\Gamma(H)$ 的一一对应关系.

证明 令

$$L^2(E^*,\mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{1.13}$$

为 $L^2(E^*,\mu)$ 的 Wiener-Itô 分解 (见第二章定理 1.5). 又令 $\mathcal{G}_n = \{I_n(f_n), f_n \in H^{\widehat{\otimes} n}\}$. 由 (1.9) 及 (1.10) 看出, 我们有

$$\bigoplus_{n=0}^{k} \mathcal{H}_n = \mathcal{P}_k = \bigoplus_{n=0}^{k} \mathcal{G}_n , \quad \forall k \geq 0,$$

其中 $\mathcal{P}_k$ 为 $\{W_f^n, 0 \leq n \leq k, f \in H\}$ 生成的 $L^2(E^*,\mu)$ 的闭子空间. 因此实际有 $\mathcal{H}_n = \mathcal{G}_n , \forall n \geq 0$. 由此立刻推得定理的结论. ∎

注 令 $\{e_j\}_{j\in \mathbb{N}}$ 为 H 的一个基, 对 $\alpha \in \Lambda$ (Λ 为至多有限项不为零的非负整数序列全体), 令 $\widehat{e}_\alpha$ 如第一章 (2.13) 定义. 由于 $|\widehat{e}_\alpha|^2 = \alpha!/|\alpha|!$, 故 $\{(\alpha!)^{-1/2} I_n(\widehat{e}_\alpha), \alpha \in \Lambda_n\}$ 构成 $\mathcal{H}_n$ 的基, 其中 $\Lambda_n = \{\alpha \in \Lambda : |\alpha| = n\}$. 与第二章 (1.46) 类似可证: $\forall \alpha \in \Lambda$,

$$\prod_j H_{\alpha_j}(We_j) = I_{|\alpha|}(\widehat{e}_\alpha) . \tag{1.14}$$

1.2 检验泛函与广义泛函空间

设 $E \hookrightarrow H \hookrightarrow E^*$ 为一 Gel′fand 三元组. 由可列 Hilbert 核空间的定义知, 存在 E 上一列单调非降相容的 Hilbert 范数 $\{|\cdot|_p\}_{p\in \mathbb{N}}$,

使得 E 为 $\{H_p\}_{p\in\mathbb{N}}$ 的投影极限, H_p 为 E 关于范数 $|\cdot|_p$ 的完备化, 且对一切 $p\in\mathbb{N}$, 存在 $p'>p$ 使得从 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 Hilbert-Schmidt 算子. 此外, 我们可要求 $|\cdot|_p\geq|\cdot|$ ($|\cdot|$ 为 H 中的范数). 今后, 我们称满足上述条件的 Hilbert 范数序列 $\{|\cdot|_p\}$ 为 E 上的 **标准范数列**, 并约定 $H_0=H$, $|\cdot|_0=|\cdot|$.

我们用 (L^2) 简记 $L^2(E^*,\mu)$. 下面我们将借助于 E 上的标准范数列通过 "二次量子化" 方法构造 (L^2) 的一族稠子空间 $(E)^\beta$, $\beta\geq 0$, 使得每个 $(E)^\beta$ 为可列 Hilbert 核空间, 且 $(E)^\beta$ 连续嵌入到 (L^2) 中. 我们将称 $(E)^\beta$ 为检验泛函空间, 称它的对偶为广义泛函空间.

设 $\{|\cdot|_p\}$ 为 E 上的一标准范数列, H_p 为 E 关于范数 $|\cdot|_p$ 的完备化. 对 $p,q\in\mathbb{N}$, $\beta\geq 0$, 令

$$\|\varphi\|_{p,q,\beta}^2\equiv\sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nq}|f_n|_p^2,\ \varphi\sim\{f_n\}, f_n\in H_p^{\widehat{\otimes}n}, \tag{1.15}$$

$$(H_{p,q,\beta})\equiv\left\{\varphi\in(L^2):\ \|\varphi\|_{p,q,\beta}<\infty\right\}. \tag{1.16}$$

又令 $(E)^\beta$ 为 $\{(H_{p,q,\beta}), p,q\in\mathbb{N}\}$ 的投影极限, 即

$$(E)^\beta=\bigcap_{p,q\geq 1}(H_{p,q,\beta}), \tag{1.17}$$

并赋予 $(E)^\beta$ 投影极限拓扑.

定理 1.4 $(E)^\beta$ 为可列 Hilbert 核空间, 且连续、稠密地嵌入到 (L^2) 中. 此外, 空间 $(E)^\beta$ 及其拓扑不依赖于 E 上的标准范数列的选取.

证明 给定 $p,q\in\mathbb{N}$. 令 $p'>p$ 使得由 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 Hilbert-Schmidt 算子, 且令 $q'>q$ 使得

$$\sum_{n=0}^{\infty}2^{n(q-q')}\|I_{pp'}\|_{\mathrm{HS}}^{2n}<\infty. \tag{1.18}$$

用 I 表示由 $(H_{p',q',\beta})$ 到 $(H_{p,q,\beta})$ 的嵌入映射. 往证 I 为 Hilbert-Schmidt 算子. 为此, 设 $\{e_i\}_{i\in\mathbb{N}}$ 及 $\{e'_i\}_{i\in\mathbb{N}}$ 分别为 H_p 及 $H_{p'}$ 的基. 对 $\alpha\in\Lambda$, 令

$$\varphi_\alpha=(|\alpha|!)^{-\beta/2}2^{-|\alpha|q/2}(\alpha!)^{-1/2}I_{|\alpha|}(\widehat{e}_\alpha), \tag{1.19}$$

类似定义 φ'_α, 则 $\{\varphi_\alpha,\alpha\in\Lambda\}$ 为 $(H_{p,q,\beta})$ 的基, $\{\varphi'_\alpha,\alpha\in\Lambda\}$ 为 $(H_{p',q',\beta})$ 的基. 于是我们有

$$\begin{aligned}\|I\|^2_{\mathrm{HS}}&=\sum_{\alpha\in\Lambda}\|\varphi'_\alpha\|^2_{p,q,\beta}\\&=\sum_{\alpha,\sigma\in\Lambda}(\varphi'_\alpha,\varphi_\sigma)^2_{p,q,\beta}\\&=\sum_{n=0}^{\infty}\sum_{\alpha,\sigma\in\Lambda_n}2^{n(q-q')}(f'_\alpha,f_\sigma)^2_p\\&\le\sum_{n=0}^{\infty}2^{n(q-q')}\|I^{\otimes n}_{pp'}\|^2_{\mathrm{HS}}\\&=\sum_{n=0}^{\infty}2^{n(q-q')}\|I_{pp'}\|^{2n}_{\mathrm{HS}}<\infty.\end{aligned}$$

因此 $(E)^\beta$ 为可列 Hilbert 核空间. 令

$$\mathcal{P}(E^*)=\Big\{\sum_{n=0}^{m}I_n(f_n),\,f_n\in E^{\widehat{\otimes}n},m\in\mathbb{N}\Big\}.$$

显然 $\mathcal{P}(E^*)\subset(E)^\beta$, $\mathcal{P}(E^*)$ 在 (L^2) 中稠, 故 $(E)^\beta$ 连续稠密嵌入到 (L^2) 中.

最后, 我们证明空间 $(E)^\beta$ 及其上拓扑不依赖 E 上的标准范数列的选取. 为此, 设 $\{|\cdot|'_k\}$ 为 E 上的另一标准范数列. 令 (H'_p) 为 E 关于范数 $|\cdot|'_p$ 的完备化, $(E)'^\beta=\bigcap_{k,l\ge1}(H'_{k,l,\beta})$, 并赋于 $(E)'^\beta$ 投影极限拓扑. 为证 $(E)^\beta=(E)'^\beta$ 及两者有相同拓扑, 只需证明: 对任意给定的 $k,l\in\mathbb{N}$, 存在 $p,q\in\mathbb{N}$, 使得 $\forall\varphi\in\mathcal{P}(E^*)$

有 $\|\varphi\|'^2_{k,l,\beta} \leq \|\varphi\|^2_{p,q,\beta}$. 为此，首先取 $p \in \mathbb{N}$ 及常数 $c > 0$, 使得 $\forall f \in E$ 有 $|f|'_k \leq c|f|_p$(因为 $|\cdot|'_k$ 关于 E 的拓扑连续，这样的 p 及 c 存在), 然后选取 q 使 $2^{(q-l)/2} \geq c$, 则对 $\varphi \in \mathcal{P}(E^*)$, $\varphi = \sum_{n=0}^N I_n(f_n)$,

$$\begin{aligned}\|\varphi\|'^2_{k,l,\beta} &= \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nl}|f_n|'^2_k \\ &\leq \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nl}c^{2n}|f_n|^2_p \\ &\leq \|\varphi\|^2_{p,q,\beta}\ .\end{aligned}$$

定理证毕. ∎

注 1 从定理 1.4 的证明容易看出，我们可以用更一般的双指标序列 $\{C_{q,n}\}$ 构造 E 的二次量子化空间. 事实上，设 $\{C_{q,n}, q \geq 1, n \geq 1\}$ 为一正实数序列，满足如下条件： (i) 对固定 n, 序列关于 q 单调非降； (ii) 对任给 $q \in \mathbb{N}, K > 0$, 存在 $q' > q$ 使得 $\sum_n C_{q,n}C_{q',n}^{-1}K^n < \infty$. 我们用 $C_{q,n}$ 代替 (1.16) 中的 $(n!)^\beta 2^{nq}$ 构造空间 $\mathcal{G}_{p,q}$, 并令 $\mathcal{G}$ 为 $\{\mathcal{G}_{p,q}\}$ 的投影极限. 则 $\mathcal{G}$ 仍为在 (L^2) 中稠的可列 Hilbert 核空间. 进一步，我们在满足上述条件 (i) 及 (ii) 的双指标序列中定义如下的等价关系： $\{C_{q,n}\} \sim \{C'_{k,n}\}$, 当且仅当 $\forall K > 0$, $q \in \mathbb{N}$, $k \in \mathbb{N}$, 存在 $p \in \mathbb{N}$, $l \in \mathbb{N}$, 使得

$$C_{q,n}{C'}^{-1}_{l,n} \leq K^n\ , \quad C'_{k,n}C^{-1}_{p,n} \leq K^n\ .$$

则用两个等价的序列构造 E 的二次量子化空间是相同的. 特别, (1.16) 中的 2^{nq} 可用 c^{nq} 代替，只要 $c > 1$.

注 2 如果存在常数 $0 < c < 1$, 使得 $|\cdot|_p \leq c|\cdot|_{p+1}$, $\forall p \in \mathbb{N}_0$, 则有

$$|f_n|_p \leq c^n|f_n|_{p+1},\ \forall n \geq 1,\ f_n \in E^{\widehat{\otimes} n}.$$

这时， $(E)^\beta$ 为 $\{H_{p,0,\beta}, p \in \mathbb{N}\}$ 的投影极限.

注 3 如果 Λ 为一定向集，E 为 $\{H_\lambda, \lambda\in\Lambda\}$ 的投影极限，则仍可由 (1.16) 定义 Hilbert 空间 $(H_{\lambda,q,\beta})$. 这时 $(E)^\beta$ 为 $\{H_{\lambda,q,\beta}, \lambda\in\Lambda, q\in\mathbb{N}\}$ 的投影极限.

对 $\beta\geq 0$, 我们用 $(E)^{-\beta}$ 表示 $(E)^\beta$ 的对偶. 对 $\beta=0$, 我们也用 (E) 表示 $(E)^0$, 并用 $(E)^*$ 表示 $(E)^{-0}$. 我们称 $(E)^\beta$ 为 **检验泛函空间**, 称 $(E)^{-\beta}$ 为 **广义泛函空间**. 由于对 $E=\mathcal{S}(\mathbb{R}^d)$ ($\mathbb{R}^d$ 上的 Schwartz 速降 C^∞ 函数空间) 这一特殊情形，(E) 及 $(E)^*$ 分别是经典白噪声分析中的 Hida 检验泛函及广义泛函空间，我们通常也把一般情形下的 (E) 及 $(E)^*$ 称为 **Hida 检验泛函空间** 及 **Hida 广义泛函空间**.

下面我们将给出 $(E)^{-\beta}$ 的具体构造. 设 H_{-p} 为 H_p 的对偶 (H 与它的对偶视为同一). 对 $p,q\in\mathbb{N}, \beta\geq 0$ 定义 $(H_{-p,-q,-\beta})$ 如下：

$$\|F\|^2_{-p,-q,-\beta}\equiv\sum_{n=0}^\infty (n!)^{1-\beta}2^{-nq}|g_n|^2_{-p},\ F\sim\{g_n\}, g_n\in H_{-p}^{\widehat{\otimes}n}, \tag{1.20}$$

$$(H_{-p,-q,-\beta})\equiv\left\{F:\|F\|_{-p,-q,-\beta}<\infty\right\}. \tag{1.21}$$

则 $(H_{-p,-q,-\beta})$ 为 $(H_{p,q,\beta})$ 的对偶，$(H_{p,q,\beta})\times(H_{-p,-q,-\beta})$ 上的典则双线性型 $\langle\langle\cdot,\cdot\rangle\rangle$ 为

$$\langle\langle\varphi,F\rangle\rangle=\sum_{n=0}^\infty n!\langle f_n,g_n\rangle\ , \tag{1.22}$$

其中 $\varphi\in(H_{p,q,\beta})$, $F\in(H_{-p,-q,-\beta})$, $\varphi\sim\{f_n\}$, $F\sim\{g_n\}$, $\langle\cdot,\cdot\rangle$ 为 $H_p^{\widehat{\otimes}n}\times H_{-p}^{\widehat{\otimes}n}$ 上的典则双线性型.

下一定理是核空间对偶一般理论的直接推论.

定理 1.5 令 $(E)^{-\beta}$ 为 Hilbert 空间序列 $\{H_{-p,-q,-\beta}, p,q\in\mathbb{N}\}$ 的归纳极限：

$$(E)^{-\beta}=\bigcup_{p,q\geq 1}(H_{-p,-q,-\beta})\ ,$$

则 $(E)^{-\beta}$ 为 $(E)^{\beta}$ 的对偶， $(E)^{-\beta}\times(E)^{\beta}$ 上的典则双线性型由 (1.22) 给出，其中 $\varphi\in(E)^{\beta}$, $F\in(E)^{-\beta}$, $\varphi\sim\{f_n\}$, $F\sim\{g_n\}$, $\langle\cdot,\cdot\rangle$ 为 $E\times E^*$ 上的典则双线性型.

注 今后, $\forall p\in\mathbb{Z}$, $q,r\in\mathbb{R}$, 我们定义如下的 Hilbert 空间:

$$
\begin{aligned}
(H_{p,q,r})=\{F\sim\{g_n\}:\ & g_n\in H_p^{\widehat{\otimes}n},\forall n\in\mathbb{N}_0,\\
&\|\varphi\|_{p,q,r}^2\equiv\sum_{n=0}^{\infty}(n!)^{1+r}2^{nq}|g_n|_p^2<\infty\}.
\end{aligned}\tag{1.23}
$$

显然， $(H_{p,q,r})$ 是 $(E)^{-|r|}$ 的稠子空间，且从 $(H_{p,q,r})$ 到 $(E)^{-|r|}$ 中的嵌入映射连续. 此外， $(H_{p,q,r})$ 与 $(H_{-p,-q,-r})$ 互为对偶.

令 $E\hookrightarrow H\hookrightarrow E^*$ 为一 Gel'fand 三元组. 我们也可按第二章在 $(E^*,\mathcal{B}(E^*),\mu)$ 上构造 Meyer-Watanabe 检验及广义泛函空间 $\mathbb{D}^{\infty}$ 及 $\mathbb{D}^{-\infty}$. 下一定理表明 Hida 检验泛函空间 (E) 比 Meyer-Watanabe 检验泛函空间 $\mathbb{D}^{\infty}$ 小.

定理 1.6 $(E)\subset\mathbb{D}^{\infty}$, (E) 在 $\mathbb{D}^{\infty}$ 中稠，且嵌入映射连续.

证明 令 $\mathcal{P}$ 表示 E^* 上 **多项式光滑泛函** 全体，即 $\mathcal{P}$ 中元素 φ 有如下形式:

$$
\varphi(x)=f(W_{\xi_1}(x),\cdots,W_{\xi_n}(x)),\ \xi_1,\cdots,\xi_n\in E,\ n\in\mathbb{N},
$$

其中 f 为 $\mathbb{R}^n$ 上的多项式. 显然有 $\mathcal{P}\subset\mathbb{D}^{\infty}\cap(E)$, 且 $\mathcal{P}$ 在 $\mathbb{D}^{\infty}$ 及 (E) 中稠. 令 $\mathcal{L}$ 为 OU 算子. 为证定理，只需证明: $\forall k\geq1,r\geq2$, 存在 $q>0$, 使得

$$
\|(I-\mathcal{L})^k\varphi\|_{L^r}\leq\|\varphi\|_{0,q,0},\ \forall\varphi\in\mathcal{P}.\tag{1.24}
$$

下面证明 (1.24). 设 $\varphi\in\mathcal{P},\varphi\sim\{f_n\}$, 令 $t=\frac{1}{2}\log(r-1)$, 则由半群 $T_t=e^{t\mathcal{L}}$ 的超压缩性 (第二章 §2 定理 3.5) 有

$$
\begin{aligned}
\|(I-\mathcal{L})^k\varphi\|_{L^r}&\leq\|e^{-t\mathcal{L}}(I-\mathcal{L})^k\varphi\|_{L^2}\\
&=\|\sum_n e^{tn}(1+n)^kI_n(f_n)\|_{L^2}\\
&\leq\|\sum_n e^{(t+k)n}I_n(f_n)\|_{L^2}\\
&=\|\varphi\|_{0,q,0},
\end{aligned}
$$

其中 q 满足 $2^{q/2}=e^{t+k}$. 定理证毕. ∎

系 1.7 $\mathbb{D}^{-\infty}\subset(E)^*$,$\mathbb{D}^{-\infty}$ 在 $(E)^*$ 中稠, 且嵌入映射连续.

1.3 经典的白噪声分析框架

在许多实际问题中, 我们的 Gel'fand 三元组 $E\hookrightarrow H\hookrightarrow E^*$ 及 E 上的标准范数列是由 H 中的一个自共轭算子 A 按如下方式生成的: 设 H 为一实可分 Hilbert 空间, 其范数为 $|\cdot|_0$,A 为 H 中的正自共轭算子, 满足如下条件: 1)$\|A^{-1}\|<1$; 2) 存在 $p_0>0$ 使得 $\|A^{-p_0}\|_{\mathrm{HS}}<\infty$. 对 $p\geq 1$, 令 $H_p=\mathcal{D}(A^p)$, 并在 H_p 上定义范数: $|x|_p=|A^px|_0$. 设 $p'>p\geq 0$, 由于 $\|A^{-1}\|<1$, 我们有 $H_{p'}\hookrightarrow H_p$, 且 $|\cdot|_p\leq|\cdot|_{p'}$. 令 $I_{p,p'}$ 表示 $H_{p'}$ 到 H_p 的嵌入映射, 则显然有

$$\|I_{p,p+p_0}\|_{\mathrm{HS}}^2=\|A^{-p_0}\|_{\mathrm{HS}}^2<\infty.$$

于是, 若令 E 为 $\{H_p,|\cdot|_p\}$ 的投影极限, 则 E 为可列 Hilbert 核空间, 且 $\{|\cdot|_p,p\geq 0\}$ 为 E 上的标准范数列. 令 E^* 为 E 的对偶空间 (将 H 的对偶 H^* 与 H 等同起来), 则 $E\hookrightarrow H\hookrightarrow E^*$ 为一 Gel'fand 三元组. 今后称这一三元组是**由 (H,A) 生成的**, 称上述的 $\{|\cdot|_p,p\geq 0\}$ 为由 A 决定的标准范数列. 一个典型的例子是: $\mathcal{S}(\mathbb{R})\hookrightarrow L^2(\mathbb{R})\hookrightarrow\mathcal{S}^*(\mathbb{R})$, 它是由 $L^2(\mathbb{R})$ 及其上的谐振子 $A=-\frac{d^2}{dt^2}+t^2+1$ 生成的 (见第一章 3.2 节的例子).

下面我们将借助于算子 A 给出 Hida 检验和广义泛函空间 (E) 及 $(E)^*$ 的一个更加直接的构造. 首先我们回忆在第一章 §2 中给出的算子 A 的二次量子化 $\Gamma(A)$ 的定义. 那里算子 $\Gamma(A)$ 是定义在 H 的对称 Fock 空间中的, 但由于 Fock 空间 $\Gamma(H)$ 与空间 (L^2) 同构 (定理 1.3), 所以我们可以把 $\Gamma(A)$ 看成是 (L^2) 中的一个正自共轭算子. 设 $\varphi\in\mathcal{D}(\Gamma(A))$, $\varphi\sim\{f_n\}$, 则 $\Gamma(A)\varphi\sim\{A^{\otimes n}f_n\}$. 更一般地, 设 $p\geq 0,\varphi\in\mathcal{D}(\Gamma(A)^p),\varphi\sim\{f_n\}$, 我们令 $\Gamma(A)^p\varphi\sim\{(A^p)^{\otimes n}f_n\}$. 由于

$$|f_n|_p=|(A^p)^{\otimes n}f_n|_0\ ,\ f_n\in\mathcal{D}((A^p)^{\otimes n}),\ p\geq 0,$$

故 (1.16) 中定义的 $H_{p,0,0}$ 正是 $\Gamma(A)^p$ 的定义域 $\mathcal{D}(\Gamma(A)^p)$. 今后，我们用 $(E)_p$ 简记 $\mathcal{D}(\Gamma(A)^p)$, 其上的范数 $\|\cdot\|_p$ 定义为

$$\|\varphi\|_p = \|\Gamma(A)^p\varphi\|_0 .$$

我们用 $(E)_{-p}$ 表示 $(E)_p$ 的对偶，$\|\cdot\|_{-p}$ 为 $(E)_{-p}$ 上的对偶范数.

另一方面，由于 $\|A^{-1}\| < 1$, 我们有

$$|f_n|_p \leq \|A^{-1}\|^n |f_n|_{p+1}, \quad n \in \mathbb{N}_0, f_n \in E^{\widehat{\otimes} n}, p \geq 0. \tag{1.25}$$

这表明定理 1.4 下面的注 2 中的条件成立. 因此，(E) 为 $\{(E)_p, p \in \mathbb{N}_0\}$ 的投影极限，$(E)^*$ 为 $\{(E)_{-p}, p \in \mathbb{N}_0\}$ 的归纳极限.

注 可以证明 $\Gamma(A)^{-p_0}$ 是 (L^2) 中的 Hilbert-Schmidt 算子.

§2. 泛函空间的刻画

本节将沿用上一节的记号. 今后，对任一实数域 $\mathbb{R}$ 上的线性拓扑空间 K, 我们用 $K_{\mathbb{C}}$ 表示它的**复化**, 即令 $K_{\mathbb{C}} = K + iK$. 若 K 为 Hilbert 空间，则对 $f_1, f_2, g_1, g_2 \in K$. $f_1 + ig_1$ 与 $f_2 + ig_2$ 在 $K_{\mathbb{C}}$ 中的内积 $(\cdot,\cdot)$ 为

$$(f_1 + ig_1, f_2 + ig_2) \equiv (f_1, f_2) + (g_1, g_2) + i[(f_2, g_1) - (f_1, g_2)] . \tag{2.1}$$

我们仍用 $|\cdot|_p$ 表示 H_p 的复化空间 $H_{p,\mathbb{C}}$ 及 $H_{p,\mathbb{C}}^{\otimes n}$ 上的范数.

今后我们将经常用到 $K_{\mathbb{C}} \times K_{\mathbb{C}}$ 上如下定义的双线性型:

$$\langle f_1 + ig_1, f_2 + ig_2 \rangle \equiv (f_1, f_2) - (g_1, g_2) - i[(f_2, g_1) + (f_1, g_2)] . \tag{2.2}$$

它与 $K_{\mathbb{C}}$ 上的内积之间有如下关系:

$$\langle F, G \rangle = (F, \overline{G}) , \quad F, G \in K_{\mathbb{C}}. \tag{2.3}$$

设 $f_n \in (H^{\widehat{\otimes} n})_{\mathbb{C}} (\cong (H_{\mathbb{C}})^{\widehat{\otimes} n})$, $f_n = g_n + ih_n$, $g_n, h_n \in H^{\widehat{\otimes} n}$. 令

$$I_n(f_n) \equiv I_n(g_n) + iI_n(h_n) ,$$

则 $(L^2)_{\mathbb{C}}$ 与 $\Gamma(H_{\mathbb{C}})$ 同构. 我们仍用 $\varphi \sim \{f_n\}$ 简记由 (1.11) 确定的同构关系.

本节将给出空间 $(E)_{\mathbb{C}}^{-\beta}(0 \le \beta \le 1)$ 及 $(E)_{\mathbb{C}}^{\beta}(0 \le \beta < \infty)$ 的刻画, 关于空间 $(E)_{\mathbb{C}}^{-\beta}(1 < \beta < \infty)$ 的刻画则要推迟到 §4 给出.

2.1 S- 变换与空间 $(E)_{\mathbb{C}}^{-\beta}(0 \le \beta < 1)$ 的刻画

下一引理推广了 (1.9).

引理 2.1 我们有

$$I_n(f^{\otimes n}) = \langle f, f\rangle^{n/2} H_n(\langle f, f\rangle^{-1/2} W_f)\ ,\ f \in H_{\mathbb{C}}. \tag{2.4}$$

证明 为证 (2.4), 不妨设 $f \in E_{\mathbb{C}}$. 令 $f = g + ih, g, h \in E$. 则 $\forall k \ge 1$

$$\begin{aligned}\langle f^{\otimes 2k}, \tau^{\widehat{\otimes} k}\rangle &= \langle f^{\otimes 2}, \tau\rangle^k = \langle (g+ih)^{\otimes 2}, \tau\rangle^k \\ &= \langle g^{\otimes 2} - h^{\otimes 2} + 2ig\widehat{\otimes} h, \tau\rangle^k \\ &= \Big((g,g) - (h,h) + 2i(h,g)\Big)^k \\ &= \langle f, f\rangle^k.\end{aligned}$$

由此利用 (1.3) 及 (A.3) 立得 (2.4). ∎

令 $f \in H_{\mathbb{C}}$, $\mathcal{E}_f$ 为与 f 联系的指数泛函:

$$\mathcal{E}_f = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes n})\ . \tag{2.5}$$

由 (2.4) 及 (A.2) 推得

$$\mathcal{E}_f = \exp\{W_f - \tfrac{1}{2}\langle f, f\rangle\}\ . \tag{2.6}$$

引理 2.2 设 $f \in E_{\mathbb{C}}$, 则 $\forall \beta \in [0,1)$, $\mathcal{E}_f \in (E)_{\mathbb{C}}^{\beta}$.

证明 对 $p, q \ge 1$, 我们有

$$\|\mathcal{E}_f\|_{p,q,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{\beta-1} 2^{nq} |f|_p^{2n} < \infty\ .$$

于是依定义，$\mathcal{E}_f \in (E)^{\beta}_{\mathbb{C}}$ ∎

定义 2.3 设 $0 \le \beta < 1$, $\Phi \in (E)^{-\beta}_{\mathbb{C}}$. 令

$$S\Phi(f) = \langle\langle \Phi, \mathcal{E}_f \rangle\rangle \ , \quad f \in E_{\mathbb{C}} \ .$$

我们称 $S\Phi$ 在 E 上的限制为 Φ 的 **S- 变换**, 称 $S\Phi$ 本身为 Φ 的 **推广了的 S- 变换**.

若 $\Phi \in (L^2)_{\mathbb{C}}$, 则由 Cameron-Martin 定理知

$$S\Phi(f) = \mathbb{E}[\Phi \mathcal{E}_f] = \int_{E^*} \Phi(x+f)\mu(dx) \ , \quad f \in E \ . \tag{2.7}$$

设 $\Phi \sim \{f_n\}$, 则由 (1.21) 得

$$S\Phi(f) = \sum_{n=0}^{\infty} \langle f_n, f^{\otimes n} \rangle \ , \quad f \in E_{\mathbb{C}} \ . \tag{2.8}$$

由第一章命题 2.14 易知，指数泛函族 $\{\mathcal{E}_f, f \in E\}$ 生成的复域 $\mathbb{C}$ 上的线性空间在 $(E)^{\beta}_{\mathbb{C}}$ 中稠，故 $(E)^{-\beta}_{\mathbb{C}}$ 中的元素被它的 S- 变换唯一决定. 一个自然的问题是：对 $0 \le \beta < 1$, 如何用 S- 变换刻画空间 $(E)^{-\beta}_{\mathbb{C}}$? 为了回答这一问题，我们首先介绍一些有关局部凸空间中的复分析的已知结果 (参看 Dineen[1]).

我们用 $\mathcal{L}_s(E^n_{\mathbb{C}})$ 表示 $E^n_{\mathbb{C}}$ 到复数域 $\mathbb{C}$ 的对称 n- 线性型全体. 对任一 $L \in \mathcal{L}_s(E^n_{\mathbb{C}})$, 令

$$\widehat{L}(f) = L(f, \cdots, f) \ , \quad f \in E_{\mathbb{C}} \ , \tag{2.9}$$

称 $\widehat{L}$ 为 L 对应的 **n-齐次多项式**. 令 $\mathcal{P}_n(E_{\mathbb{C}})$ 表示 $E_{\mathbb{C}}$ 上的 n- 齐次多项式全体. 则由极化公式 (第一章 (2.11)) 知，$L \mapsto \widehat{L}$ 为 $\mathcal{L}_s(E^n_{\mathbb{C}})$ 到 $\mathcal{P}_n(E_{\mathbb{C}})$ 上的线性单射.

定义 2.4 设 U 为 $E_{\mathbb{C}}$ 的一非空开子集. U 上的一复值函数称为 **G- 全纯的**, 如果 $\forall \eta \in U, \forall \xi \in E_{\mathbb{C}}$, 映射 $\lambda \mapsto F(\eta + \lambda\xi)$ 在 $0 \in \mathbb{C}$ 的某个开邻域内为全纯函数. 在 U 上连续的 G- 全纯函数称为在

U 上**全纯的**. 在 $\xi_0 \in E_{\mathbb{C}}$ 的某个开邻域全纯的函数称为**在 ξ_0 处全纯**. 在全空间 $E_{\mathbb{C}}$ 上 G- 全纯 (全纯) 的函数称为 **G- 整解析函数 (整解析函数)**, 简称为 **G- 整函数 (整函数)**.

复分析的一个定理告诉我们: U 上局部有界的 G- 全纯函数连续, 从而在 U 上全纯.

我们用 $H_G(U)$ 及 $H(U)$ 分别表示 U 上 G- 全纯及全纯函数全体. 设 $F \in H_G(U)$ 对每个 $\eta \in U$, 存在唯一的对称 n- 线性型序列 $\{F_\eta^{(n)}, n \in \mathbb{N}\}$, $F_\eta^{(n)} \in \mathcal{L}_s(E_{\mathbb{C}}^n)$, 使得

$$F(\eta+\xi) = F(\eta) + \sum_{n=1}^{\infty} \frac{1}{n!} \widehat{F}_\eta^{(n)}(\xi) , \tag{2.10}$$

这里 ξ 属于 $E_{\mathbb{C}}$ 中原点的某个开邻域. 如果 $F \in H_G(E_{\mathbb{C}})$, 则 (2.10) 对一切 $\xi \in E_{\mathbb{C}}$ 成立.

下一引理是多复变函数论中的一个熟知结果.

引理 2.5 设 $n \in \mathbb{N}, n \geq 2$, f 为 $\mathbb{R}^n$ 上一复值函数. 假定对每个 $1 \leq k \leq n$ 及 $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$, 映射 $x_k \longmapsto f(x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n)$ 在 $\mathbb{C}$ 上有整解析延拓, 则 f 在 $\mathbb{C}^n$ 上有整解析延拓.

定义 2.6 设 F 为 E 上一复值函数, $0 \leq \beta < 1$. 如果 F 满足:

(C.1) 对一切 $f, g \in E$, 映射 $\lambda \longmapsto F(g+\lambda f)$ 在 $\mathbb{C}$ 上有整解析延拓;

(C.2) 存在常数 $C, K > 0$ 及 $p \in \mathbb{N}_0$, 使得

$$|F(zf)| \leq C \exp\{K|z|^{2/(1-\beta)} |f|_p^{2/(1-\beta)}\} , \forall f \in E, z \in \mathbb{C} , \tag{2.11}$$

则称 F 为 **U_β- 泛函**.

引理 2.7 设 $0 \leq \beta < 1$. 每个 U_β- 泛函 F 在 $E_{\mathbb{C}}$ 上有唯一整解析延拓. 此外, 若 F 满足 (2.11), 则对一切 $0 < \rho < 1, \xi \in E_{\mathbb{C}}$ 有

$$|F(\xi)| \leq C' \exp(K' |\xi|_p^{2/(1-\beta)}) , \tag{2.12}$$

其中 $C' = C(1-\rho)^{-\frac{1+\beta}{2}}$, $K' = (2e^2)^{\frac{1}{1-\beta}} K \rho^{-\frac{1+\beta}{1-\beta}}$.

证明 首先证明 F 在 $E_{\mathbb{C}}$ 上有唯一的 G- 整解析延拓. 为此, 我们仍用 F 表示定义 2.6 条件 (C.1) 中得到的 $E_{\mathbb{C}}$ 上的函数 $F(g_0+\lambda g_1)$, 其中 $g_0, g_1 \in E$, $\lambda \in \mathbb{C}$. 令 $g_2, g_3 \in E$, 考虑如下映射:

$$(x_1, x_2, x_3) \longmapsto F(g_0 + x_1 g_1 + x_2 g_2 + x_3 g_3)\ ,\ (x_1, x_2, x_3) \in \mathbb{R}^3\ .$$

由条件 (C.1) 及引理 2.5 知, 该函数在 $\mathbb{C}^3$ 上有整解析延拓. 特别, 这表明延拓所得的 F 为 G- 整函数. 显然, G- 整解析延拓是唯一的.

下面我们证明 (2.12), 这将蕴含 F 的局部有界性, 从而 F 为整函数. 由 (2.10) 得

$$F(\lambda\xi) = F(0) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \widehat{F}_0^{(n)}(\xi)\ . \tag{2.13}$$

我们有如下的 Cauchy 公式 (例如见 Nachbin[1]):

$$\frac{1}{n!}\widehat{F}_0^{(n)}(\xi) = \frac{1}{2\pi i}\int_{|z|=R} \frac{F(z\xi)}{z^{n+1}} dz\ . \tag{2.14}$$

令 $f \in E$, $|f|_p = 1$, $R = (\frac{n(1-\beta)}{2K})^{\frac{1-\beta}{2}}$, 则由 (2.14) 及 (2.11) 得

$$\begin{aligned}\Big|\frac{1}{n!}\widehat{F}_0^{(n)}(f)\Big| &\le C R^{-n} e^{K(R|f|_p)^{\frac{2}{1-\beta}}} \\ &= C\Big(\frac{2eK}{n(1-\beta)}\Big)^{\frac{n(1-\beta)}{2}}\ .\end{aligned} \tag{2.15}$$

故由 $\widehat{F}_0^{(n)}$ 的齐性知: $\forall f \in E$

$$\begin{aligned}\Big|\frac{1}{n!}\widehat{F}_0^{(n)}(f)\Big| &= |f|_p^n \Big|\frac{1}{n!}\widehat{F}_0^{(n)}(|f|_p^{-1} f)\Big| \\ &\le C\Big(\frac{2eK}{n(1-\beta)}\Big)^{\frac{n(1-\beta)}{2}} |f|_p^n\ .\end{aligned} \tag{2.16}$$

由此进一步利用极化公式 (第一章 (2.11)) 得: $\forall f_1,\cdots,f_n\in E$,

$$\Big|\frac{1}{n!}F_0^{(n)}(f_1,\cdots,f_n)\Big|\le C\frac{n^n}{n!}\Big(\frac{2eK}{n(1-\beta)}\Big)^{\frac{n(1-\beta)}{2}}\prod_{j=1}^n|f_j|_p\,. \tag{2.17}$$

由于 $F_0^{(n)}$ 为 $E_{\mathbb{C}}^n$ 上的对称 n- 线性型, 故由 (2.17) 推得 (利用不等式 $(x+y)^2\le 2(x^2+y^2)$): $\forall\xi_1,\cdots,\xi_n\in E_{\mathbb{C}}$,

$$\begin{aligned}\Big|\frac{1}{n!}F_0^{(n)}(\xi_1,\cdots,\xi_n)\Big| &\le C\frac{n^n}{n!}\Big(\frac{2eK}{n(1-\beta)}\Big)^{\frac{n(1-\beta)}{2}}2^{\frac{n}{2}}\prod_{j=1}^n|\xi_j|_p\\ &=C\Big(\frac{1}{n!}\Big)^{\frac{1-\beta}{2}}\Big(\frac{n^n}{n!}\Big)^{\frac{1+\beta}{2}}\Big(\frac{2eK}{1-\beta}\Big)^{\frac{n(1-\beta)}{2}}2^{\frac{n}{2}}\prod_{j=1}^n|\xi_j|_p\\ &\le C\Big(\frac{1}{n!}\Big)^{\frac{1-\beta}{2}}\Big[2e^2\Big(\frac{2K}{1-\beta}\Big)^{1-\beta}\Big]^{\frac{n}{2}}\prod_{j=1}^n|\xi_j|_p\,,\end{aligned} \tag{2.18}$$

这里我们用到了 $n^n/n!\le e^n$. 最后, 利用 Hölder 不等式 (取 $s=2/(1-\beta)$, $t=2/(1+\beta)$, 作为一对共轭指数), 由 (2.13) 及 (2.18) 立刻推得 (2.12). ∎

引理 2.8 设 $0\le\beta<1$, F 为一 U_β- 泛函, 满足 (2.11). 令 $p'>p$ 使得从 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 Hilbert-Schmidt 算子, 且令 $q\in\mathbb{R}$ 使得 $2^q>e^2(\frac{2K}{1-\beta})^{1-\beta}\|I_{pp'}\|_{\mathrm{HS}}^2$, 则存在唯一的 $\Phi\in(H_{-p',-q,-\beta})_{\mathbb{C}}$ 使得 F 为 Φ 的 S- 变换. 此外我们有

$$\|\Phi\|_{-p',-q,-\beta}\le C[1-2^{-q}e^2\Big(\frac{2K}{1-\beta}\Big)^{1-\beta}\|I_{pp'}\|_{\mathrm{HS}}^2]^{-1/2}\,. \tag{2.19}$$

证明 我们采用上一引理证明中的记号. 由 (2.18) 知, $\forall n\ge 1$, $F_0^{(n)}$ 关于它的每个变量 ξ_j 在 $E_{\mathbb{C}}$ 上连续. 故由核定理 (第一章定理 3.1 7), 存在 $g_n\in(E_{\mathbb{C}}^*)^{\widehat{\otimes}n}$ 使得

$$\langle g_n,\xi_1\widehat{\otimes}\cdots\widehat{\otimes}\xi_n\rangle=\frac{1}{n!}F_0^{(n)}(\xi_1,\cdots,\xi_n),\ \xi_1,\cdots,\xi_n\in E_{\mathbb{C}}^*\,. \tag{2.20}$$

令 $p' > p$ 使得 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 Hilbert-Schmidt 算子，并令 $\{e_k, k \in \mathbb{N}\}$ 为 $H_{p'}$ 的一个基，且 $e_k \in E, \forall k \in \mathbb{N}$. 由于 $\{e_{k_1} \otimes \cdots \otimes e_{k_n} : k_i \in \mathbb{N}, 1 \le i \le n\}$ 为 $H_{p',\mathbb{C}}^{\otimes n}$ 的基，我们有 (利用 (2.20),(2.17) 及 $n^n/n! \le e^n$)

$$
\begin{aligned}
|g_n|_{-p'}^2 &= \sum_{k_1,\cdots,k_n} \left|\langle g_n, e_{k_1} \otimes \cdots \otimes e_{k_n}\rangle\right|^2 \\
&= \sum_{k_1,\cdots,k_n} (n!)^{-2} \left|F_0^{(n)}(e_{k_1}, \cdots, e_{k_n})\right|^2 \\
&\le C^2 (n!)^{\beta-1} \left[e^2 \left(\frac{2K}{1-\beta}\right)^{1-\beta}\right]^n \left(\sum_{k=1}^{\infty} |I_{pp'} e_k|_p^2\right)^n \\
&= C^2 (n!)^{\beta-1} \left[e^2 \left(\frac{2K}{1-\beta}\right)^{1-\beta}\right]^n \|I_{pp'}\|_{\mathrm{HS}}^{2n} \, . \qquad (2.21)
\end{aligned}
$$

令 $g_0 = F(0)$, $\Phi \sim (g_n)$, 由 (2.21) 得 (2.19), 故 $\Phi \in (H_{-p',-q,-\beta})_{\mathbb{C}}$. 特别， $\Phi \in (E)_{\mathbb{C}}^{-\beta}$, 且对 $f \in E$, 由 (2.20) 及 (2.13) 得

$$
\begin{aligned}
S\Phi(f) &= \sum_{n=0}^{\infty} \langle g_n, f^{\otimes n}\rangle \\
&= F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \widehat{F}_0^{(n)}(f) = F(f) \, .
\end{aligned}
$$

这表明 F 为 Φ 的 S- 变换. ∎

下一定理通过 S- 变换刻画了 $(E)_{\mathbb{C}}^{-\beta}, 0 \le \beta < 1$.

定理 2.9 设 $0 \le \beta < 1$. 映射 $F: E \longrightarrow \mathbb{C}$ 为 $(E)_{\mathbb{C}}^{-\beta}$ 中一元素的 S- 变换，当且仅当 F 为一 U_β- 泛函.

证明 充分性由引理 2.8 得知，往证必要性. 设 $\Phi \in (E)_{\mathbb{C}}^{-\beta}$, 则存在 $p, q \in \mathbb{N}$, 使得 $\Phi \in (H_{-p,-q,-\beta})_{\mathbb{C}}$. 令 $F(f) = S\Phi(f)$, $f \in E$, 则 Φ 的推广了的 S- 变换 (仍记为 F) 就是 F 到 $E_{\mathbb{C}}$ 上的整解析延

拓. 设 $\Phi \sim \{g_n\}$, 则由 (2.8) 得

$$
\begin{aligned}
| F(zf) | &\leq \sum_{n=0}^{\infty} | \langle g_n, (zf)^{\otimes n} \rangle | \\
&\leq \sum_{n=0}^{\infty} |z|^n |g_n|_{-p} |f|_p^n \\
&\leq \Big(\sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |g_n|_{-p}^2 \Big)^{1/2} \Big(\sum_{n=0}^{\infty} (\frac{1}{n!})^{1-\beta} 2^{qn} |z|^{2n} |f|_p^{2n} \Big)^{1/2} \\
&= \|\Phi\|_{-p,-q,-\beta} \Big(\sum_{n=0}^{\infty} (\frac{1}{n!})^{1-\beta} 2^{qn} |z|^{2n} f|_p^{2n} \Big)^{1/2}. \qquad (2.22)
\end{aligned}
$$

令 $0 < \rho < 1$. 由 Hölder 不等式 (以 $1/\beta$ 与 $1/(1-\beta)$ 作为一对共轭指数) 得

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left(\frac{1}{n!} \right)^{1-\beta} 2^{qn} |z|^{2n} |f|_p^{2n} \\
&\leq \Big(\sum_{n=0}^{\infty} \rho^n \Big)^{\beta} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} 2^{\frac{qn}{1-\beta}} |z|^{\frac{2n}{1-\beta}} |f|_p^{\frac{2n}{1-\beta}} \rho^{-\frac{\beta n}{1-\beta}} \Big)^{1-\beta} \\
&= (1-\rho)^{-\beta} \exp\{(1-\beta) 2^{\frac{q}{1-\beta}} \rho^{-\frac{\beta}{1-\beta}} |z|^{\frac{2}{1-\beta}} |f|_p^{\frac{2}{1-\beta}}\}.
\end{aligned}
$$

这表明 F 满足定义 2.6 中条件 (C.2), 从而 F 为一 U_β- 泛函. ∎

注 今后我们用 $S^{-1}F$ 表示与 U_β- 泛函 F 对应的广义泛函.

下面两个定理是定理 2.9 的重要推论.

定理 2.10 设 $0 \leq \beta < 1$, $\{F_n, n \in \mathbb{N}\}$ 为一列 U_β- 泛函, 如果满足下列条件:

(1) $\forall f \in E$, $\{F_n(f), n \in \mathbb{N}\}$ 为 $\mathbb{C}$ 中 Cauchy 列;

(2) 存在 $p \in \mathbb{N}_0$ 及 $C, K > 0$ 使得

$$
| F_n(zf) | \leq C \exp K(|z||f|_p)^{\frac{2}{1-\beta}}, \ \forall f \in E, \forall z \in \mathbb{C}, \forall n \geq 1, \quad (2.23)
$$

则 $\{S^{-1}F_n, n \in \mathbb{N}\}$ 在 $(E)_{\mathbb{C}}^{-\beta}$ 中强收敛.

证明 由引理 2.8 知，存在 $p', q \in \mathbb{C}$, $\rho \in (0,1)$, 使得

$$\|S^{-1}F_n\|_{-p',-q,-\beta} \leq C(1-\rho)^{1/2} . \tag{2.24}$$

条件 (1) 意味着：$\forall f \in E$, 序列 $\{\langle\langle S^{-1}F_n, \mathcal{E}_f\rangle\rangle, n \in \mathbb{N}\}$ 为 $\mathbb{C}$ 中的 Cauchy 列. 但是，由 $\{\mathcal{E}_f, f \in E\}$ 张成的线性子空间在 $(H_{p',q,\beta})$ 中稠，故由 (2.24) 推知 $\{S^{-1}F_n, n \in \mathbb{N}\}$ 在 $(H_{-p',-q,-\beta})_{\mathbb{C}}$ 中收敛. ∎

注 1 由不等式 (2.22) 容易看出该定理的逆命题也成立.

注 2 由于 $(E)_{\mathbb{C}}^{-\beta}$ 中的序列强、弱收敛彼此等价 (见第一章定理 3.12), 该定理叙述中的强收敛可改为弱收敛.

定理 2.11 设 $0 \leq \beta < 1$,$(\Omega, \mathcal{F}, \nu)$ 为一测度空间，$\omega \longmapsto \Phi_\omega$ 为 Ω 到 $(E)_{\mathbb{C}}^{-\beta}$ 中的映射，$F_\omega = S\Phi_\omega$ 为 Φ_ω 的 S- 变换. 如果 F_ω 满足下列条件：

(1) $\forall f \in E$, $\omega \longmapsto F_\omega(f)$ 为 $(\Omega, \mathcal{F})$ 上的可测映射；

(2) 存在 $K > 0$, $p \in \mathbb{N}$ 及 Ω 上非负 ν- 可积函数 $C(\omega)$, 使得对 ν-a.e. ω, 有

$$| F_\omega(zf) | \leq C(\omega) \exp K(|z||f|_p)^{\frac{2}{1-\beta}}, \quad \forall f \in E, \forall z \in \mathbb{C}, \tag{2.25}$$

则存在 $p', q \in \mathbb{N}$, 使得 $\omega \longmapsto \Phi_\omega$ 在 $(H_{-p',-q,-\beta})_{\mathbb{C}}$ 中 Bochner 可积，且有

$$S\left(\int_\Omega \Phi_\omega d\nu(\omega)\right)(f) = \int_\Omega S\Phi_\omega(f) d\nu(\omega) \ , \ \forall f \in E . \tag{2.26}$$

证明 由引理 2.8 知：存在 $p', q \in \mathbb{N}$, $\rho \in (0,1)$, 使得

$$\|\Phi_\omega\|_{-p',-q,-\beta} \leq C(\omega)(1-\rho)^{-1/2} \ , \ \nu\text{-a.e.}\, \omega . \tag{2.27}$$

但是条件 (1) 蕴含：$\forall f \in E$, $\omega \longmapsto \langle\langle \Phi_\omega, \mathcal{E}_f\rangle\rangle$ 为可测函数，从而 $\forall \varphi \in (H_{p',q,\beta})$, $\omega \longmapsto \langle\langle \Phi_\omega, \varphi\rangle\rangle$ 为可测函数，故由 (2.27) 及条件 (2) 知，作为 $(H_{-p',-q,-\beta})_{\mathbb{C}}$ 值泛函，$\omega \longmapsto \Phi_\omega$ 为 Bochner 可积，且有 (2.26)(因为 $SG(f) = \langle\langle G, \mathcal{E}_f\rangle\rangle$). ∎

2.2 局部 S- 变换与空间 $(E)_{\mathbb{C}}^{-1}$ 的刻画

现在转向 $(E)_{\mathbb{C}}^{-1}$ 的刻画. 设 $f \in E_{\mathbb{C}}$, 则当且仅当 $2^q|f|_p^2 < 1$ 时 $\mathcal{E}_f \in (H_{p,q,1})_{\mathbb{C}}$. 事实上

$$\|\mathcal{E}_f\|_{p,q,1}^2 = \sum_{n=0}^{\infty}(n!)^2 2^{nq}|(n!)^{-1}f^{\otimes n}|_p^2 = \sum_{n=0}^{\infty} 2^{nq}|f|_p^{2n} .$$

因此, 当且仅当 $f = 0$ 时才有 $\mathcal{E}_f \in (E)_{\mathbb{C}}^1$. 这样一来, 我们就不能象前面那样对 $(E)_{\mathbb{C}}^{-1}$ 的任何元素定义 S- 变换了. 但是, 由于

$$(E)_{\mathbb{C}}^{-1} = \bigcup_{p,q\in \mathbb{N}} (H_{-p,-q,-1})_{\mathbb{C}} ,$$

我们可以定义 $(E)_{\mathbb{C}}^{-1}$ 元素的"局部 S- 变换". 下面我们经常使用如下记号:

$$U_{p,q} = \{\xi \in E_{\mathbb{C}} : \ 2^q|\xi|_p^2 < 1\} , \quad p \in \mathbb{Z}, q \in \mathbb{R} .$$

定义 2.12 设 $\Phi \in (E)_{\mathbb{C}}^{-1}$, 令 $p, q \in \mathbb{N}$, 使得 $\Phi \in (H_{-p,-q,-1})_{\mathbb{C}}$. 假定 $\Phi \sim \{g_n, n \in \mathbb{N}_0\}$. 令

$$S\Phi(\xi) = \langle\langle \Phi, \mathcal{E}_\xi \rangle\rangle = \sum_{n=0}^{\infty}\langle g_n, \xi^{\otimes n}\rangle , \xi \in U_{p,q} .$$

显然 $S\Phi$ 为 U_{pq} 中的全纯函数, 我们称它为 Φ 的 **局部 S- 变换**.

今后我们用 $\mathrm{Hol}_0(E_{\mathbb{C}})$ 表示在 $0 \in E_{\mathbb{C}}$ 处全纯函数全体 (见定义 2.4). 于是 $\forall \Phi \in (E_{\mathbb{C}})^{-1}$, 有 $S\Phi \in \mathrm{Hol}_0(E_{\mathbb{C}})$. 设 $F_1, F_2 \in \mathrm{Hol}_0(F_{\mathbb{C}})$. 如果存在 $0 \in E_{\mathbb{C}}$ 的某个开邻域 U 使得 F_1 与 F_2 在 U 上一致, 则称 F_1 与 F_2 等价. 我们将用 $F_1 \sim F_2$ 表示这一等价关系.

下一定理通过局部 S- 变换刻画了 $(E)_{\mathbb{C}}^{-1}$.

定理 2.13 (1) 设 $\Phi \in (E)_{\mathbb{C}}^{-1}$, 则 $S\Phi \in \mathrm{Hol}_0(E_{\mathbb{C}})$.

(2) 设 $F \in \mathrm{Hol}_0(E_{\mathbb{C}})$, 则存在唯一的 $\Phi \in (E)_{\mathbb{C}}^{-1}$, 使得 $S\Phi \sim F$. 更确切地, 假定存在 $p, q \in \mathbb{N}$, $C > 0$, 使得 F 在 $U_{p,q}$ 上全纯, 且 $\forall \xi \in U_{p,q}$, $|F(\xi)| \leq C$. 令 $p' > p$ 使得 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 Hilbert-Schmidt 算子, 且令 $q' \in \mathbb{R}$ 使得 $\rho = 2^{-(q'+2q+2)}e^2\|I_{pp'}\|_{\mathrm{HS}}^2 < 1$, 则与 F 相应的 Φ 属于 $(H_{-p',-q',-1})_{\mathbb{C}}$, 且有如下范数估计:

$$\|\Phi\|_{-p',-q',-1} \leq C(1-\rho)^{-1/2} . \tag{2.28}$$

证明 容易验证 (1), 往证 (2). 设 F 满足定理中的条件. 则对满足 $|\xi|_p = 1$ 的 $\xi \in E_{\mathbb{C}}$ 及满足 $|z| < 2^{-q}$ 的 $z \in \mathbb{C}$, 有

$$F(z\xi) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \widehat{F}_0^{(n)}(\xi) . \tag{2.29}$$

由 Cauchy 公式 (2.14)(取 $R = 2^{-(q+1)}$) 得

$$\left| \frac{1}{n!} \widehat{F}_0^{(n)}(\xi) \right| \leq C 2^{-n(q+1)} .$$

利用 $\widehat{F}_0^{(n)}$ 的齐性及极化公式 (第一章 (2.11)) 可得

$$\left| \frac{1}{n!} F_0^{(n)}(\xi_1, \cdots, \xi_n) \right| \leq C 2^{-n(q+1)} e^n \prod_{j=1}^{n} |f_j|_p . \tag{2.30}$$

这里用到 $(n!)^{-1}n^n \leq e^n$. 故由核定理知, $\forall n \geq 1$, 存在 $g_n \in (E_{\mathbb{C}}^*)^{\widehat{\otimes} n}$, 使得 (2.20) 成立. 此外, 与引理 2.8 的证明类似可证

$$\begin{aligned} |g_n|_{-p'}^2 &\leq C^2 (e 2^{-(q+1)} \|I_{pp'}\|_{\mathrm{HS}})^{2n} \\ &= C^2 2^{q'n} \rho^n . \end{aligned} \tag{2.31}$$

令 $\Phi \sim \{g_n, n \in \mathbb{N}_0\}$, 其中 $g_0 = F(0)$. 显然 $\Phi \in (H_{-p',-q',-1})_{\mathbb{C}}$, 且有 (2.28). 此外, 容易看出 $S\Phi$ 与 F 在 $U_{p',q\vee q'}$ 上一致. ∎

与定理 2.10 及 2.11 的证明完全类似，我们可以证明定理 2.13 的如下两个重要推论.

定理 2.14 设 $F_n \in \mathrm{Hol}_0(E_{\mathbb{C}}), n \in \mathbb{N}$. 如果满足下列两个条件:

(1)存在 $p \in \mathbb{N}, q \in \mathbb{R}$ 及 $C > 0$, 使得每个 F_n 在 $U_{p,q}$ 上全纯，且 $|F_n(\xi)| \leq C, \forall \xi \in U_{p,q}$;

(2) 对一切 $\xi \in U_{p,q}$, $\{F_n(\xi), n \in \mathbb{N}\}$ 为 $\mathbb{C}$ 中 Cauchy 列,

则 $\{S^{-1}F_n\}$ 在 $(E)_{\mathbb{C}}^{-1}$ 中强收敛.

定理 2.15 设 $(\Omega, \mathcal{F}, \nu)$ 为一测度空间，$\omega \mapsto \Phi_\omega$ 为 Ω 到 $(E)_{\mathbb{C}}^{-1}$ 中的映射. 假定存在 $p \in \mathbb{N}, q \in \mathbb{R}$, 使得 Φ_ω 的局部 S- 变换 F_ω 在 $U_{p,q}$ 中有定义且满足下列条件:

(1) $\forall \xi \in U_{p,q}, \omega \mapsto F_\omega(\xi)$ 为可测映射;

(2) 存在非负 ν- 可积函数 $C(\omega)$, 使得对 ν-a.e. ω,

$$|F_\omega(\xi)| \leq C(\omega),\ \forall \xi \in U_{p,q},$$

则存在 $p', q' \in \mathbb{N}$, 使得 $\omega \mapsto \Phi_\omega$ 在 $(H_{-p',q',-1})_{\mathbb{C}}$ 中 Bochner 可积，且有

$$S\Big(\int_\Omega \Phi_\omega d\nu(\omega)\Big)(\xi) = \int_\Omega S\Phi_\omega(\xi) d\nu(\omega)\ ,\ \forall \xi \in U_{p,q}. \tag{2.32}$$

2.3 检验泛函空间的两种刻画

前面我们用 S- 变换和局部 S- 变换分别刻画了空间 $(S)_{\mathbb{C}}^{-\beta}(0 \leq \beta < 1)$ 和空间 $(E)_{\mathbb{C}}^{-1}$, 现在研究检验泛函空间 $(E)_{\mathbb{C}}^{\beta}(0 \leq \beta < \infty)$ 的刻画. 首先，由于 $(E)_{\mathbb{C}}^{\beta}$ 是 $(E)_{\mathbb{C}}^{*}$ 的子空间，我们可以通过 S- 变换刻画 $(E)_{\mathbb{C}}^{\beta}$.

定理 2.16 E 上的一复值泛函 F 为 $(E)_{\mathbb{C}}^{\beta}$ 中某元素的 S- 变换，当且仅当它满足定义 2.6 中的条件 (C.1) 及下述条件 (C.3):

(C.3) $\forall p \in \mathbb{N}, \forall \epsilon > 0$, 存在 $C_{p,\epsilon} > 0$, 使得

$$|F(zf)| \leq C_{p,\epsilon}\exp\{\epsilon|z|^{\frac{2}{1+\beta}}|f|_{-p}^{\frac{2}{1+\beta}}\}\ ,\ \forall f \in E,\ z \in \mathbb{C}. \tag{2.33}$$

证明 必要性. 设 $\varphi \in (E)^{\beta}_{\mathbb{C}}$, $F = S\varphi$. 显然 F 满足 (C.1). 对 $p \in \mathbb{N}_0$, 与 (2.34) 的证明类似可证: $\forall f \in E, z \in \mathbb{C}$

$$| F(zf) | \le \|\varphi\|_{p,q,\beta} \Big(\sum_{n=0}^{\infty} \Big(\frac{1}{n!}\Big)^{1+\beta} 2^{-qn} |z|^{2n} |f|_{-p}^{2n} \Big)^{1/2} . \tag{2.34}$$

对给定 $\epsilon > 0$, 取 q 充分大, 使得 $\rho = 2^{-q}(\frac{2\epsilon}{1+\beta})^{-1} < 1$. 由 (2.34) 得

$$| F(zf) | \le \|\varphi\|_{p,q,\beta} \Big(\sum_{n=0}^{\infty} \rho^n \Big)^{1/2} \exp\{\epsilon |z|^{\frac{2}{1+\beta}} |f|_{-p}^{\frac{2}{1+\beta}}\} , \tag{2.35}$$

这里利用了不等式 $(n!)^{-(1+\beta)} x^n \le \exp\{(1+\beta) x^{\frac{1}{1+\beta}}\}$. 因此 (C.3) 成立.

充分性. 设 F 满足 (C.1) 和 (C.3). 在 (2.14) 中令

$$R = \Big(\frac{n(1+\beta)}{2\epsilon}\Big)^{\frac{1+\beta}{2}} ,$$

与引理 2.7 的证明类似可证: $\forall f_1, \cdots, f_n \in E$

$$\begin{aligned} \Big|\frac{1}{n!} F_0^{(n)}(f_1, \cdots, f_n)\Big| &\le C_{p,\epsilon} \frac{n^n}{n!} \Big(\frac{2e\epsilon}{n(1+\beta)}\Big)^{\frac{n(1+\beta)}{2}} \prod_{j=1}^{n} | f_j |_{-p} \\ &\le C_{p,\epsilon} \Big(\frac{1}{n!}\Big)^{\frac{1+\beta}{2}} \Big[e^2 \Big(\frac{2\epsilon}{1+\beta}\Big)^{1+\beta}\Big]^{\frac{n}{2}} \prod_{j=1}^{n} | f_j |_{-p} . \end{aligned} \tag{2.36}$$

与引理 2.8 的证明类似可证: 存在 $\varphi \in (E)^{\beta}_{\mathbb{C}}$, $\varphi \sim \{f_n\}$, 使得 $S\varphi = F$, 且对一切 $p, q \in \mathbb{N}$, 当 ϵ 充分小时有

$$|f_n|_p^2 \le C_{p,\epsilon}^2 (n!)^{-(1+\beta)} \Big[e^2 \Big(\frac{2\epsilon}{1+\beta}\Big)^{1+\beta}\Big]^n \|I_{pp'}\|_{\mathrm{HS}}^{2n} ,$$

$$\begin{aligned} \|\varphi\|_{p,q,\beta}^2 &= \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |f_n|_p^2 \\ &\le C_{p,\epsilon}^2 \sum_{n=0}^{\infty} \Big(2^q e^2 \Big(\frac{2\epsilon}{1+\beta}\Big)^{1+\beta} \|I_{pp'}\|_{\mathrm{HS}}^2\Big)^n < \infty , \end{aligned}$$

这表明 $\varphi \in (E)_{\mathbb{C}}^{\beta}$. ∎

下面我们将给出检验泛函空间 $(E)_{\mathbb{C}}^{\beta}$ 的另一刻画. 这一刻画比通过 S- 变换的刻画更加直接和明了, 也便于应用. 为此我们首先引进若干概念和记号.

定义 2.17 令 $0 < \alpha \leq 2$. 设 φ 为定义在 $E_{\mathbb{C}}^*$ 上的复值函数. 如果 $\forall p \in \mathbb{N}_0$, φ 为 $H_{-p,\mathbb{C}}$ 上的整函数 (见定义 1.9), 且 $\forall \epsilon > 0$, 存在 $C_{p,\epsilon} > 0$, 使得

$$|\varphi(x)| \leq C_{p,\epsilon} \exp\{\epsilon |x|_{-p}^{\alpha}\}, \forall x \in H_{-p,\mathbb{C}}, \tag{2.37}$$

则称 φ 为 $E_{\mathbb{C}}^*$ 上 **增长阶至多为 α 的最小型整函数**.

我们用 $\mathcal{A}^{\alpha}(E_{\mathbb{C}}^*)$ 表示 $E_{\mathbb{C}}^*$ 上增长阶至多为 α 的最小型整函数全体, 并用 $\mathcal{A}^{\alpha}(E^*)$ 表示 $\mathcal{A}^{\alpha}(E_{\mathbb{C}}^*)$ 在 E^* 上的限制, 即

$$\mathcal{A}^{\alpha}(E^*) = \{\varphi|_{E^*} : \varphi \in \mathcal{A}^{\alpha}(E_{\mathbb{C}}^*)\}. \tag{2.38}$$

这里 $\varphi|_{E^*}$ 表示 φ 在 E^* 上的限制.

$(E)_{\mathbb{C}}^{\beta}$ 作为 $(L^2)_{\mathbb{C}}$ 的子空间, 它的每个元素在 E^* 上是几乎处处定义的, 因此 $(E)_{\mathbb{C}}^{\beta}$ 实际上是 μ- 等价类泛函空间. 我们通常把一个等价类与它的一个代表元素等同起来. 在这一意义下, 我们有

定理 2.18 设 $0 \leq \beta < \infty$, 则 $(E)_{\mathbb{C}}^{\beta} = \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$. 此外, 令 $p_0 \in \mathbb{N}$ 使得从 H_{p_0} 到 H 的嵌入映射为迹算子, 则 $\forall p \geq p_0, \epsilon > 0$, $\exists\, q \in \mathbb{N}$, $C_{q,\epsilon} > 0$, 使得

$$|\widetilde{\varphi}(x)| \leq \|\varphi\|_{p,q,\beta} C_{p,\epsilon} \exp\left\{\epsilon |x|_{-p}^{\frac{2}{1+\beta}}\right\}. \tag{2.39}$$

其中 $\widetilde{\varphi}$ 为 φ 的连续版本.

证明 设 $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $\varphi \sim \{f_n\}$, 即

$$\varphi(x) = \sum_{n=0}^{\infty} \langle f_n, : x^{\otimes n} : \rangle , \; x \in E^* , \tag{2.40}$$

这里级数在 L^2- 意义下收敛. 对 $y,z\in H_{-p,,\mathbb{C}}$, 我们有

$$\begin{aligned}|(z+iy)^{\otimes n}|_{-p}&\le(|z|_{-p}+|y|_{-p})^n\\&=[(|z|_{-p}+|y|_{-p})^{\frac{2}{1+\beta}}]^{\frac{1+\beta}{2}n}\\&\le[2^{\frac{(1-\beta)^+}{1+\beta}}(|z|_{-p}^{\frac{2}{1+\beta}}+|y|_{-p}^{\frac{2}{1+\beta}})]^{\frac{1+\beta}{2}n}\ ,\end{aligned}\tag{2.41}$$

这里我们用了 C_r- 不等式: $(a+b)^r\le(2^{r-1}\vee1)(a^r+b^r),r>0$, $a,b\ge0$. 令 $p_0\in\mathbb{N}$, 使得从 H_{p_0} 到 H 的嵌入映射为迹算子, 则由 Minlos-Sazanov 定理知: 对一切 $p\ge p_0$, Gauss 测度 μ 以 H_{-p} 为支集. 于是对 $p\ge p_0$, 当 q 足够大时, 可使

$$C_1^2=\int_{E^*}\exp\Big\{(1+\beta)2^{\frac{(1-\beta)^+-q}{1+\beta}}|y|_{-p}^{\frac{2}{1+\beta}}\Big\}\mu(dy)<\infty\ .\tag{2.42}$$

由 (1.4)(它对 $x\in E^*_{\mathbb{C}}$ 也成立) 我们有

$$\begin{aligned}&\sum_{n=0}^{\infty}|\langle f_n,:z^{\otimes n}:\rangle|\le\sum_{n=0}^{\infty}|f_n|_p\int_{E^*}|(z+iy)^{\otimes n}|_{-p}\,\mu(dy)\\&\le\sum_{n=0}^{\infty}(n!)^{\frac{1+\beta}{2}}2^{nq/2}|f_n|_p\\&\quad\times\int_{E^*}(n!)^{-\frac{1+\beta}{2}}\Big[2^{\frac{(1-\beta)^+-q}{1+\beta}}(|z|_{-p}^{\frac{2}{1+\beta}}+|y|_{-p}^{\frac{2}{1+\beta}})\Big]^{\frac{1+\beta}{2}n}\mu(dy)\\&\le\|\varphi\|_{p,q,\beta}\Big[\sum_{n=0}^{\infty}\Big(\int_{E^*}\cdots\mu(dy)\Big)^2\Big]^{1/2}\\&\le\|\varphi\|_{p,q,\beta}\Big(\int_{E^*}\sum_{n=0}^{\infty}(n!)^{-(1+\beta)}\\&\quad\times\Big[2^{\frac{(1-\beta)^+-q}{1+\beta}}(|z|_{-p}^{\frac{2}{1+\beta}}+|y|_{-p}^{\frac{2}{1+\beta}})\Big]^{(1+\beta)n}\mu(dy)\Big)^{1/2}\\&\le\|\varphi\|_{p,q,\beta}\Big[\int_{E^*}\exp\{(1+\beta)[2^{\frac{(1-\beta)^+-q}{1+\beta}}(|z|_{-p}^{\frac{2}{1+\beta}}+|y|_{-p}^{\frac{2}{1+\beta}})]\mu(dy)\Big]^{1/2}\\&=\|\varphi\|_{p,q,\beta}C_1\exp\Big\{\frac{1+\beta}{2}2^{\frac{(1-\beta)^+-q}{1+\beta}}|z|_{-p}^{\frac{2}{1+\beta}}\Big\}\ .\end{aligned}\tag{2.43}$$

于是由 (2.43) 知：对 $p \geq p_0$，如下定义的函数 $\tilde{\varphi}$ 是 $H_{-p,\mathbb{C}}$ 上的整函数：

$$\tilde{\varphi}(z) = \sum_{n=0}^{\infty} \langle f_n : z^{\otimes n} : \rangle \,, \tag{2.44}$$

且 $\forall \epsilon > 0, \forall p \geq p_0$，当 q 足够大时有

$$| \tilde{\varphi}(z) | \leq \|\varphi\|_{p,q,\beta} C_{p,\epsilon} \exp\{\epsilon |z|_{-p}^{\frac{2}{1+\beta}}\} \,. \tag{2.45}$$

另一方面，对 $0 \leq p \leq p_0$，有 $H_{-p,\mathbb{C}} \subset H_{-p_0,\mathbb{C}}$ 及 $|z|_{-p_0} \leq |z|_{-p}$，从而 $\tilde{\varphi}$ 是 $H_{-p,\mathbb{C}}$ 上的整函数，且 (2.45) 成立．这表明 $\tilde{\varphi} \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*_{\mathbb{C}})$．由于 $\tilde{\varphi}(x) = \varphi(x)$，$\mu$-a.e. $x \in E^*$，故 $\varphi = \tilde{\varphi}|_{E^*} \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$．

反之，设 $\varphi \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*_{\mathbb{C}})$．不妨在定义 2.17 中取 $p_0 \in \mathbb{N}$ 使 $\mu(H_{-p_0}) = 1$．则 $\forall p \geq p_0, \forall \epsilon > 0$，由 (2.37) 得 (利用 C_r- 不等式)：

$$\begin{aligned} | \varphi(x+z) | &\leq C_{p,\epsilon} \exp\left\{\epsilon |x+z|_{-p}^{\frac{2}{1+\beta}}\right\} \\ &\leq C_{p,\epsilon} \exp\left\{\epsilon 2^{\frac{(1-\beta)^+}{1+\beta}} |x|_{-p}^{\frac{2}{1+\beta}}\right\} \exp\left\{\epsilon 2^{\frac{(1-\beta)^+}{1+\beta}} |z|_{-p}^{\frac{2}{1+\beta}}\right\} \,. \end{aligned} \tag{2.46}$$

令

$$\psi(z) = \int_{E^*} \varphi(x+z)\mu(dx) \,, \; z \in H_{-p,\mathbb{C}} \,, \tag{2.47}$$

由 Fernique 定理 (I.4.20)，我们可在 (2.46) 中取 $0 < \epsilon_1 \leq \epsilon$ 足够小以使上述积分存在，且 ψ 为 $H_{-p,\mathbb{C}}$ 上的整函数．由 (2.46) 知

$$| \psi(z) | \leq C'_{p,\epsilon} \exp\left\{\epsilon 2^{\frac{(1-\beta)^+}{1+\beta}} |z|_{-p}^{\frac{2}{1+\beta}}\right\} \,. \tag{2.48}$$

由于 $\varphi|_{E^*}$ 显然属于 $(L^2)_{\mathbb{C}}$，故由 (2.7) 得

$$S\, \varphi|_{E^*}(f) = \psi(f) \,, \; f \in E \,. \tag{2.49}$$

(2.47) 和 (2.48) 蕴含 $S\,\varphi|_{E^*}$ 满足定理 2.16 的条件，从而 $\varphi|_{E^*} \in (S)^{\beta}_{\mathbb{C}}$． ∎

注 由上述定理知，$(E)^{\beta}_{\mathbb{C}}$ 中的每个元素都有它的唯一连续版本．今后，凡涉及 $(E)^{\beta}_{\mathbb{C}}$ 中元素，都假定取的是它的连续版本．

系 2.19 设 $0 \leq \beta < \infty$, $\varphi_n \in (E)^{\beta}_{\mathbb{C}}$, $n \in \mathbb{N}$．如果在 $(E)^{\beta}_{\mathbb{C}}$ 中 $\varphi_n \to \varphi \in (E)^{\beta}_{\mathbb{C}}$, 则 $\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \forall x \in E^*$.

证明 设 $x \in H_{-p}, p \geq p_0$, 其中 p_0 满足 $\mu(H_{p_0}) = 1$, 则由 (2.43) 得

$$|\varphi_n(x) - \varphi(x)| \leq \|\varphi_n - \varphi\|_{p,q,\beta} C_1 \exp\{\alpha(q)|x|_{-p}^{\frac{2}{1+\beta}}\} ,$$

于是 $\varphi_n(x) \to \varphi(x)$. ∎

系 2.20 设 $0 \leq \beta < \infty$, 则 $(E)^{\beta}_{\mathbb{C}}$ 对乘积运算封闭.

2.4 广义泛函的若干例子

下面我们给出广义泛函的若干例子，其中许多例子将在今后被用到.

例 2.21(广义指数泛函) 设 $z \in E^*_{\mathbb{C}}$. 令 $\mathcal{E}_z \sim \{\frac{1}{n!} z^{\otimes n}\}$, 则 $\mathcal{E}_z \in (E)^*_{\mathbb{C}}$. 事实上，设 $z \in H_{-p,\mathbb{C}}$, 则 $\forall q \in \mathbb{N}$

$$\|\mathcal{E}_z\|^2_{-p,-q,0} = \sum_{n=0}^{\infty} (n!)^{-1} 2^{-nq} |z|^{2n}_{-p} = \exp\left\{2^{-q}|z|^2_{-p}\right\}. \tag{2.50}$$

我们称 $\mathcal{E}_z$ 为 (广义) 指数泛函，它是 Hida 广义泛函．它的 S- 变换为

$$S\mathcal{E}_z(f) = e^{\langle z,f\rangle}, \quad \forall f \in E.$$

例 2.22 (参数为复值的 Gauss 测度) 首先设 $\lambda > 0$. 令

$$\mu^{(\lambda)}(B) = \mu(\lambda^{-1}B) \ , \ B \in \mathcal{B}(E^*) \ . \tag{2.51}$$

则 $\forall f \in E$, 令

$$\begin{aligned}
G(f) &= \int_{E^*} \mathcal{E}_f(x)\mu^{(\lambda)}(dx) \\
&= \int_{E^*} \exp\{\langle x, f\rangle - \tfrac{1}{2}|f|^2\}\mu^{(\lambda)}(dx) \\
&= \int_{E^*} \exp\{\langle x, \lambda f\rangle - \tfrac{1}{2}|f|^2\}\mu(dx) \\
&= \int_{E^*} \mathcal{E}_{\lambda f}(x)\mu(dx)\exp\{\tfrac{\lambda^2-1}{2}|f|^2\} \\
&= \exp\{\tfrac{\lambda^2-1}{2}|f|^2\} .
\end{aligned}$$

由定理 2.9 知, G 为 $(E)^*$ 中一元素的 S- 变换. 我们用 $F(\lambda)$ 表示 $S^{-1}G$, 则 $F(\lambda)$ 可视为 $\mu^{(\lambda)}$ 关于 μ 的 **广义 Radon–Nikodym 导数**, 它是 Hida 广义泛函. 在这个意义下, 我们可以把参数为 λ^2 的 Gauss 测度看成为 Hida 广义泛函.

现设 $\lambda \in \mathbb{C}$. 令 $G(f) = \exp\{\frac{\lambda^2-1}{2}|f|^2\}, f \in E$. 则 G 仍为 $(E)^*_{\mathbb{C}}$ 中某一元素的 S- 变换, 我们仍用 $F(\lambda)$ 表示 $S^{-1}G$. 下面我们将给出它的混沌分解. 由于

$$\begin{aligned}
\exp\{\tfrac{\lambda^2-1}{2}|f|^2\} &= \sum_{k=0}^{\infty} \frac{(\lambda^2-1)^k}{k!2^k}|f|^{2k} \\
&= \sum_{k=0}^{\infty} \frac{(\lambda^2-1)^k}{k!2^k}\langle \tau^{\widehat{\otimes} k}, f^{\otimes 2k}\rangle,
\end{aligned}$$

对照 S- 变换的表达式 (2.8), 我们得到 $F(\lambda)$ 的如下具体表达:

$$F(\lambda) \sim \{g_n, n \geq 0\},$$

$$g_{2k+1} = 0 , \quad g_{2k} = \frac{(\lambda^2-1)^k}{k!2^k}\tau^{\widehat{\otimes} k} , \; k \geq 0 . \tag{2.52}$$

令 $p \in \mathbb{N}$, 使得从 H_p 到 H 的嵌入映射 I_{0p} 为 H-S 算子, 并

令 $\{e_i, i \in \mathbb{N}\}$ 为 H_p 的基. 则由 (1.2) 得

$$\begin{aligned}|\tau|_{-p}^2 &= \sum_{i,j=1}^{\infty} \langle \tau, e_i \otimes e_j \rangle^2 = \sum_{i,j=1}^{\infty} (e_i, e_j)^2 \\ &\leq \sum_{i,j=1}^{\infty} |e_i|^2 |e_j|^2 = \|I_{0p}\|_{\mathrm{H.S}}^4 .\end{aligned} \tag{2.53}$$

于是当 $q \in \mathbb{R}$ 使得 $2^q > |\lambda^2 - 1| \|I_{0p}\|_{\mathrm{H.S}}^2$ 时, $F(\lambda) \in (H_{-p,-q,-\beta})_{\mathbb{C}}$.

例 2.23(δ- **泛函**) 仿照 Schwartz 广义函数理论, $\forall y \in E^*$, 我们希望定义一个广义泛函 δ_y 如下: $\forall \varphi \in (E)_{\mathbb{C}}$, $\langle\langle \varphi, \delta_y \rangle\rangle = \varphi(y)$. 如果存在这样的 δ_y, 则对 $f \in E$,

$$S\delta_y(f) = \langle\langle \mathcal{E}_f, \delta_y \rangle\rangle = \mathcal{E}_f(y) = \exp\{\langle y, f\rangle - \tfrac{1}{2}|f|^2\}. \tag{2.54}$$

由于 (2.54) 右端确实是某个 Hida 广义泛函的 S- 变换, 我们用 δ_y 表示这一广义泛函 (称它为 δ- **泛函**). 由于 $\{\mathcal{E}_f, f \in E\}$ 生成的线性空间在 (E) 中稠, 故由 (2.54) 及系 2.19 知

$$\langle\langle \varphi, \delta_y \rangle\rangle = \varphi(y)\ ,\ \forall \varphi \in (E)_{\mathbb{C}},\ \forall y \in E_{\mathbb{C}}^*. \tag{2.55}$$

更一般地, $\forall y \in E_{\mathbb{C}}^*$, (2.54) 右端是某个 Hida 广义泛函的 S- 变换, 我们仍用 δ_y 表示这一广义泛函.

下面我们通过 δ_y 的 S- 变换求出 δ_y 的混沌分解. 由 (2.54) 得

$$\begin{aligned}S\delta_y(f) &= \exp\{\langle y, f\rangle - \tfrac{1}{2}|f|^2\} \\ &= \sum_{l=0}^{\infty} \frac{\langle y^{\otimes l}, f^{\otimes l}\rangle}{l!} \sum_{l=0}^{\infty} \frac{(-1)^k |f|^{2k}}{k! 2^k} \\ &= \sum_{l,k=0}^{\infty} \frac{(-1)^k}{l! k! 2^k} \langle y^{\otimes l} \widehat{\otimes} \tau^{\widehat{\otimes} k}, f^{\otimes(2k+l)}\rangle \\ &= \sum_{n=0}^{\infty} \langle \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(n-2k)! k! 2^k} y^{\otimes(n-2k)} \widehat{\otimes} \tau^{\widehat{\otimes} k}, f^{\otimes n}\rangle,\end{aligned}$$

这表明 $\delta_y \sim \{g_n\}$. 其中

$$g_n = \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(n-2k)!k!2^k} y^{\otimes(n-2k)} \widehat{\otimes} \tau^{\widehat{\otimes} k}. \tag{2.56}$$

注 由 (2.54) 知 $\delta_0 = F(0)$.

例 2.24(Schwartz 广义泛函与 $\mathbf{W}_f$ 的复合) 设 $f \in H$, $f \neq 0$, $T \in \mathcal{S}^*(\mathbb{R})$. 我们将定义 T 与 W_f 的复合, 使之为通常意义下函数复合的合理推广. 首先设 T 为一有界 Borel 函数. 由于 $T(W_f) \in (L^2)$, 它的 S- 变换可通过 (2.7) 计算如下:

$$\begin{aligned}
ST(W_f)(\xi) &= \int_{E^*} T(W_f)(x+\xi)\mu(dx) \\
&= \int_{E^*} T\Big(W_f + \langle f, \xi\rangle\Big)\mu(dx) \\
&= \int_{\mathbb{R}} T(t + \langle f, \xi\rangle) p_{|f|^2}(t)dt \\
&= T * p_{|f|^2}(\langle f, \xi\rangle) ,
\end{aligned} \tag{2.57}$$

其中 “$*$” 表示卷积, $p_{|f|^2}(t)$ 是方差为 $|f|^2$ 的正态分布密度, 即

$$p_{|f|^2}(t) = \frac{1}{\sqrt{2\pi}|f|} \exp\Big\{ - \frac{t^2}{2|f|^2} \Big\} .$$

现设 $T \in \mathcal{S}^*(\mathbb{R})$. 由于 $p_{|f|^2} \in \mathcal{S}(\mathbb{R})$, $T * p_{|f|^2}$ 有意义, 且为缓增的连续函数, 故存在 $c_1, c_2 > 0$, 使得

$$\Big|T * p_{|f|^2}(x)\Big| \leq c_1 e^{c_2|x|^2} .$$

于是 $\xi \mapsto T * p_{|f|^2}(\langle f, \xi\rangle)$ 为 E 上一 U_0- 泛函, 从而由定理 2.9 知, 它是某个 Hida 广义泛函的 S- 变换. 我们称该广义泛函为 T 与 W_f 的复合, 仍记为 $T(W_f)$.

如果 $a \in \mathbb{R}$, $T = \delta_a$, 通常称 $\delta_a(W_f)$ 为 **Donsker δ- 泛函**(参看第二章 4.4 节例 1).

注 如果 $T_n \in \mathcal{S}^*(\mathbb{R}), n \geq 1$, 且 $\{T_n\}$ 在 $\mathcal{S}^*(\mathbb{R})$ 中收敛于 T, 则由定理 2.10 证明知，序列 $\{T_n(W_f)\}$ 在 (E^*) 中收敛于 $T(W_f)$. 特别，对任何 $T \in \mathcal{S}^*(\mathbb{R})$, 我们可选取 $\{T_n\} \subset \mathcal{S}(\mathbb{R})$, 使 $\{T_n\}$ 在 $\mathcal{S}^*(\mathbb{R})$ 中收敛于 T, 于是 $T(W_f)$ 为序列 $\{T_n(W_f\}, n \geq 1)$ 在 $(E)^*$ 中的极限. 这表明，我们上面定义的 T 与 W_f 的复合是合理的.

例 2.25 (Brown 运动的局部时) 考虑经典的白噪声分析框架： $E = \mathcal{S}(\mathbb{R}), H = L^2(\mathbb{R}), E^* = \mathcal{S}^*(\mathbb{R})$. 设 $s \geq 0$, 我们用 W_s 简记 $W_{\mathbf{1}_{[0,s]}}$. 易见 $\{W_s, s \geq 0\}$ 为一 Gauss 过程，且 $\mathbb{E}(W_t - W_s)^2 = |t-s|$. 于是 $\{W_s, s \geq 0\}$ 的连续版本即为概率空间 $(\mathcal{S}^*(\mathbb{R}), \mathcal{B}(\mathcal{S}^*(\mathbb{R})), \mu)$ 上的 Brown 运动. 设 $a \in \mathbb{R}, \{W_s, s \geq 0\}$ 在 a 处的局部时 L_t^a 可以形式地表示为

$$L_t^a = \int_0^t \delta(W_s - a)ds = \int_0^t \delta_a(W_s)ds. \tag{2.58}$$

我们将证明上式中的积分在 Bochner 意义下存在. 事实上，设 $\xi \in \mathcal{S}(\mathbb{R})$, 由 (2.57) 得

$$S\delta_a(W_s)(\xi) = \frac{1}{\sqrt{2\pi s}} \exp\Big\{-\frac{1}{2s}\Big(\int_0^s \xi(u)du - a\Big)^2\Big\}.$$

从而有

$$\begin{aligned}
|S\delta_a(W_s)(z\xi)| &\leq \frac{e^{-\frac{a^2}{2s}}}{\sqrt{2\pi s}} \exp\{\tfrac{1}{2s}(2|a||z|\sqrt{s}|\xi| + |z|^2 s|\xi|^2)\} \\
&\leq \frac{e^{-\frac{a^2}{2s}}}{\sqrt{2\pi s}} \exp\{\tfrac{1}{2s}(a^2 + 2|z|^2 s|\xi|^2)\} \\
&= \frac{1}{\sqrt{2\pi s}} \exp\{|z|^2|\xi|^2\}, \qquad z \in \mathbb{C}.
\end{aligned}$$

这里 $|\xi|^2 = \int_{\mathbb{R}} |\xi(s)|^2 ds$. 于是由定理 2.11 知，(2.58) 中的积分在 Bochner 意义下存在.

例 2.26 (Poisson 测度) 仍考虑经典的白噪声分析框架: $E=\mathcal{S}(\mathbb{R}), H=L^2(\mathbb{R}), E^*=\mathcal{S}^*(\mathbb{R})$. 令 $\{P_t^{(i)}, t\geq 0\}, i=1,2$ 为某概率空间 $(\Omega,\mathcal{F},\mathbb{P})$ 上两个相互独立的 Poisson 过程, 令

$$P_t(\omega)=\begin{cases} P_t^{(1)} & t\geq 0, \\ P_{-t}^{(2)} & t<0. \end{cases}$$

容易证明, $\forall\xi\in\mathcal{S}(\mathbb{R})$, 我们有

$$\mathbb{E}\left[e^{i\int_{\mathbb{R}}\xi(t)dP_t}\right]=\exp\left\{\int_{\mathbb{R}}(e^{i\xi(t)}-1)dt\right\}. \tag{2.59}$$

如果用 π 表示 $\dot{P}$ 在 $\mathcal{S}^*(\mathbb{R})$ 上的分布, 则称 π 为 Poisson 测度. 于是由 (2.59) 得

$$\int_{\mathcal{S}^*(\mathbb{R})}e^{i\langle x,\xi\rangle}\pi(dx)=\exp\left\{\int_{\mathbb{R}}(e^{i\xi(t)}-1)dt\right\}. \tag{2.60}$$

令

$$F(f)=\exp\left\{\int_{\mathbb{R}}(e^{f(t)}-1)dt-\frac{1}{2}\int_{\mathbb{R}}f(t)^2dt\right\},\ f\in\mathcal{S}(\mathbb{R})_{\mathbb{C}}, \tag{2.61}$$

显然 F 为 $\mathcal{S}(\mathbb{R})_{\mathbb{C}}$ 上的整解析函数, 从而由定理 2.13 知, 可将 Poisson 测度 π (或它关于 Guass 测度 μ 的广义 Radon-Nikodym 导数 $d\pi/d\mu$) 看成 $(E)^{-1}$ 的一个元素, 其局部 S- 变换为 F. 由 (2.61) 知, 当 $\beta\in[0,1)$ 时, F 不是 U_β- 泛函, 从而 $d\pi/d\mu\notin(E)^{-\beta}$.

例 2.27(多维 Brown 运动自交局部时) 设 $d\geq 2$. 考虑经典的白噪声分析框架 $(E)\hookrightarrow(L^2)\hookrightarrow(E)^*$, 其中 $E=S(\mathbb{R})^d$, $H=L^2(\mathbb{R})^d, E^*=S^*(\mathbb{R})^d$. 令 $\{B_t\}=\{B_t^1,\cdots,B_t^d\}$ 为 d- 维 Brown 运动在 (E^*,μ) 上的实现, 即对任何 $f\in S(\mathbb{R})^d$, 有

$$\int_{-\infty}^{\infty}B_t(x)f(t)dt=\langle x,f\rangle,\ \mu\text{-a.e. } x.$$

令 δ_0 为在 0 处的 Dirac δ- 函数，则 $\{B_t\}$ 在 0 处的局部时 $L_t^0(B)$ 可以表示成如下的 Bochner 积分：

$$L_t^0(B) = \int_0^t \delta_0(B_s)ds.$$

自然会猜想，$\{B_t\}$ 的自交局部时可以定义为如下形式的积分：

$$\int_{0\leq u<v\leq t} \delta_0(B_v - B_u)dudv.$$

可以证明，对 $d = 2$，上述积分有意义，但当 $d \geq 3$，上述积分在 $(E)^*$ 中不存在．下面我们讨论如何将这一积分进行"重正化"使之有意义．

首先，令 $\Lambda^{(n)}$ 表示 d- 重指标集的如下子集：

$$\Lambda^{(n)} = \left\{\alpha = (\alpha_1, \cdots, \alpha_d) : \alpha_j \in \mathbb{N}_0, 1 \leq j \leq d, \sum_{j=1}^d \alpha_j = n\right\}.$$

易知任一 $\varphi \in L^2(E^*, \mu)$ 有如下的混沌分解：

$$\varphi = \sum_{n=0}^\infty \sum_{\alpha\in\Lambda^{(n)}} I_\alpha(f_\alpha),$$

$$\|\varphi\|^2 = \sum_{n=0}^\infty \sum_{\alpha\in\Lambda^{(n)}} \alpha! |f_\alpha|^2,$$

其中 $f_\alpha \in \otimes_{i=1}^d \widehat{L^2(\mathbb{R}^{\alpha_i})}$,

$$I_\alpha(f_\alpha) = \int f_\alpha(t_1, \cdots, t_n)dB_{t_1}^1 \cdots dB_{t_{\alpha_1}}^1 \cdots dB_{t_{n-\alpha_d+1}}^d \cdots dB_{t_n}^d.$$

一般地，任何 $\varphi \in (E)^*$ 有类似形式展开，其中 $f_\alpha \in \otimes_{i=1}^d \widehat{S^*(\mathbb{R}^{\alpha_i})}$. 如同上一节，我们用 $|\cdot|_0$ 及 $|\cdot|_p$ 分别表示 $L^2(\mathbb{R}^k)$ 及 $S_p(\mathbb{R}^k)$ 中的范数 (参见第一章 3.2 节).

设 $a = (a_1, \cdots, a_d) \in \mathbb{R}^d, \delta_a$ 为在 a 处的 δ- 函数．容易证明 $\delta_a(B_v - B_u)$ 的 S- 变换为 (参见例 2.24)

$$S\delta_a(B_v - B_u)(\xi) = \frac{1}{[2\pi(v-u)]^{d/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^d \left(a_j - \int_u^v \xi_j(r)dr\right)^2\right\}.$$

由此可以具体求出 $\delta_a(B_v - B_u)$ 的混沌分解： $\delta_a(B_v - B_u) \sim \{\psi_\alpha^{(a)}(v,u), \alpha \in \Lambda\}$, 其中 Λ 为 d- 重指标集,

$$
\begin{aligned}
&\psi_\alpha^{(a)}(u,v)(x) \\
&= [2(v-u)]^{-d/2}\pi^{-d/4}e^{-\sum_{j=1}^d x_j^2}\prod_{j=1}^d \frac{1}{\sqrt{\alpha_j!}}e_{\alpha_j}(x_j)\Big(\frac{\mathbf{1}_{[u,v]}}{\sqrt{u-v}}\Big)^{\otimes|\alpha},
\end{aligned}
$$

这里 e_k 为 k- 阶 Hermite 函数. 由此进一步可以证明： $\forall p > 0$,

$$\delta_a(B_u - B_v) \in (E)_{-p}.$$

若 $a \neq 0$, 则 $\forall t \geq 0, p > 0$, 有

$$G_d^a(t) \equiv \int_{0 \leq u < v \leq t} \delta_a(B_v - B_u)dudv \in (E)_{-p}.$$

何声武、杨文强等 [1] 证明了下述结果：对一切 $t > 0$,

$$G_d(t) = \int_{0 \leq u < v < t}\Big[\delta_0(B_v - B_u) - \sum_{n=0}^{d-2}\sum_{\alpha \in \Lambda^{(n)}} I_\alpha\Big(\psi_\alpha^{(0)}(u,v)\Big)\Big]dudv$$

为一 Hida 广义泛函，称为 d- 维 Brown 运动的自交局部时.

令 $\Delta_k^{(t)} = \{(t_1, \cdots, t_{k+1}) \in [0,t]^{k+1} : t_1 < \cdots < t_{k+1}\}$. $\{B_t\}$ 的 $k+1$ 重相交局部时可以形式地表示为

$$\delta_d^k(t) = \int_{\Delta_k^{(t)}} \prod_{i=1}^k \delta_0(B_{t_{i+1}} - B_{t_i})dt_1 \cdots dt_{k+1}.$$

对 $d = 2$, 在 Imkeller-Yan[2] 中证明了 $\delta_2^k(t)$ 的下述重正化

$$\bar{\delta}_2^k(t) = \int_{\Delta_k^{(t)}} \prod_{i=1}^k \Big[\delta_0(B_{t_{i+1}} - B_{t_i}) - \frac{1}{2\pi(t_{i+1} - t_i)}\Big]dt_1 \cdots dt_{k+1}$$

为一 Hida 广义泛函.

§ 3. 泛函的乘积与 Wick 积

3.1 泛函的乘积

在第二章 §1 中，我们给出了两个 Wiener 泛函 $I_m(f_m)$ 与 $I_n(g_n)$ 的乘积公式，其中 $f_m \in H^{\widehat{\otimes} m}, g_n \in H^{\widehat{\otimes} n}$, $H = L^2(T, \mathcal{B}, \lambda)$. 在乘积公式中，出现了 f_m 与 g_n 的"缩合"(contraction): $f_m \widehat{\otimes}_k g_n$, $0 \le k \le m \wedge n$. 现在我们要对一般 Hilbert 空间元素张量积定义缩合.

假定 $\xi_i \in H_{\mathbb{C}}$, $1 \le i \le m$, $\eta_i \in H_{\mathbb{C}}$, $1 \le i \le n$. 令

$$f_m = \otimes_{i=1}^m \xi_i \ , \quad g_n = \otimes_{i=1}^n \eta_i \ . \tag{3.1}$$

对 $0 \le k \le m \wedge n$, 我们定义 f_m 与 g_n 的缩合 $f_m \otimes_k g_n$ 为

$$f_m \otimes_k g_n = \prod_{i=1}^{k} \langle \xi_{m-k+i}, \eta_{n-k+i} \rangle \Big(\otimes_{i=1}^{m-k} \xi_i \Big) \otimes \Big(\otimes_{i=1}^{n-k} \eta_i \Big) \ , \tag{3.2}$$

则 $f_m \otimes_k g_n \in H_{\mathbb{C}}^{\otimes m+n-2k}$.

现设 $\{e_i, i \ge 1\}$ 为 $H_{\mathbb{C}}$ 的一个基. 令

$$e_\alpha = \otimes_{j=1}^n e_{\alpha_j} \ , \quad \alpha = (\alpha_1, \cdots \alpha_n) \ , \tag{3.3}$$

则 $\{e_\alpha \, , \alpha \in \Lambda_n\}$ 为 $H_{\mathbb{C}}^{\otimes n}$ 的一个基. 容易直接验证：对 (3.1) 中的 f_m 与 g_n, 我们有

$$f_m \otimes_k g_n = \sum_{\sigma \in \Lambda_{m-k}, \delta \in \Lambda_{n-k}} \ \sum_{\alpha \in \Lambda_k} (f_m, e_\sigma \otimes e_\alpha)(g_n, e_\delta \otimes e_\alpha) e_\sigma \otimes e_\delta \, . \tag{3.4}$$

容易证明：对一般的 $f_m \in H_{\mathbb{C}}^{\otimes m}$ 及 $g_n \in H_{\mathbb{C}}^{\otimes n}$, (3.4) 式右边的级数在 $H_{\mathbb{C}}^{\otimes n+m-2k}$ 中收敛，且有

$$| \, f_m \otimes_k g_n \, | \le | \, f_m \, | \, | \, g_n \, | \ . \tag{3.5}$$

于是我们可以用 (3.4) 式定义 f_m 与 g_n 的 **缩合**$f_m\otimes_k g_n$. 当 $k=0$ 时，我们用 $\langle f_m,g_n\rangle$ 表示 $f_m\otimes_0 g_n$. 这样我们得到 $H_{\mathbb{C}}^{\otimes m}\times H_{\mathbb{C}}^{\otimes n}$ 到 $H_{\mathbb{C}}^{\otimes m+n-2k}$ 中的一双线性连续映射. 特别，这蕴含由 (3.4) 式定义的缩合不依赖于 $H_{\mathbb{C}}$ 的基的选取. 此外，如果 $f_m\in H^{\otimes m}$, $g_n\in H^{\otimes n}$, 我们可以只取 H 的基来定义 f_m 与 g_n 的缩合，这与取 $H_{\mathbb{C}}$ 的基定义它们的缩合是一致的.

现在设 $f_m\in H_{\mathbb{C}}^{\widehat{\otimes} m}$, $g_n\in H_{\mathbb{C}}^{\widehat{\otimes} n}$, 我们用 $f_m\widehat{\otimes}_k g_n$ 表示 $f_m\otimes_k g_n$ 的对称化，则有

$$|f_m\widehat{\otimes}_k g_n|\le|f_m||g_n|, \tag{3.6}$$

这是因为对称化不使范数变大.

引理 3.1 设 $\beta\ge 0$, $m,n\in\mathbb{N}_0$, 则有

$$\sum_{k=0}^{m\wedge n}k!\binom{m}{k}\binom{n}{k}\left[(m+n-2k)!\right]^{\frac{1+\beta}{2}}\le(m!n!)^{\frac{1+\beta}{2}}2^{(m+n)(1+\beta)}. \tag{3.7}$$

证明 (3.7) 的左边可以改写为

$$\sum_{k=0}^{m\wedge n}(k!)^{-\beta}\left[\binom{m}{k}\binom{n}{k}\right]^{\frac{1-\beta}{2}}\binom{m+n-2k}{n-k}^{\frac{1-\beta}{2}}(m!n!)^{\frac{1+\beta}{2}}.$$

于是当 $0\le\beta<1$,

$$\begin{aligned}(3.7)\text{左边}&\le\sum_{k=0}^{m\wedge n}\binom{m+n}{2k}^{\frac{1-\beta}{2}}\binom{m+n-2k}{n-k}^{\frac{1-\beta}{2}}(m!n!)^{\frac{1+\beta}{2}}\\&\le\left[\sum_{k=0}^{m\wedge n}\binom{m+n}{2k}\right]^{\frac{1-\beta}{2}}\left[\sum_{k=0}^{m\wedge n}\binom{m+n-2k}{n-k}\right]^{\frac{1+\beta}{2}}(m!n!)^{\frac{1+\beta}{2}}\\&\le 2^{(m+n)(1-\beta)/2}2^{(m+n)(1+\beta)/2}(m!n!)^{(1+\beta)/2}\\&=2^{m+n}(m!n!)^{(1+\beta)/2}.\end{aligned} \tag{3.8}$$

若 $\beta \geq 1$, 则

$$
\begin{aligned}
(3.7)\text{ 左边} &\leq \sum_{k=0}^{m\wedge n} \binom{m+n-2k}{n-k}^{\frac{1+\beta}{2}} (m!n!)^{\frac{1+\beta}{2}} \\
&\leq \Big[\sum_{k=0}^{m\wedge n} \binom{m+n-2k}{n-k}\Big]^{\frac{1+\beta}{2}} (m!n!)^{\frac{1+\beta}{2}} \\
&\leq 2^{(m+n)(1+\beta)/2}(m!n!)^{(1+\beta)/2} . \qquad (3.9)
\end{aligned}
$$

由 (3.8) 及 (3.9) 立得 (3.7) . ∎

现设 $f_m \in H_{\mathbb{C}}^{\widehat{\otimes} m}$, $g_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$. 与第二章命题 1.8 类似可证

$$
I_m(f_m)I_n(g_n) = \sum_{k=0}^{m\wedge n} k!\binom{m}{k}\binom{n}{k} I_{m+n-2k}(f_m \widehat{\otimes}_k g_n) , \qquad (3.10)
$$

其中 $f_m \widehat{\otimes}_k g_n$ 为 $f_m \otimes_k g_n$ 的对称化.

下面设 $p \in \mathbb{N}_0$, $q \in \mathbb{R}_+$, $\beta \geq 0$, $(H_{p,q,\beta})$ 如 (1.16) 所定义.

定理 3.2 设 $\epsilon > 0$, $q' = q + 2(1+\beta) + \epsilon$, $\varphi, \psi \in (H_{p,q',\beta})_{\mathbb{C}}$, 则 $\varphi\psi \in (H_{p,q,\beta})_{\mathbb{C}}$, 且有

$$
\|\varphi\psi\|_{p,q,\beta} \leq (1-2^{-\epsilon})^{-1} \|\varphi\|_{p,q',\beta} \|\psi\|_{p,q',\beta} . \qquad (3.11)
$$

证明 设 $\varphi = \sum_m I_m(f_m)$, $\psi = \sum_n I_n(g_n)$. 由 (3.10), (1.16) 及

(3.7) 得

$$
\begin{aligned}
\|\varphi\psi\|_{p,q,\beta} &\leq \sum_{m,n=0}^{\infty}\sum_{k=0}^{m\wedge n} k!\binom{m}{k}\binom{n}{k}\|I_{m+n-2k}(f_m\widehat{\otimes}_k g_n)\|_{p,q,\beta} \\
&\leq \sum_{m,n=0}^{\infty}\sum_{k=0}^{m\wedge n} k!\binom{m}{k}\binom{n}{k}[(m+n-2k)!]^{\frac{1+\beta}{2}}2^{\frac{(m+n-2k)q}{2}}|f_m|_p|g_n|_p \\
&\leq \sum_{m,n=0}^{\infty}(m!n!)^{(1+\beta)/2}2^{(m+n)(1+\beta)+(m+n)q/2}|f_m|_p|g_n|_p \\
&= \Big(\sum_{m=0}^{\infty}2^{-(\epsilon m)/2}(m!)^{\frac{1+\beta}{2}}2^{m(1+\beta+(q+\epsilon/2)}|f_m|_p\Big) \\
&\quad\times\Big(\sum_{n=0}^{\infty}2^{-\epsilon n/2}(n!)^{(1+\beta)/2}2^{n(1+\beta+(q+\epsilon)/2)}|g_n|_p\Big) \\
&\leq (1-2^{-\epsilon})^{-1}\|\varphi\|_{p,q'\beta}\|\psi\|_{p,q'\beta}.
\end{aligned}
$$

∎

注 设 $\varphi,\psi\in(H_{p,q,\beta})_{\mathbb{C}}$, $q>2(1+\beta)$. 若 $\varphi\sim\{f_m\}$, $\psi\sim\{g_n\}$, 则由 (3.10) 及定理 3.2 的证明知 $\varphi\psi\sim\{h_l\}$,

$$
h_l=\sum_{m+n=l}\sum_{k=0}^{\infty}k!\binom{m+k}{k}\binom{n+k}{k}f_{m+k}\widehat{\otimes}_k g_{n+k}\ , \tag{3.12}
$$

其中 (3.12) 右端级数在 $H_{p,\mathbb{C}}^{\widehat{\otimes} l}$ 中收敛.

系 3.3 设 $0\leq\beta<\infty$, 则 $\{\varphi,\psi\}\longmapsto\varphi\psi$ 为 $(E)_{\mathbb{C}}^{\beta}\times(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 中的连续映射.

注 设 $\varphi\in(E)_{\mathbb{C}}^{\beta}$, $F\in(E)_{\mathbb{C}}^{-\beta}$, 则由系 3.3 知, 存在唯一的 $G\in(E)_{\mathbb{C}}^{-\beta}$, 使得

$$
\langle\langle G,\psi\rangle\rangle=\langle\langle\varphi\psi,F\rangle\rangle\ ,\ \forall\psi\in(E)_{\mathbb{C}}^{\beta}. \tag{3.13}
$$

我们称 G 为 φ 与 F 的乘积, 记为 φF.

3.2 广义泛函的 Wick 积

对两个广义泛函，一般不能定义它们的乘积，但我们可以定义它们的 "Wick 积". 首先考虑两个广义指数泛函 (见例 2.21)$\mathcal{E}_y$ 及 $\mathcal{E}_z$, 其中 $y, z \in E^*_{\mathbb{C}}$. 我们称 $\mathcal{E}_{y+z}$ 为 $\mathcal{E}_y$ 与 $\mathcal{E}_z$ 的 Wick 积. 这时，我们有

$$S\mathcal{E}_{y+z}(\xi) = S\mathcal{E}_y(\xi)S\mathcal{E}_z(\xi) \ , \forall \xi \in E \ . \tag{3.14}$$

一般说来，设 $0 \le \beta < 1, \varphi, \psi \in (E)^{-\beta}_{\mathbb{C}}$. 令

$$F(\xi) = S\varphi(\xi)S\psi(\xi) \ , \ \xi \in E \ . \tag{3.15}$$

则由定理 2.9 知 F 为 U_β- 泛函. 我们称 $S^{-1}F$ 为 φ 与 ψ 的 **Wick 积**, 记为 $\varphi \diamond \psi$.

下一定理给出了 Wick 积的混沌分解表达式.

定理 3.4 设 $0 \le \beta < 1$. 令 $\varphi, \psi \in (E)^{-\beta}_{\mathbb{C}}$, $\varphi \sim \{f_n\}, \psi \sim \{g_n\}$. 则 $\varphi \diamond \psi \sim \{h_n\}$. 其中

$$h_l = \sum_{m+n=l} f_m \widehat{\otimes} g_n \ . \tag{3.16}$$

证明 依 Wick 积的定义，我们有

$$\begin{aligned}
\sum_{l=0}^{\infty}(h_l, \xi^{\otimes l}) &= S(\varphi \diamond \psi)(\xi) \\
&= S\varphi(\xi)S\psi(\xi) \\
&= \sum_{m=0}^{\infty}(f_m, \xi^{\otimes m})\sum_{n=0}^{\infty}(g_n, \xi^{\otimes n}) \\
&= \sum_{l=0}^{\infty}\Big(\sum_{m+n=l} f_m \widehat{\otimes} g_n, \xi^{\otimes l}\Big) \ , \ \forall \xi \in E \ ,
\end{aligned}$$

故有 (3.16). ∎

例 令 $F(\lambda)$, δ_y 及 $\mathcal{E}_y$ 如 §4 中定义, 则由 (2.54) 知, $\delta_y = \mathcal{E}_y \diamond F(0)$, $\mathcal{E}_y = \delta_y \diamond F(\sqrt{2})$.

现在假定 $\beta \geq 1, \varphi, \psi \in (E)_{\mathbb{C}}^{-\beta}, \varphi \sim \{f_n\}, \psi \sim \{g_n\}$. 这时 φ 和 ψ 的 S- 变换不一定存在. 所以我们一般不能通过 (3.15) 来定义 φ 与 ψ 的 Wick 积. 但是, (3.16) 式总是有意义的. 自然要问: 是否序列 $\{h_n\}$ 对应于 $(E)_{\mathbb{C}}^{-\beta}$ 中一个元素? 下一定理表明, 答案是肯定的.

定理 3.5 设 $p \in \mathbb{Z}, q, r \in \mathbb{R}, \varphi, \psi \in (H_{p,q,r})_{\mathbb{C}}$, $\varphi \sim \{f_n\}, \psi \sim \{g_n\}$. 令 $\{h_n\}$ 由 (3.15) 定义. 则 $\forall \epsilon > (1+r)^+$, $\{h_n\}$ 对应于 $(H_{p,q-\epsilon,r})_{\mathbb{C}}$ 中一元素, 记为 $\varphi \diamond \psi$. 我们称它为 φ 与 ψ 的 **Wick 积**. 此外, 我们有

$$\|\varphi \diamond \psi\|_{p,q-\epsilon,r} \leq (1 - 2^{-\epsilon+(1+r)^+})^{-1/2} \|\varphi\|_{p,q,r} \|\psi\|_{p,q,r} . \qquad (3.17)$$

证明 由 (3.16) 得

$$\begin{aligned}
&(n!)^{1+r} 2^{n(q-\epsilon)} |h_n|_p^2 \\
&\leq (n!)^{1+r} 2^{n(q-\epsilon)} \Big(\sum_{k+j=n} |f_k|_p |g_j|_p \Big)^2 \\
&= 2^{-n\epsilon} \Big(\sum_{k+j=n} \binom{n}{k}^{\frac{1+r}{2}} 2^{\frac{nq}{2}} (k!)^{\frac{1+r}{2}} (j!)^{\frac{1+r}{2}} |f_k|_p \, g_j|_p \Big)^2 \\
&\leq 2^{n[-\epsilon-(1+r)^+]} \Big(\sum_{k=0}^{n} 2^{kq} (k!)^{1+r} |f_k|_p^2 \Big) \Big(\sum_{j=0}^{n} 2^{jq} (j!)^{1+r} |g_j|_p^2 \Big) \\
&\leq 2^{n[-\epsilon+(1+r)^+]} \|\varphi\|_{p,q,r}^2 \|\psi\|_{p,q,r}^2 ,
\end{aligned}$$

由此立得 (3.17). ∎

系 3.6 $(E)_{\mathbb{C}}^{-\beta}$ 及 $(E)_{\mathbb{C}}^{\beta}$ 对 Wick 积封闭, 且 $\{\varphi, \psi\} \longmapsto \varphi \diamond \psi$ 为从 $(E)_{\mathbb{C}}^{-\beta} \times (E)_{\mathbb{C}}^{-\beta}$ 到 $(E)_{\mathbb{C}}^{-\beta}$ 及从 $(E)_{\mathbb{C}}^{\beta} \times (E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 中的对称双线性连续映射.

下一定理对 $(E)_{\mathbb{C}}^{-1}$ 情形给出了更精细的结果.

定理 3.7 设 $p \in \mathbb{Z}, q \in \mathbb{R}$. 则 $\forall \epsilon > 0$ 有

$$\|\varphi \diamond \psi\|_{p,q,-1} \le (1 - 2^{-\epsilon})^{-1/2} \|\varphi\|_{p,q+\epsilon,-1} \|\psi\|_{p,q,-1}. \tag{3.18}$$

证明 令 $\varphi \sim \{f_n\}, \psi \sim \{g_n\}, \varphi \diamond \psi \sim \{h_n\}$. 则由 (3.16) 得

$$\begin{aligned}
\|\varphi \diamond \psi\|_{p,q,-1}^2 &= \sum_{n=0}^{\infty} 2^{nq} |h_n|_p^2 \\
&\le \sum_{n=0}^{\infty} 2^{nq} \left(\sum_{k=0}^{n} |f_k|_p |g_{n-k}|_p \right)^2 \\
&= \sum_{n=0}^{\infty} 2^{nq} \sum_{k,j=0}^{n} |f_k|_p |f_j|_p |g_{n-k}|_p |g_{n-j}|_p \\
&= \sum_{k,j=0}^{\infty} 2^{\frac{k+j}{2}q} |f_k|_p |f_j|_p \sum_{n \ge k,j} 2^{(n-\frac{k+j}{2})q} |g_{n-k}|_p |g_{n-j}|_p \\
&\le \sum_{k,j=0}^{\infty} 2^{\frac{k+j}{2}q} |f_k|_p |f_j|_p \sum_{n \ge k} 2^{(n-k)q} |g_{n-k}|_p^2 \\
&= \left(\sum_{k=0}^{\infty} 2^{kq/2} |f_k|_p \right)^2 \|\psi\|_{p,q-1}^2 \\
&\le \left(\sum_{k=0}^{\infty} 2^{-k\epsilon} \right) \|\varphi\|_{p,q+\epsilon,-1}^2 \|\psi\|_{p,q,-1}^2.
\end{aligned}$$

(3.18) 得证. ∎

注 在经典的白噪声分析框架下 (见 1.3 节), 由 (1.25) 推知 (3.18) 可改写成

$$\|\varphi \diamond \psi\|_{p,0,-1} \le (1 - \|A^{-1}\|^{2\epsilon})^{-1/2} \|\varphi\|_{p+\epsilon,0,-1} \|\psi\|_{p,0,-1}.$$

这是 Vage[1] 中一个不等式的改进形式 (也见 Holden-Øksendal-Ubøe-Zhang[1]).

3.3 应用于 Feynman 积分

考虑 Schrödinger 方程

$$\frac{\partial\psi}{\partial t}=i\left(\frac{\Delta}{2}-V\right)\psi,\quad \psi(0,x)=f(x), \tag{3.19}$$

其中 Δ 为 $I\!R^d$ 中的 Laplace 算子, V 为 $I\!R^d$ 上的一实值 Borel 函数 (代表位势). 1948 年 Feynman 猜测方程 (3.19) 的基本解 (称为传播子 (propagator)) 可以表示为

$$G(t,x,y)=N^{-1}\int_{\Gamma_{x,y}}\exp\left\{i\int_0^t\left[\frac{1}{2}|\dot{\gamma}(s)|^2-V(\gamma(s))\right]ds\right\}\mathcal{D}(\gamma), \tag{3.20}$$

其中 $\Gamma_{x,y}$ 为连接 y 和 x 的适当的路径空间, $\mathcal{D}(\gamma)$ 是 $\Gamma_{x,y}$ 上的"平坦测度",N 为重正化常数. 量子物理中通常称 (3.20) 中的"积分"为 Feynman (路径) 积分. 但是, 由于这样的"平坦测度"$\mathcal{D}(\gamma)$ 在无穷维空间 $\Gamma_{x,y}$ 中并不存在, 所以 (3.20) 中的积分没有严格的数学定义. 如何给出 Feynman 积分的一个严格的数学定义是一个悬而未决的难题.

下面我们介绍如何从白噪声分析角度研究 Feynman 积分. 为此, 先考虑如下热方程:

$$\frac{\partial u}{\partial t}=\left(\frac{\lambda}{2}\Delta-iV\right)u,\quad u(0,x)=f(x), \tag{3.21}$$

其中 $\lambda>0$. 在一定条件下方程 (3.21) 的解由如下的 Feynman-Kac 公式给出

$$u(t,x)=I\!E\left[f(\sqrt{\lambda}B_t+x)\exp\left\{-i\int_0^t V(\sqrt{\lambda}B_s+x)ds\right\}\right], \tag{3.22}$$

其中 $\{B_t\}$ 为从 0 出发的标准 Brown 运动. 如果利用 Hida 广义泛函 $F(\sqrt{\lambda})$(见例 2.22), 我们可以将 (3.22) 改写成

$$\begin{aligned} u(t,x)&=\int f(B_t+x)e^{-i\int_0^t V(B_s+x)ds}d\mu^{(\sqrt{\lambda})}\\ &=\langle\langle F(\sqrt{\lambda})f(B_t+x)e^{-i\int_0^t V(B_s+x)ds},\ 1\rangle\rangle, \end{aligned} \tag{3.23}$$

这里第一个等式是完全形式上的，为使第二个等式有意义，必须说明式中的三项乘积有意义. 令 $f(x)=\delta_y(x)$, 则由 (3.23) 我们得到方程 (3.21) 的基本解表达式:

$$q^{\lambda}(t,x,y)=\langle\langle F(\sqrt{\lambda})\delta(B_t-y+x)e^{-i\int_0^t V(B_s+x)ds},\ 1\rangle\rangle.$$

假定 q^{λ} 关于 λ 可解析延拓，则可以猜想方程 (3.19) 的基本解可表示为

$$G(t,x,y)=\langle\langle F(\sqrt{i})\delta(B_t-y+x)e^{-i\int_0^t V(B_s+x)ds},\ 1\rangle\rangle. \qquad (3.24)$$

为了使 (3.24) 右边有意义，我们要证明在对 V 加一定条件下，式中三项乘积是一个 Hida 广义泛函. 为此，我们先将 $V(B_s+x)$ 表成如下的 Bochner 积分:

$$V(B_s+x)=\int_{\mathbb{R}^d}dz\,V(z)\delta(B_s+x-z).$$

对 $\forall x\in \mathbb{N}$, 令 $\Delta_n=\{(t_1,\cdots,t_n):0<t_1<\cdots<t_n<t\}$, 则有

$$\begin{aligned}\exp\Big\{-i\int_0^t V(B_s+x)ds\Big\}&=\sum_{n=0}^{\infty}(-i)^n\int_{\Delta_n}d^nt\prod_{i=1}^{n}V(B_{t_i}+x)\\&=\sum_{n=0}^{\infty}(-i)^n\int_{\Delta_n}d^nt\prod_{i=1}^{n}\int_{\mathbb{R}^d}dz_iV(z_i)\delta(B_{t_i}+x-z_i).\end{aligned}$$

在 Khandekar-Streit[1] 中，作者给出了如下乘积作为 Hida 广义泛函的合理定义:

$$F(\sqrt{i})\delta(B_t-y+x)\prod_{i=1}^{n}\delta(B_{t_i}+x-z_i), \qquad (3.25)$$

并利用前面的级数展开证明：在 $d=1$ 的情形下，当 $V(y)dy$ 是一有紧支撑的符号测度时， (3.24) 右边的三项乘积是一个 Hida 广

义泛函. 这里 $d=1$ 是一本质性假定, 因为在计算中涉及如下的积分

$$M_n=\int_{\Delta_n}d^n t\prod_{i=1}^{n+1}|t_i-t_{i-1}|^{-d/2}.$$

只有当 $d=1$ 时上述积分才收敛, 且这时有 $\sum_n M_n<\infty$.

关于 (3.25) 中乘积的合理定义也可参看 Yan[10], 由 (3.24) 给出的 G 确实是 (3.19) 的基本解已由解析 Feynman 积分方法得到证实 (见 Yan[9],Theorem5.4).

§4. 广义泛函空间的矩刻画与正广义泛函

本节将给出广义泛函空间 $(E)_{\mathbb{C}}^{-\beta}\,(0\le\beta<+\infty)$ 的统一刻画 — "矩刻画", 并利用这一刻画证明当 $0\le\beta\le1$ 时 $(E)^{-\beta}$ 中的正广义泛函可以表现为测度.

4.1 重正化算子

下面将要定义的重正化算子是证明广义泛函空间矩刻画定理的一个关键. "重正化" 概念来源于量子物理. 在白噪声分析框架下, 所谓重正化就是把普通乘积变成 Wick 积. 于是, 若设 $\varphi=\prod_{i=1}^n W_{\xi_i}$, $\xi_i\in H_{\mathbb{C}}$, 则 φ 的重正化 (记为 $R\varphi$) 就是

$$W_{\xi_1}\diamond\cdots\diamond W_{\xi_n}=I_n(\widehat{\otimes}_{i=1}^n\xi_i).$$

现在我们试图将算子 R 的定义推广到一切检验泛函空间 $(E)_{\mathbb{C}}^{\beta}(0\le\beta<\infty)$ 中, . 为此, 我们先观察如下事实: 设 $\xi_i\in E_{\mathbb{C}}$, $1\le i\le n$, $\varphi=\prod_{i=1}^n W_{\xi_i}$. 令 $\mathcal{E}_x$ 为 §4 中定义的广义指数泛函, 则有

$$\begin{aligned}\langle\langle R\varphi,\mathcal{E}_x\rangle\rangle&=\langle\langle I_n(\widehat{\otimes}_{i=1}^n\xi_i),\frac{1}{n!}I_n(x^{\otimes n})\rangle\rangle\\&=\prod_{i=1}^n\langle\xi_i,x\rangle=\varphi(x)\ ,\ \forall x\in E^*\ .\end{aligned}\tag{4.1}$$

另一方面，由例 2.23 关于 δ- 泛函 δ_x 的定义及定理 3.4 下面的例子，我们有

$$\langle\langle\varphi,\mathcal{E}_x\diamond F(0)\rangle\rangle=\langle\langle\varphi,\delta_x\rangle\rangle=\varphi(x)\ ,\ \forall x\in E^*\ . \tag{4.2}$$

比较 (4.1) 及 (4.2) 式得

$$\langle\langle R\varphi,\mathcal{E}_x\rangle\rangle=\langle\langle\varphi,\mathcal{E}_x\diamond F(0)\rangle\rangle\ . \tag{4.3}$$

由于 $\{\mathcal{E}_x,x\in E^*\}$ 张成的线性空间在每个 $(E)^{-\beta}$ 中稠 $(0\leq\beta<\infty)$，故由定理 3.5 推知

$$\langle\langle R\varphi,G\rangle\rangle=\langle\langle\varphi,G\diamond F(0)\rangle\rangle\ ,\ \forall G\in(E)_{\mathbb{C}}^{-\beta}\ . \tag{4.4}$$

进一步，我们考虑如下形式的泛函：

$$\varphi=f(W_{\xi_1},\cdots,W_{\xi_n})\ ,\ n\in\mathbb{N},\xi_1,\cdots,\xi_n\in E_{\mathbb{C}}\ . \tag{4.5}$$

其中 f 为 n 个变量的多项式，即 φ 为多项式光滑泛函．我们用 $\mathcal{P}$ 表示多项式光滑泛函全体．显然有 $\mathcal{P}\subset\bigcap_{\beta\geq 0}(E)_{\mathbb{C}}^{\beta}$，且 $\mathcal{P}$ 在每个 $(E)_{\mathbb{C}}^{\beta}$ 中稠．算子 R 的定义可以自然推广到 $\mathcal{P}$ 上，使之成为一个线性映射．于是 (4.4) 对一切 $\varphi\in\mathcal{P}$ 成立．

下一定理表明：$\forall\beta\in[0,\infty)$，$R$ 可以延拓成为从 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 上的连续线性算子．我们称 R 为**重正化算子**．

定理 4.1 设 $0\leq\beta<\infty$．则在 $\mathcal{P}$ 上定义的满足 (4.4) 的线性算子 R 可以唯一地延拓成为从 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 上的连续线性算子，且 (4.4) 成立．其逆映射 R^{-1} 也是从 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 上的连续线性算子，并且有

$$\langle\langle R^{-1}\varphi,G\rangle\rangle=\langle\langle\varphi,G\diamond F(\sqrt{2})\rangle\rangle\ ,\ \forall\varphi\in(E)_{\mathbb{C}}^{\beta},\ G\in(E)_{\mathbb{C}}^{-\beta}\ . \tag{4.6}$$

证明 设 $p>0$, $q\geq 0$, 使得 $2^q>\|I_{0p}\|_{\mathrm{HS}}^2$．令 $\epsilon>(1-\beta)^+$，则对任何 $\varphi\in\mathcal{P}$，由 (4.4) 及 (3.17) 得

$$\begin{aligned}\|R\varphi\|_{p,q,\beta}&=\sup_{\|G\|_{-p,-q,-\beta}=1}\left|\langle\langle R\varphi,G\rangle\rangle\right|\\&\leq\|\varphi\|_{p,q+\epsilon,\beta}\sup_{\|G\|_{-p,-q,-\beta}=1}\|G\diamond F(0)\|_{-p,-(q+\epsilon),-\beta}\\&\leq\|\varphi\|_{p,q+\epsilon,\beta}(1-2^{-\epsilon+(1-\beta)^+})^{-1/2}\|F(0)\|_{-p,-q,-\beta}\ .\end{aligned}$$

于是 R 为从 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 中的连续线性算子.

下面证明 R 为 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 上的单射, 且 (4.6) 成立. 首先, 由 (1.3) 及 (1.6) 知, R 为从 $\mathcal{P}$ 到 $\mathcal{P}$ 上的单射. 此外, 设 $\varphi = I_n(\otimes_{i=1}^{n}\xi_i)$, 其中 $\xi_i \in H_{\mathbb{C}}$, $1 \le i \le n$. 则由 (4.4) 有 (注意 $F(\sqrt{2}) \diamond F(0) = 1$)

$$
\begin{aligned}
\langle\langle R^{-1}\varphi, \mathcal{E}_x\rangle\rangle &= \langle\langle R^{-1}\varphi, \mathcal{E}_x \diamond F(\sqrt{2}) \diamond F(0)\rangle\rangle \\
&= \langle\langle \varphi, \mathcal{E}_x \diamond F(\sqrt{2})\rangle\rangle \, .
\end{aligned}
$$

由于 $\{\mathcal{E}_x, x \in E_{\mathbb{C}}^*\}$ 张成的线性空间在每个 $(E)_{\mathbb{C}}^{-\beta}$ 中稠 $(0 \le \beta < \infty)$, 故由定理 3.5 推知 (4.6) 对一切 $\varphi \in \mathcal{P}$ 及 $G \in (E)_{\mathbb{C}}^{-\beta}$ 成立. 进一步, 与算子 R 情形类似, R^{-1} 也可延拓成为从 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 上的连续线性算子, 并且 (4.6) 成立. ∎

注 1 设 $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $\varphi \sim \{f_n\}, R\varphi \sim \{g_n\}, R^{-1}\varphi \sim \{h_n\}$. 请读者自行验证如下两个公式:

$$
g_n = \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^k \frac{(n+2k)!}{k!2^k}\langle \tau^{\otimes k}, f_{n+2k}\rangle \, , \tag{4.7}
$$

$$
h_n = \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!2^k}\langle \tau^{\otimes k}, f_{n+2k}\rangle \, . \tag{4.8}
$$

这里两个级数都在 $E_{\mathbb{C}}^{\widehat{\otimes} n}$ 中收敛, $\langle \tau^{\otimes k}, f_{n+2k}\rangle$ 为 $E_{\mathbb{C}}^{\widehat{\otimes} n}$ 中的元素, 使得 $\forall w_n \in E^{*\widehat{\otimes} n}_{\mathbb{C}}$, 有

$$
\langle\langle \tau^{\otimes k}, f_{n+2k}\rangle, w_n\rangle = \langle \tau^{\otimes k} \otimes w_n, f_{n+2k}\rangle \, .
$$

注 2 设 $\varphi \in (E)_{\mathbb{C}}^{\beta}$. 令 $p_0 \in \mathbb{N}$, 使得从 H_{p_0} 到 H 的嵌入映射为迹算子. 则由 (1.4),(1.5) 及定理 2.18 可以证明:

$$
R\varphi(x) = \int_{E^*} \varphi(x \pm iy)\mu(dy) \ , \ x \in H_{-p_0} \, , \tag{4.9}
$$

$$
R^{-1}\varphi(x) = \int_{E^*} \varphi(x + y)\mu(dy) \ , \ x \in H_{-p_0} \, . \tag{4.10}
$$

注 3 设 $p>0,\ q\geq 0$, 使得 $2^q>\|I_{0p}\|_{\mathrm{HS}}^2$. 令 $\epsilon>(1-\beta)^+$, 则由定理 4.1 的证明知, R 可以延拓成为从 $(H_{p,q+\epsilon,\beta})_{\mathbb{C}}$ 到 $(H_{p,q,\beta})_{\mathbb{C}}$ 中的有界线性算子.

4.2 广义泛函空间的矩刻画

在本小节我们恒假定 $0\leq\beta<\infty$.

定义 4.2 设 $\Phi\in(E)_{\mathbb{C}}^{-\beta}$. 令 $M_0^\Phi=\langle\langle\Phi,1\rangle\rangle$. 对 $n\geq 1$, 令 M_n^Φ 为如下定义的 E^n 上的对称 n- 线性型:

$$M_n^\Phi(f_1,\cdots,f_n)=\langle\langle\Phi,W_{f_1}\cdots W_{f_n}\rangle\rangle\ ,\ f_1,\cdots,f_n\in E\ . \tag{4.11}$$

我们称 M_n^Φ 为 Φ 的 **n- 阶矩**.

下一定理给出了 $(E)_{\mathbb{C}}^{-\beta}$ 的矩刻画.

定理 4.3 设 $0\leq\beta<\infty$, $M_0\in\mathbb{C}$, $\forall n\geq 1, M_n$ 为 E^n 上的对称 n- 线性型, $\widehat{M}_n$ 为 M_n 对应的 n- 齐次多项式. 为了存在某个 $\Phi\in(E)_{\mathbb{C}}^{-\beta}$, 使得 $\forall n\geq 0$, M_n 为 Φ 的 n- 阶矩, 必须且只需存在 $p\geq 0, C>0, K>0$, 使得 $\forall n\in\mathbb{N}, \forall f\in E$,$\widehat{M}_n(f)$ 有如下估计:

$$|\,\widehat{M}_n(f)\,|\leq KC^n(n!)^{\frac{1+\beta}{2}}\mid f\mid_p^n\ . \tag{4.12}$$

证明 必要性. 设存在 $\Phi\in(E)_{\mathbb{C}}^{-\beta}$, 使得 $M_n=M_n^\Phi$. 则存在 $p\geq 0, q\geq 0$, 使得 $\Phi\diamond F(\sqrt{2})\in(H_{-p,-q,-\beta})_{\mathbb{C}}$. 故由 (4.11) 及 (4.6) 得

$$\begin{aligned}|\,\widehat{M}_n(f)\,|&=\left|\langle\langle\Phi,R^{-1}\Big(I_n(f^{\otimes n})\Big)\rangle\rangle\right|\\&=\left|\langle\langle\Phi\diamond F(\sqrt{2}),I_n(f^{\otimes n})\rangle\rangle\right|\\&\leq\|\Phi\diamond F(\sqrt{2})\|_{-p,-q,-\beta}\|I_n(f^{\otimes n})\|_{p,q,\beta}\\&\leq\|\Phi\diamond F(\sqrt{2})\|_{-p,-q,-\beta}(n!)^{\frac{1+\beta}{2}}2^{nq}|f|_p^n\ ,\end{aligned}$$

由此推得 (4.12).

充分性. 假设存在 $p \ge 0, C > 0, K > 0$, 使得 (4.12) 成立. 令 $p' > p$, 使得从 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 HS 算子, 则由核定理 (第一章定理 3.17) 知, 存在 $M^{(n)} \in H_{-p',\mathbb{C}}^{\widehat{\otimes}n}$, 使得

$$M_n(f_1, \cdots, f_n) = \langle M^{(n)}, f_1 \otimes \cdots \otimes f_n \rangle \ , \ n \ge 1 \ , \tag{4.13}$$

并由 (4.12) 推得 (参看引理 2.7 及 2.8 的证明)

$$\| M^{(n)} \|_{-p'}^2 \le K^2 e^{2n} C^{2n} (n!)^{1+\beta} \| I_{pp'} \|_{\mathrm{HS}}^{2n} \ . \tag{4.14}$$

令 $\Psi \sim \{(n!)^{-1} M^{(n)}\}$, 其中 $M^{(0)} \equiv M_0$, 则 $\Psi \in (E)_{\mathbb{C}}^{-\beta}$. 事实上, 由 (4.14) 知, 当 $q > 0$ 使得 $2^q > e^2 C^2 \| I_{pp'} \|_{\mathrm{HS}}^2$ 时, 有 $\Psi \in (H_{-p',-q,-\beta})_{\mathbb{C}}$. 令 $\Phi = \Psi \diamond F(0)$, 则 $\langle\langle \Phi, 1 \rangle\rangle = M_0$, 并由 (4.13) 及 (4.4) 推得: $\forall n \ge 1, f_k \in E_{\mathbb{C}}, 1 \le k \le n$, 我们有

$$\begin{aligned}
\langle\langle \Phi, W_{f_1} \cdots W_{f_n} \rangle\rangle &= \langle\langle \Psi \diamond F(0), R^{-1} I_n(\widehat{\otimes}_{k=1}^n f_k) \rangle\rangle \\
&= \langle\langle \Psi, I_n(\widehat{\otimes}_{k=1}^n f_k) \rangle\rangle \\
&= \langle M^{(n)}, \widehat{\otimes}_{k=1}^n f_k \rangle \\
&= M_n(f_1, \cdots, f_n) \ .
\end{aligned}$$

这表明 M_n 是 Φ 的 n- 阶矩. 定理证毕. ∎

注 设 M_n 为 E^n 上的连续对称 n- 线性型. 由核定理, 存在 $M^{(n)} \in E^{*\widehat{\otimes}n}$, 使得 (4.13) 成立. 我们称 $M^{(n)}$ 为 M_n 的 **核**.

下一定理通过广义泛函的矩估计 ((4.12) 式) 给出广义泛函的范数估计.

定理 4.4 设 $0 \le \beta < \infty, \Phi \in (E)_{\mathbb{C}}^{-\beta}$, $\{M_n, n \ge 0\}$ 为其矩序列. 假定 (4.12) 成立. 令 $p' > p$ 使得 $\|I_{pp'}\|_{\mathrm{HS}} < \infty$, 则当 $\epsilon > (1-\beta)^-$ 及 $q > \log_2[\|I_{0p'}\|_{\mathrm{HS}}^2 \vee e^2 C^2 \|I_{pp'}\|_{\mathrm{HS}}^2]$, 我们有

$$\begin{aligned}
&\|\Phi\|_{-p',-(q+\epsilon),-\beta} \\
&\le K[(1-2^{-\epsilon+(1-\beta)^-})(1-2^{-q}\|I_{0p'}\|_{\mathrm{HS}}^2)(1-2^{-q}e^2C^2\|I_{pp'}\|_{\mathrm{HS}}^2)]^{-1/2}.
\end{aligned} \tag{4.15}$$

证明 令 $M^{(n)}$ 为 M_n 的核， $\Psi \sim \{(n!)^{-1}M^{(n)}\}$，则由 (4.14) 得

$$\|\Psi\|_{-p',-q',-\beta} \leq K(1-2^{-q}e^2C^2\|I_{pp'}\|_{\mathrm{HS}}^2)^{-1/2} . \tag{4.16}$$

由定理 4.3 知 $\Phi = \Psi \diamond F(0)$. 于是由 (4.16) 及 (3.17) 推得 (4.15). ∎

下面两个定理是定理 4.4 的重要推论，其证明从略 (参见定理 2.10 及 2.11 的证明).

定理 4.5 设 $0 \leq \beta < \infty$,$\{\Phi_k, k \in \mathbb{N}\}$ 为 $(E)_{\mathbb{C}}^{-\beta}$ 中元素序列，如果满足下列条件：

1) $\forall f \in E, \forall n \geq 0, \{\langle\langle \Phi_k, W_f^n \rangle\rangle, k \in \mathbb{N}\}$ 为 $\mathbb{C}$ 中 Cauchy 列；

2) 存在 $p \in \mathbb{N}_0; K > 0$ 及 $C > 0$, 使得

$$|\langle\langle \Phi_k, W_f^n \rangle\rangle| \leq KC^n(n!)^{\frac{1+\beta}{2}}|f|_p^n, \ \forall f \in E, n \geq 0, k \geq 1, \tag{4.17}$$

则 $\{\Phi_k, k \in \mathbb{N}\}$ 在 $(E)_{\mathbb{C}}^{-\beta}$ 中强收敛.

定理 4.6 设 $0 \leq \beta < \infty, (\Omega, \mathcal{F}, \nu)$ 为一测度空间， $\omega \longmapsto \Phi_\omega$ 为 Ω 到 $(E)_{\mathbb{C}}^{-\beta}$ 中的映射. 如果满足下列条件：

1) $\forall f \in E, n \geq 0, \omega \longmapsto \langle\langle \Phi_\omega, W_f^n \rangle\rangle$ 为 $(\Omega, \mathcal{F})$ 上的可测映射；

2) 存在常数 $K > 0, C > 0, p \in \mathbb{N}_0$ 及 Ω 上非负 ν- 可积函数 $G(\omega)$, 使得对 ν-a.e. ω,

$$|\langle\langle \Phi_\omega, W_f^n \rangle\rangle| \leq KG(\omega)C^n(n!)^{\frac{1+\beta}{2}}|f|_p^n, \forall f \in E, n \geq 0 , \tag{4.18}$$

则存在 $p', q \in \mathbb{N}$, 使得 $\omega \longmapsto \Phi_\omega$ 在 $(H_{-p',-q,-\beta})$ 中 Bochner 可积，且有

$$\langle\langle \int_\Omega \Phi_\omega d\nu(\omega), W_f^n \rangle\rangle = \int_\Omega \langle\langle \Phi_\omega, W_f^n \rangle\rangle d\nu(\omega), \forall f \in E, n \geq 0 . \tag{4.19}$$

4.3 正广义泛函的测度表示

在经典的 Schwartz 广义函数论中，正广义函数与一类 Borel 测度相对应. 下面将把此结果推广到无穷维情形，并将证明：对

一切 $\beta \in [0,1]$, 任一正广义泛函 $\Phi \in (E)^{-\beta}$ 对应于 $(E^*, \mathcal{B}(E^*))$ 上的一有限测度 ν, 使得

$$\int_{E^*} \varphi(x)\nu(dx) = \langle\langle \Phi, \varphi \rangle\rangle \ , \ \forall \varphi \in (E)^{\beta} \ . \tag{4.20}$$

此外，对 $0 \le \beta < +\infty$ 情形，我们还将给出能按 (4.20) 对应一正广义泛函的测度 ν 的刻画.

定义 4.7 设 $\Phi \in (E)^{-\beta}$. 如果对一切非负检验泛函 $\varphi \in (E)^{\beta}$, 有 $\langle\langle \Phi, \varphi \rangle\rangle \ge 0$, 则称 Φ 为 **正广义泛函**.

今后我们用 $(E)^{\beta}_{+}$ 表示 $(E)^{\beta}$ 中非负元素全体，用 $(E)^{-\beta}_{+}$ 表示 $(E)^{-\beta}$ 中正广义泛函全体.

例 设 $y \in E^*$, 则 $\delta_y \in (E)^*_+$. 这是因为 $\forall \varphi \in (E)_+$, $\langle\langle \varphi, \delta_y \rangle\rangle = \varphi(y) \ge 0$.

下一引理是 Berezansky-Kondratiev[1] 中第五章定理 2.2 的一个简单推论，其证明从略.

引理 4.8 设 $0 \le \beta \le 1$, $M_0 \in \mathbb{R}$, $\forall n \ge 1, M_n$ 为 E^n 上的对称 n- 线性型，$\widehat{M_n}$ 为 M_n 对应的 n- 齐次多项式，使得 $\forall n \in \mathbb{N}$, $\forall f \in E$,$\widehat{M_n}(f)$ 有 (4.12) 的估计. 如果序列 $\{M_n, n \ge 0\}$ 为正定的，即 $M_0 \ge 0$, 且对任何 $n \ge 1, g_k \in E^{\widehat{\otimes} k}, 1 \le k \le n$, 有

$$\sum_{k,j=1}^{n} \langle M^{(k+j)}, g_k \otimes g_j \rangle \ge 0 \ , \tag{4.21}$$

其中 $M^{(n)}$ 为 M_n 的核 (见定理 4.3 的注), 则存在 $(E^*, \mathcal{B}(E^*))$ 上的一有限测度 ν, 使得 $\forall n \ge 0, M_n$ 为 ν 的 n- 阶矩，即有

$$\int_{E^*} W_{f_1} \cdots W_{f_n} \nu(dx) = M_n(f_1, \cdots, f_n) \ , \ \forall f_1, \cdots, f_n \in E \ . \tag{4.22}$$

下一定理表明：当 $0 \le \beta \le 1$ 时 $(E)^{-\beta}$ 中的正广义泛函可以通过 $(E^*, \mathcal{B}(E^*))$ 上一有限测度来表现.

定理 4.9 设 $0 \leq \beta \leq 1, \Phi \in (E)_+^{-\beta}$. 则存在 $(E^*, \mathcal{B}(E^*))$ 上的一有限测度 ν, 使得 (4.20) 成立.

证明 令

$$\varphi(x) = \sum_{k=0}^{n} \langle g_k, x^{\otimes k} \rangle , \quad x \in E^* .$$

则由定理 2.18 知, $\varphi \in (E)^{\beta}$. 故由 Φ 的正性得

$$\sum_{k,j=1}^{n} \langle M^{(k+j)}, g_k \otimes g_j \rangle = \langle\langle \Phi, \varphi^2 \rangle\rangle \geq 0 ,$$

其中 $M^{(n)}$ 为 M_n^{Φ} 的核. 由引理 4.8 知, 存在 $(E^*, \mathcal{B}(E^*))$ 上一有限测度 ν, 使得 $\forall n \geq 0, M_n^{\Phi}$ 为 ν 的 n- 阶矩, 即有 (4.22). 现设 $\varphi \in (E)^{\beta}$. 选取 $\mathcal{P}$ 中元素序列 $\{\varphi_n, n \geq 1\}$, 使得 $\{\varphi_n\}$ 在 $(E)^{\beta}$ 中收敛于 φ. 由系 3.3 知, $\{\varphi_n^2\}$ 在 $(E)^{\beta}$ 中收敛于 φ^2. 假定 $\Phi \in (H_{-p,-q,-\beta})$. 由于 $\{\varphi_n^2\}$ 在 $(H_{p,q,\beta})$ 中收敛于 φ^2, 故有

$$\begin{aligned} \lim_{n\to\infty} \int_{E^*} \varphi_n^2 d\nu &= \lim_{n\to\infty} \langle\langle \Phi, \varphi_n^2 \rangle\rangle \\ &= \langle\langle \Phi, \varphi^2 \rangle\rangle < \infty , \end{aligned}$$

这蕴含序列 $\{\varphi_n, n \geq 1\}$ 关于测度 ν 为一致可积的. 另一方面, 我们有

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \langle\langle \varphi_n, \delta_x \rangle\rangle = \langle\langle \varphi, \delta_x \rangle\rangle = \varphi(x) , \forall x \in E^* .$$

因此由 (4.22) 得

$$\int_{E^*} \varphi d\nu = \lim_{n\to\infty} \int_{E^*} \varphi_n d\nu = \lim_{n\to\infty} \langle\langle \Phi, \varphi_n \rangle\rangle = \langle\langle \Phi, \varphi \rangle\rangle .$$

定理证毕. ∎

注 若 $0 \leq \beta < 1$, 则 $\forall f \in E$, 指数泛函 $\mathcal{E}_f \in (E)^{\beta}$. 这时我们也可以借助于 Minlos 定理来证明定理 4.9. 但对 $\beta = 1$ 情形, 我们

只能用引理 4.8 来证明它．这里给出的证明统一适用于这两种情形．

在实际应用中一个重要问题是：如何刻画 $(E^*,\mathcal{B}(E^*))$ 上那些可以通过 (4.20) 式表现为正广义泛函的测度？下一定理回答了这一问题．

定理 4.10 设 $0\le\beta<+\infty$, ν 为 $(E^*,\mathcal{B}(E^*))$ 上的一有限测度．为要存在某 $\Phi\in(E)^{-\beta}_{+}$, 使得 (4.20) 成立，必须且只需 ν 满足如下条件：

(C.1) 存在 $p\ge 0, C>0, K>0$, 使得

$$\int_{E^*}\langle f,x\rangle^{2n}\nu(dx)\le KC^{2n}((2n)!)^{\frac{1+\beta}{2}}|f|_p^{2n}\ ,\ \forall f\in E\ ,\forall n\ge 0.\qquad(4.23)$$

证明 由定理 4.3, 条件 (C.1) 的必要性是显然的，往证充分性．设条件 (C.1) 成立．由 Schwarz 不等式及 $(2n)!\le 2^{2n}(n!)^2$ 得 (注意 $\nu(E^*)\le K$)

$$\begin{aligned}\int_{E^*}|\langle f,x\rangle|^n\ \nu(dx)&\le\nu(E^*)^{1/2}\Big[\int_{E^*}\langle f,x\rangle_{-p}^{2n}\nu(dx)\Big]^{1/2}\\&\le K(C2^{\frac{1+\beta}{2}})^n(n!)^{\frac{1+\beta}{2}}|f|_p^n\ .\end{aligned}\qquad(4.24)$$

令 M_n 为 ν 的 n- 阶矩，即

$$M_n(f_1,\cdots,f_n)=\int_{E^*}\prod_{i=1}^n\langle f_i,x\rangle\nu(dx)\ ,\ \forall f_1,\cdots,f_n\in E\ .$$

则由 (4.24) 及极化公式推得： $\forall f_1,\cdots,f_n\in E$,

$$\begin{aligned}|M_n(f_1,\cdots,f_n)|&\le KC_1^n\frac{n^n}{n!}(n!)^{\frac{1+\beta}{2}}\prod_{j=1}^n|f_j|_p\\&\le K(eC_1)^n(n!)^{\frac{1+\beta}{2}}\prod_{j=1}^n|f_j|_p\ ;\end{aligned}\qquad(4.25)$$

其中 $C_1=C2^{\frac{1+\beta}{2}}$. 故由定理 4.3, 存在 $\Phi\in(E)^{-\beta}$, 使得 M_n 为 Φ 的 n- 阶矩, 即有

$$\langle\langle\Phi,\prod_{i=1}^{n}\langle f_i,\cdot\rangle\rangle\rangle=\int_{E^*}\prod_{i=1}^{n}\langle f_i,x\rangle\nu(dx)\ .$$

于是进一步有

$$\langle\langle\Phi,\varphi\rangle\rangle=\int_{E^*}\varphi d\nu\ ,\ \forall\varphi\in\mathcal{P}\ ,\tag{4.26}$$

其中 $\mathcal{P}$ 为多项式光滑泛函全体. 由此根据定理 4.9 的证明推知 (4.20) 成立. 特别, 这蕴含 $\Phi\in(E)_{+}^{-\beta}$, 证毕. ∎

下一定理给出了条件 (C.1) 的一个等价形式, 它似乎对应用更加方便.

定理 4.11 任取 $p_0\in\mathbb{N}$ 使得从 H_{p_0} 到 H 的嵌入映射为迹算子 (即使得 $\mu(H_{-p_0})=1$). 设 ν 为 $\big(E^*,\mathcal{B}(E^*)\big)$ 上的一有限测度. 则定理 4.10 中的条件 (C.1) 等价于如下条件:

(C.1)$'$ 存在 $p\geq p_0,\epsilon>0$ 使得

$$\int_{E^*}\exp\Big\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\Big\}\nu(dx)<\infty.\tag{4.27}$$

证明 设 (C.1) 成立. 令 $p'>p_0,p'>p$, 使得 $H_{p'}$ 到 H_p 的嵌入映射 $I_{pp'}$ 为 Hilbert-Schmidt 算子. 令 $\{e_j,j\geq1\}\subset E$ 为 $H_{p'}$ 的一个基, 则由 (4.23) 有

$$\begin{aligned}\int_{E^*}|x|_{-p'}^{2n}\nu(dx)&=\int_{E^*}\Big(\sum_{j=1}^{\infty}\langle x,e_j\rangle^2\Big)^n\nu(dx)\\&=\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\int_{E^*}\langle x,e_{k_1}\rangle^2\cdots\langle x,e_{k_n}\rangle^2\nu(dx)\\&\leq K(eC_1)^{2n}[(2n)!]^{\frac{1+\beta}{2}}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\prod_{j=1}^{n}|\,e_{k_j}\,|_p^2\\&=K(eC_1)^{2n}[(2n)!]^{\frac{1+\beta}{2}}\|I_{pp'}\|_{\mathrm{HS}}^{2n}\ .\end{aligned}\tag{4.28}$$

于是，当 $0<\epsilon<\frac{1}{2}(eC\|I_{pp'}\|_{\mathrm{HS}})^{\frac{2}{1+\beta}}$ 时，我们由 (4.28) 得

$$\begin{aligned}
\int_{E^*}\exp\left\{\epsilon|x|_{-p'}^{\frac{2}{1+\beta}}\right\}\nu(dx) &= \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\int_{E^*}|x|_{-p'}^{\frac{2n}{1+\beta}}\nu(dx)\\
&\le \sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}[\nu(E^*)]^{\frac{\beta}{1+\beta}}\left(\int_{E^*}|x|_{-p'}^{2n}\nu(dx)\right)^{\frac{1}{1+\beta}}\\
&\le [\nu(E^*)]^{\frac{\beta}{1+\beta}}\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\left(eC\|I_{pp'}\|_{\mathrm{HS}}\right)^{\frac{2n}{1+\beta}}[(2n!)]^{\frac{1}{2}}\\
&\le [\nu(E^*)]^{\frac{\beta}{1+\beta}}\sum_{n=0}^{\infty}[2\epsilon(eC\|I_{pp'}\|_{\mathrm{HS}})^{\frac{2}{1+\beta}}]^n<\infty.
\end{aligned}$$

因此，(C.1)⇒(C.1)$'$ 得证.

反之，设 (C.1)$'$ 成立．则由定理 2.18 知，如下定义的 $(E)^\beta$ 上的泛函是连续的：

$$L(\varphi)=\int_{E^*}\varphi(x)\nu(dx)\ ,\ \varphi\in(E)^\beta.$$

于是，存在 $\Phi\in(E)_+^{-\beta}$，使得 (4.20) 成立．因此，由定理 4.10 知条件 (C.1) 成立. ∎

定义 4.12 设 $0\le\beta<+\infty,\nu$ 为 $(E^*,\mathcal{B}(E^*))$ 上的一有限测度，满足定理 4.10 中的条件 (C.1)$'$(或定理 4.10 中的条件 (C.1)). 我们称与 ν 联系的正广义泛函 Φ(即满足 (4.20) 式的 Φ) 为 ν 关于 Gauss 测度 μ 的 **(广义)Radon-Nikodym 导数**, 记为 $d\nu/d\mu$.

今后我们用 $\mathcal{M}^\beta(E^*)$ 表示 $(E^*,\mathcal{B}(E^*))$ 上满足条件 (C.1) 或 (C.1)$'$ 的测度全体.

定理 4.13 设 $0\le\beta<+\infty,\nu$ 为 $(E^*,\mathcal{B}(E^*))$ 上的一有限测度.

1) 假定 ν 满足定理 4.10 中的条件 (C.1). 令 $p'>p$ 满足 $\|I_{pp'}\|_{\mathrm{HS}}<\infty$. 如果 $\epsilon>(1-\beta)^+$ 且

$$q>\log_2(\|I_{0p'}\|_{\mathrm{HS}}^2)\vee e^2C^22^{1+\beta}\|I_{pp'}\|_{\mathrm{HS}}^2,$$

则有

$$\|\frac{d\nu}{d\mu}\|_{-p',-(q+\epsilon),-\beta} \leq K\nu(E^*)^{1/2}(1-2^{-\epsilon+(1-\beta)^+})^{-1/2}$$
$$\times\left[(1-2^{-q}\|I_{0p'}\|_{\mathrm{HS}}^2)(1-2^{1+\beta-q}e^2C^2\|I_{pp'}\|_{\mathrm{HS}}^2)\right]^{-1/2}. \tag{4.29}$$

2) 假定 ν 满足定理 4.11 中的条件 (C.1)$'$. 如果 q 足够大，使得 (2.42) 成立，且使 $\frac{1+\beta}{2}2^{\frac{(1-\beta)^+-q}{1+\beta}} \leq \epsilon$, 则有

$$\|\frac{d\nu}{d\mu}\|_{-p,-q,-\beta} \leq \int_{E^*}\exp\left\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\right\}\nu(dx)$$
$$\times\left(\int_{E^*}\exp\left\{(1+\beta)2^{\frac{(1-\beta)^+-q}{1+\beta}}|x|_{-p}^{\frac{2}{1+\beta}}\right\}\mu(dx)\right)^{1/2}. \tag{4.30}$$

证明 1) 由 (4.24) 及定理 4.4 直接推得. 往证 2). 设 $\varphi \in (E)^\beta$. 由 (2.43) 得

$$|\varphi(x)| \leq \|\varphi\|_{p,q,\beta}C_1\exp\left\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\right\},$$

其中 C_1 由 (2.42) 给出. 于是我们有

$$\left|\langle\langle\frac{d\nu}{d\mu},\varphi\rangle\rangle\right| = \left|\int_{E^*}\varphi(x)\nu(dx)\right|$$
$$\leq \|\varphi\|_{p,q,\beta}C_1\int_{E^*}\exp\left\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\right\}\nu(dx),$$

由此推得 (4.30). ∎

作为定理 4.13 及定理 4.9 的一个推论，我们有

定理 4.14 设 $0 \leq \beta \leq 1, \{\nu_n, n \geq 1\}$ 为 $(E^*, \mathcal{B}(E^*))$ 上一列有限测度. 如果

(1) $\forall f \in E, k \in \mathbb{N}_0, \{\nu_n(W_f^k), n \geq 1\}$ 为 $\mathbb{R}$ 中的 Cauchy 列;

(2) 下列两个 (等价) 条件之一成立:

(2a) 存在 $p \geq 0, K > 0, C > 0$, 使得

$$\left| \int_{E^*} \langle f, x\rangle^{2k} \nu_n(dx) \right| \leq K^2 C^{2k} \Big((2k)! \Big)^{\frac{2-\beta}{2}} |f|_p^{2k},$$
$$\forall f \in E, k \geq 1, n \geq 1, \tag{4.31}$$

(2b) 存在 $p > 0, \epsilon > 0, K > 0$, 使得

$$\int_{E^*} \exp\left\{ \epsilon |x|_{-p}^{\frac{2}{1+\beta}} \right\} \nu_n(dx) \leq K, \ \forall n \geq 1, \tag{4.32}$$

则存在 $(E^*, \mathcal{B}(E^*))$ 上一有限测度 ν, 使得

$$\lim_{n\to\infty} \int_{E^*} \varphi d\nu_n = \int_{E^*} \varphi d\nu, \ \forall \varphi \in (E)^\beta. \tag{4.33}$$

证明 一方面, 条件 (1) 蕴含 (4.33) 对一切 $\varphi \in \mathcal{P}$ 成立. 另一方面, 由定理 4.13 知, 条件 (2a) 或 (2b) 蕴含 $\{d\nu_n/d\mu, n \geq 1\}$ 在 $(E)^{-\beta}$ 中有界, 故 $\{d\nu_n/d\mu, n \geq 1\}$ 在 $(E)^{-\beta}$ 中强收敛于某元素 Φ. 显然 Φ 为正的, 故由定理 4.9,Φ 对应于 $(E^*, \mathcal{B}(E^*))$ 上的一有限测度 ν, 即 $\Phi = d\nu/d\mu$. 于是 (4.33) 成立. ∎

设 ν_1, ν_2 为 $(E^*, \mathcal{B}(E^*))$ 上两个有限测度. 令

$$\nu_1 * \nu_2(B) = \int_{E^*} \nu_1(B - x)\nu_2(dx), \ \forall B \in \mathcal{B}(E^*), \tag{4.34}$$

称 $\nu_1 * \nu_2$ 为 ν_1 与 ν_2 的 **卷积**.

下一定理表明, $\mathcal{M}^\beta(E^*)$ 对卷积封闭.

定理 4.15 设 $0 \leq \beta < +\infty, \nu_1, \nu_2 \in \mathcal{M}^\beta(E^*)$, 则 $\nu_1 * \nu_2 \in \mathcal{M}^\beta(E^*)$, 且有

$$\frac{d(\nu_1 * \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} \diamond \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), \tag{4.35}$$

其中 $F(\sqrt{2})$ 如例 2.22 定义.

证明 设 $\Phi \in (E)^{-\beta}$. 令 M_n^{Φ} 为它的 n- 阶矩，则 $\forall f \in E$ 有

$$\widehat{M_n^{\Phi}}(f) = \langle\langle \Phi, W_f^n \rangle\rangle = \langle\langle \Phi \diamond F(\sqrt{2}), I_n(f^{\otimes n}) \rangle\rangle. \tag{4.36}$$

我们分别用 M_n^1, M_n^2 及 M_n 分别表示 ν_1, ν_2 及 $\nu_1 * \nu_2$ 的 n- 阶矩. 则 $\forall f \in E$, 由 (4.36) 及 (3.16) 得

$$\begin{aligned}
\widehat{M_n}(f) &= \int_{E^*} \langle f, z \rangle^n (\nu_1 * \nu_2)(dx) \\
&= \int_{E^*} \int_{E^*} \langle f, x+y \rangle^n \nu_1(dx)\nu_2(dy) \\
&= \int_{E^*} \int_{E^*} (\langle f, x \rangle + \langle f, y \rangle)^n \nu_1(dx)\nu_2(dy) \\
&= \sum_{k=0}^{n} \binom{n}{k} \widehat{M_k^1}(f) \widehat{M_{n-k}^2}(f) \\
&= \sum_{k=0}^{n} \binom{n}{k} \langle\langle \frac{d\nu_1}{d\mu} \diamond F(\sqrt{2}), I_k(f^{\otimes k}) \rangle\rangle \\
&\qquad \cdot \langle\langle \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), I_{n-k}(f^{\otimes n-k}) \rangle\rangle \\
&= \langle\langle \frac{d\nu_1}{d\mu} \diamond F(\sqrt{2}) \diamond \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), I_n(f^{\otimes n}) \rangle\rangle \\
&= \langle\langle \frac{d\nu_1}{d\mu} \diamond \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), W_f^n \rangle\rangle,
\end{aligned}$$

由此立得定理结论. ∎

定理 4.16 设 $0 \le \beta < +\infty, \nu \in \mathcal{M}^{\beta}(E^*), y \in E^*$, 令

$$\nu_y(B) = \nu(B-y), \ \forall B \in \mathcal{B}(E^*),$$

则 $\nu_y \in \mathcal{M}^{\beta}(E^*)$, 且有

$$\frac{d\nu_y}{d\mu} = \mathcal{E}_y \diamond \frac{d\nu}{d\mu}. \tag{4.37}$$

特别有 $d\mu_y/d\mu = \mathcal{E}_y$.

证明 我们有

$$\begin{aligned}
\widehat{M_n^{\nu_y}}(f) &= \int_{E^*}\langle f,x\rangle^n \nu_y(dx) = \int_{E^*}\langle f,x+y\rangle^n \nu(dx) \\
&= \int_{E^*}\Big(\langle f,x\rangle+\langle f,y\rangle\Big)^n \nu(dx) \\
&= \sum_{k=0}^{n}\binom{n}{k}\widehat{M_k^{\nu}}(f)\langle f,y\rangle^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}\langle\langle \frac{d\nu}{d\mu}\diamond F(\sqrt{2}), I_k(f^{\otimes k})\rangle\rangle\langle\langle \mathcal{E}_y, I_{n-k}(f^{\otimes n-k})\rangle\rangle \\
&= \langle\langle \frac{d\nu}{d\mu}\diamond F(\sqrt{2})\diamond \mathcal{E}_y, I_n(f^{\otimes n})\rangle\rangle.
\end{aligned}$$

由此推得 (4.37). ∎

4.4 应用于 $P(\phi)_2$- 量子场

本节将简要介绍白噪声分析在量子场论中的一个应用——证明 $P(\phi)_2$ 场可通过一个正 Hida 广义泛函来表示 (详见 Potthoff-Streit[2]).

令 $\mathcal{H}$ 为一复 Hilbert 空间, 其内积为 $(\cdot,\cdot)$. 所谓 $I\!R^d$ 上的一个典则场, 是指 $S(I\!R^d)$ 上的一对取值于 $\mathcal{H}$ 中的自共轭算子的线性映射 (ϕ,π), 满足如下 **典则对易关系**:

$$\begin{aligned}
&[\phi(f),\phi(g)] = [\pi(f),\pi(g)] = 0, \\
&[\pi(f),\phi(g)] = \frac{1}{i}(f,g)_0,
\end{aligned} \tag{4.38}$$

其中 $f,g\in S(I\!R^d)$, $(\cdot,\cdot)_0$ 表示 $L^2(I\!R^d)$ 中的内积. 场的动力学由 $\mathcal{H}$ 中的一正的自共轭算子 H(称为 **Hamilton 算子**) 来描述. 通常假定它满足如下对易关系:

$$[H,\phi(f)] = \frac{1}{i}\pi(f). \tag{4.39}$$

如果采用 **Weyl 算子** $U(f)=e^{i\phi(f)}$ 及 $V(f)=e^{i\pi(f)}$, 则 (4.38) 可改写成如下的 **Weyl 关系**:

$$\begin{gathered}U(f)U(g)=U(f+g), \quad V(f)V(g)=V(f+g), \\ U(f)V(g)=V(g)U(f)e^{-i(f,g)_0}.\end{gathered} \tag{4.40}$$

这时 (4.39) 可改写成

$$[H,U(f)]=U(f)\pi(f)+\frac{1}{2}(f,f)_0U(f), \tag{4.41}$$

或者等价地,

$$[U(f),[H,U(g)]]=-(f,g)_0U(f+g). \tag{4.42}$$

此外, 我们还假定存在唯一的 (归一化的)H 的实特征向量 $\Omega\in\mathcal{H}$, 使得 $H\Omega=0$. Ω 称为真空态. 最后假定存在一反酉算子 T(满足 $T^2=I$), 称为时间反转, 使得

$$\begin{gathered}TU(f)T^{-1}=U(-f), \quad TV(f)T^{-1}=V(f), \\ THT^{-1}=H, \quad T\Omega=\Omega.\end{gathered} \tag{4.43}$$

我们称三元组 (ϕ,H,Ω) 为一个 **典则量子场**.

如果我们把 ϕ 及 π 看成算子值分布 $\phi(x),\pi(x)$, 即形式上有

$$\phi(f)=\int\phi(x)f(x)dx, \quad \pi(f)=\int\pi(x)f(x)dx,$$

则可如下定义随时间变化的场:

$$\begin{aligned}\phi(t,x)&=\exp(itH)\phi(x)\exp(-itH), \quad t\in\mathbb{R},\\ \pi(t,x)&=\exp(itH)\pi(x)\exp(-itH), \quad t\in\mathbb{R}.\end{aligned}$$

上述三元组 (ϕ,H,Ω) 连同由时变场生成的代数上的一 Poincaré 群构成相对论性不变的典则量子场论. 构造非平凡的 4- 维典则量子场论至今仍是一个公开问题.

定义 4.17 令 (ϕ,H,Ω) 为一典则量子场. 如果对任意 $f,g\in S(\mathbb{R}^d)$, 有 $U(f)\Omega\in\mathcal{D}(H^{1/2})$, 则由 (4.40) 及 (4.41) 可以推得

$$(U(f)\Omega,HU(g)\Omega)=\frac{1}{2}(f,g)_0(U(f)\Omega,U(g)\Omega). \tag{4.44}$$

称 (4.44) 式为 **Araki关系**.

定义 4.18 如果存在常数 $\alpha,\beta,\gamma\geq 0$ 及 $p\in\mathbb{N}_0$, 使得 $\forall f\in S(\mathbb{R}^d)$, 作为 $\mathcal{D}_f=\{F(\phi(f))\Omega : F\in S(\mathbb{R})\}$ 上的二次型成立

$$\pm\phi(f)\leq\alpha H+\beta|f|_p^2+\gamma, \tag{4.45}$$

其中 $|\cdot|_p$ 为由 d- 维谐振子在 $L^2(\mathbb{R}^d)$ 上定义的范数 (参见第一章 3.2 节), 则称 ϕ- 界成立.

定理 4.19 假定 (ϕ,H,Ω) 满足 Araki 关系且有 ϕ- 界, 则存在一 (正的)Hida 广义泛函 $\Xi\in\left(S(\mathbb{R}^d)\right)^*$, 使得

$$(\Omega,e^{i\phi(f)}\Omega)=\langle\langle\Xi,e^{iW_f}\rangle\rangle,\quad f\in S(\mathbb{R}^d). \tag{4.46}$$

该定理的证明关键是利用 (4.45) 式得到如下估计: 存在常数 K(依赖 α,β,γ) 使得 $\forall n\in\mathbb{N}$

$$|(\Omega,\phi(f)^n\Omega)|\leq K^n\sqrt{n!}|f|_p^n,\quad f\in S(\mathbb{R}^d). \tag{4.47}$$

由此推得 $f\mapsto(\Omega,e^{i\phi(f)}\Omega)e^{\frac{1}{2}(f,f)_0}$ 是 U_0- 泛函, 从而由定理 2.9 知存在 $\Xi\in(S(\mathbb{R}^d))^*$, 使 (4.46) 成立. Ξ 的正性则由它的矩序列的正定性推得 (见引理 4.8).Ξ 的存在性亦可直接利用 (4.47) 从定理 4.3 推得.

下面应用定理 4.19 在白噪声框架下来说明 $P(\phi)_2$ 场可以用一个正 Hida 广义泛函表示. 令 $m^2>0,\omega_0$ 为 $L^2(\mathbb{R})$ 上的由拟微分算子 $\sqrt{-\Delta+m^2}$ 所决定的自共轭算子. 设 $l>0$, 令 Δ_D 为 $L^2([-l,l])$ 上的带 Dirichlet 边界条件的 Laplace 算子, 并令 $\omega_D=\sqrt{-\Delta_D+m^2}$, 我们仍用 ω_D 表示它到 $L^2(\mathbb{R})$ 上的自然延拓. 下

面用 ω 统一表示 ω_0 或 ω_D. 令 μ 为 $I\!R$ 上的标准 Gauss 测度. $\forall f \in S(I\!R)$, 我们定义 $L^2(\mu)_{\mathbb{C}}$ 上的自共轭算子如下:

$$\phi_l(f) = D^*_{\omega^{-1/2} f} + D_{\omega^{-1/2} f},$$

$$H_0 = \frac{1}{2} \sum_k D^*_{\omega^{1/2} e_k} D_{\omega^{1/2} e_k},$$

其中 $\{e_k, k \in I\!N\}$ 为 $L^2(I\!R)$ 的一标准正交基, 且每个 e_k 属于 $S(I\!R)$. 我们定义场 ϕ_l 关于 ω 的 Wick 幂:

$$:\phi_l(f)^n := \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2k-1)!! \phi_l(f)^{n-2k} |\omega^{-1/2} f|_0^{2k}.$$

设 P 为一实多项式. 假定 P 满足如下两条件之一:

$$\text{(H1)} \qquad P(u) = \lambda \sum_{k=0}^{2n} a_k u^k,\ n \in I\!N,\ \lambda, a_{2n} > 0, \lambda/m^2 \text{ 足够小},$$

$$\text{(H2)} \qquad P(u) = \sum_{k=0}^{n} a_k u^{2k} + bu,\ a_n > 0.$$

设 $\{\delta_{t,n}, n \in I\!N\} \subset S(I\!R)$ 为在 $S^*(I\!R)$ 中定义在 t 处的 Dirac δ- 函数 δ_t 的序列. 令

$$V_{l,n} = \int_{-l}^{l} : P\Big(\phi_l(\delta_{l,n})\Big) : dt.$$

可证 (见 Simon[1]): $\lim_{n\to\infty} V_{l,n} = V_l$, 且 V_l 属于 $L^p(\mu), \forall p > 1$. 于是作为乘积算子, V_l 定义了 $L^2(\mu)$ 上的一自共轭算子 (仍记为 V_l). 我们用 $\widehat{H}_l$ 表示 $L^2(\mu)$ 中本质自共轭算子 $H_0 + V_l$ 的自共轭延拓, 并令

$$H_l = \widehat{H}_l - \inf \operatorname{spec} \widehat{H}_l.$$

则 H 有唯一真空态 $\Omega_l \in L^2(\mu)$. 可以验证, $\forall l > 0, (\phi_l, H_l, \Omega_l)$ 满足 Araki 关系并有 ϕ- 界:

$$\pm \phi_l(f) \le \alpha H_l + \beta |f|_1^2 + \gamma,$$

其中 α,β 和 γ 不依赖于 $l>0$. 于是由定理 4.19, 存在 $\Xi_l\in(S(\mathbb{R}))^*_+$, 使得

$$(\Omega_l, e^{i\phi_l(f)}\Omega_l)=\langle\langle\Xi_l, e^{iW_f}\rangle\rangle,\ f\in S(\mathbb{R}),$$

且与 (4.47) 相应的估计中的 K 也不依赖于 l. 此外, 由量子场论中的一已知结果, 存在一复 Hilbert 空间 $\mathcal{K}$ 及 $S(\mathbb{R})$ 上取值于 $\mathcal{K}$ 中自共轭算子的线性映射 ϕ 及 $\mathcal{K}$ 中某实的单位向量 Ω, 使得 $\forall f\in S(\mathbb{R})$,

$$\lim_{l\to\infty}\left(\Omega_l, e^{i\phi_l(f)}\Omega_l\right)=\left(\Omega, e^{i\phi(f)}\Omega\right)_{\mathcal{K}}.$$

我们称 $(\phi,\mathcal{K},\Omega)$ 为 $P(\phi)_2$- **量子场**. 由定理 2.10 知, 存在 $\Xi\in(S(\mathbb{R}))^*_+$, 使得

$$\left(\Omega, e^{i\phi(f)}\Omega\right)_{\mathcal{K}}=\langle\langle\Xi, e^{iW_f}\rangle\rangle. \tag{4.48}$$

第五章 广义泛函空间中的线性算子

§1. 广义泛函的分析运算

上一章我们建立了白噪声分析的一般框架. 在本节我们将对广义泛函引入几种基本的分析运算: 刻度 (Scaling) 变换, 推移和 Sobolev 微分, 梯度与散度. 这些分析运算对白噪声分析的实际应用是很关键的. 本节恒假定 $0 \leq \beta < \infty$.

1.1 刻度变换

由第四章定理 2.18 知, $(E)^{\beta}_{\mathbb{C}} = \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$. 对 $\varphi \in (E)^{\beta}_{\mathbb{C}}$, 我们用 $\widetilde{\varphi}$ 表示 φ 到 $E^*_{\mathbb{C}}$ 上的整解析延拓. 设 $\lambda \in \mathbb{C}$, 我们在 $(E)^{\beta}_{\mathbb{C}}$ 上定义如下算子 σ_λ:

$$\sigma_\lambda \varphi(x) = \widetilde{\varphi}(\lambda x), \ x \in E^*. \tag{1.1}$$

显然 $\sigma_\lambda \varphi \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$. 于是 σ_λ 为 $(E)^{\beta}_{\mathbb{C}}$ 到 $(E)^{\beta}_{\mathbb{C}}$ 中的线性算子. 我们称 σ_λ 为 **刻度变换**.

为了研究刻度变换, 我们需要引进一个数乘 λ 的 **二次量子化算子** $\Gamma(\lambda)$. 这一算子将在今后常被用到.

设 $\lambda \in \mathbb{C}, \varphi \in (E)^{-\beta}_{\mathbb{C}}, \varphi \sim \{f_n\}$. 令

$$\Gamma(\lambda) \sim \{\lambda^n f_n\}. \tag{1.2}$$

容易看出, $\Gamma(\lambda)$ 是 $(E)^{-\beta}_{\mathbb{C}}$ 及 $(E)^{\beta}_{\mathbb{C}}$ 上的连续线性算子. 事实上, $\forall \lambda \neq 0$, 我们有

$$\|\Gamma(\lambda)\|_{p,q-2\log_2|\lambda|,\pm\beta} = \|\varphi\|_{p,q,\pm\beta}, \ \forall p \in \mathbb{Z}, q \in \mathbb{R}. \tag{1.3}$$

下一定理表明 σ_λ 为 $(E)^{\beta}_{\mathbb{C}}$ 上的连续线性算子, 且给出了 $\sigma_\lambda \varphi$ 的混沌分解. 此外, 给出了 σ_λ 的对偶算子 σ^*_λ 的表达式. 这里及

今后，我们用 A^* 表示从一拓扑线性空间到另一拓扑线性空间的连续线性算子 A 的对偶 (见附录 B).

定理 1.1 设 $\lambda \in \mathbb{C}, \varphi \in (E)^{\beta}_{\mathbb{C}}, \varphi \sim \{f_n\}$. 令 $\sigma_\lambda \varphi \sim \{h_n\}$, 则有

$$h_n = \frac{\lambda^n}{n!} \sum_{l=0}^{\infty} (\lambda^2 - 1)^l \frac{(n+2l)!}{l! 2^l} \langle \tau^{\widehat{\otimes} l}, f_{n+2l} \rangle, \tag{1.4}$$

其中级数在 $E_{\mathbb{C}}^{\widehat{\otimes} n}$ 中绝对收敛. 此外，我们有

$$\langle\langle \sigma_\lambda \varphi, G \rangle\rangle = \langle\langle \varphi, \Gamma(\lambda) G \diamond F(\lambda) \rangle\rangle, \ \forall G \in (E)^{-\beta}_{\mathbb{C}}. \tag{1.5}$$

其中 $F(\lambda)$ 如第四章例 2.22 定义. 特别，σ_λ 为 $(E)^{\beta}_{\mathbb{C}}$ 到 $(E)^{\beta}_{\mathbb{C}}$ 中的连续线性算子，其对偶算子为

$$\sigma_\lambda^* G = \Gamma(\lambda) G \diamond F(\lambda), \quad \forall G \in (E)^{-\beta}_{\mathbb{C}}, \tag{1.6}$$

它是从 $(S)^{-\beta}_{\mathbb{C}}$ 到 $(S)^{-\beta}_{\mathbb{C}}$ 中的连续线性算子.

证明 首先，我们证明如下公式：$\forall x \in E^*$, 有

$$:(\lambda x)^{\otimes n}: = n! \sum_{k=0}^{[n/2]} \frac{(\lambda^2 - 1)^k \lambda^{n-2k}}{2^k k! (n-2k)!} :x^{\otimes (n-2k)}: \widehat{\otimes} \tau^{\otimes k}. \tag{1.7}$$

这一公式与附录中公式 (A.9) 类似，其证明也完全类似. 事实上，对任何 $t \in \mathbb{R}, f \in E$, 我们有

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{t^n}{n!} \langle f^{\otimes n}, :(\lambda x)^{\otimes n}: \rangle = \mathcal{E}_{tf}(\lambda x) \\
&= \mathcal{E}_{t\lambda f}(x) \exp\left\{ \tfrac{(\lambda^2 - 1)t^2}{2} \langle f, f \rangle \right\} \\
&= \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \langle f^{\otimes j}, :x^{\otimes j}: \rangle \sum_{k=0}^{\infty} \frac{(\lambda^2 - 1)^k t^{2k}}{2^k k!} \langle f^{\otimes 2k}, \tau^{\otimes k} \rangle,
\end{aligned}$$

比较 t^n 的系数即得 (1.7).

设 $p \in \mathbb{N}$ 使得 H_p 到 H 的嵌入映射 I_{0p} 为 Hilbert–Schmidt 算子，则由第四章 (2.53) 知

$$|\langle \tau^{\widehat{\otimes} k}, f_{n+2l}\rangle|_p \leq \|I_{0p}\|_{\mathrm{HS}}^2 \, |f_{n+2l}|_p,$$

由此容易验证 (1.4) 中的级数在 $E_{\mathbb{C}}^{\widehat{\otimes} n}$ 中绝对收敛. 于是由 (1.7) 知

$$\sum_{n=0}^{\infty}\langle h_n, :x^{\otimes n}:\rangle = \tilde{\varphi}(\lambda x).$$

这表明 $\sigma_\lambda \varphi \sim \{h_n\}$.

现设 $G \in (E)_{\mathbb{C}}^{-\beta}$,$G \sim \{g_n\}$. 则由 (1.4) 得

$$n!\langle h_n, g_n\rangle = \sum_{l=0}^{\infty}(n+2l)!\langle \tfrac{(\lambda^2-1)^l}{2^l l!}\tau^{\widehat{\otimes} l}\widehat{\otimes}\lambda^n g_n, f_{n+2l}\rangle.$$

由此利用第四章 (2.52) 及 Wick 积的表达式 (3.16) 便得 (1.5).

由 (1.5)、(1.3) 及第四章 (3.17) 易证 (参见第四章定理 4.1 的证明): 若 $p \in \mathbb{N}$, $q \geq 0$, 使得 $2^q > |\lambda^2 - 1|\|I_{0p}\|_{\mathrm{HS}}^2$, 则 $\forall \epsilon > (1-\beta)^+$ 有

$$\begin{aligned}&\|\sigma_\lambda \varphi\|_{p, q-2\log_2 |\lambda|, \beta}\\ &\leq (1 - 2^{-\epsilon+(1-\beta)^+})^{-1/2}\|\varphi\|_{p,q+\epsilon,\beta}\|F(\lambda)\|_{-p,-q,-\beta}.\end{aligned} \tag{1.8}$$

由此推知 σ_λ 为 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 的连续线性算子. 由 (1.5) 知 σ_λ 的对偶算子 σ_λ^* 由 (1.6) 给出. ∎

注 1 由 (1.8) 知，若 $p \in \mathbb{N}, q \geq 0$, 使得 $2^q > |\lambda^2 - 1|\|I_{0p}\|_{\mathrm{HS}}^2$, 则 $\forall \epsilon > (1-\beta)^+$, σ_λ 可以延拓为从 $(H_{p,q+\epsilon,\beta})_{\mathbb{C}}$ 到 $(H_{p,q-2\log_2|\lambda|,\beta})_{\mathbb{C}}$ 的有界线性算子.

注 2 对比第四章 (4.6) 及本节 (1.5), 我们有

$$R^{-1} = \Gamma((\sqrt{2})^{-1})\sigma_{\sqrt{2}}. \tag{1.9}$$

由此推知

$$R=\sigma_{(\sqrt{2})^{-1}}\Gamma(\sqrt{2}). \tag{1.10}$$

注 3 由第四章 (3.5) 容易证明：$\forall\varphi\in(E)^{\beta}_{\mathbb{C}},\lambda\longmapsto\sigma_{\lambda}\varphi$ 为 $\mathbb{C}$ 到 $(E)^{\beta}_{\mathbb{C}}$ 中的连续映射，并且 $\forall G\in(E)^{-\beta}_{\mathbb{C}}$，$\lambda\longmapsto\langle\langle\sigma_{\lambda}\varphi,G\rangle\rangle$ 是 $\mathbb{C}$ 上的整函数.

1.2 推移算子与 Sobolev 微分

设 $\varphi\in(E)^{\beta}_{\mathbb{C}},y\in E^{*}_{\mathbb{C}}$. 令

$$\tau_{y}\varphi(x)=\tilde{\varphi}(y+x),\ x\in E^{*}. \tag{1.11}$$

显然 $\tau_{y}\varphi\in(E)^{\beta}_{\mathbb{C}}$. 我们称 τ_{y} 为 **推移算子**. 由第四章 (1.4) 我们有

$$\begin{aligned}:(x+y)^{\otimes n}:&=\int_{E^{*}}(x+y+iz)^{\otimes n}\mu(dz)\\&=\sum_{k=0}^{n}\binom{n}{k}\int_{E^{*}}(x+iz)^{\otimes k}\widehat{\otimes}y^{\otimes n-k}\mu(dz)\\&=\sum_{k=0}^{n}\binom{n}{k}:x^{\otimes k}:\widehat{\otimes}y^{\otimes n-k}\ .\end{aligned} \tag{1.12}$$

于是，采用与定理 1.1 的类似证明，我们得到

定理 1.2 设 $y\in E^{*}_{\mathbb{C}},\varphi\in(E)^{\beta}_{\mathbb{C}}$, $\varphi\sim\{f_{n}\}$. 令 $\tau_{y}\varphi\sim\{h_{n}\}$, 则有

$$h_{n}=\sum_{k=0}^{n}\binom{k+n}{n}\langle y^{\otimes k},f_{n+k}\rangle, \tag{1.13}$$

其中级数在 $E^{\widehat{\otimes}n}_{\mathbb{C}}$ 中绝对收敛. 此外，我们有

$$\langle\langle\tau_{y}\varphi,G\rangle\rangle=\langle\langle\varphi,G\diamond\mathcal{E}_{y}\rangle\rangle\,,\forall G\in(E)^{-\beta}_{\mathbb{C}}. \tag{1.14}$$

特别，τ_{y} 为 $(E)^{\beta}_{\mathbb{C}}$ 到 $(E)^{\beta}_{\mathbb{C}}$ 的连续线性算子，其对偶算子为

$$\tau^{*}_{y}G=G\diamond\mathcal{E}_{y}\ ,\ \forall G\in(E)^{-\beta}_{\mathbb{C}}. \tag{1.15}$$

此外，对任意 $\varphi \in (E)_{\mathbb{C}}^{\beta}, y \longmapsto \tau_y\varphi$ 为 $E_{\mathbb{C}}^*$ 到 $(E)_{\mathbb{C}}^{\beta}$ 的连续映射.

注 (1) 设 $p \in \mathbb{Z}, q \in \mathbb{R}, r > -1$, 则由 (1.14) 及第四章 (3.17) 推知，$\forall y \in H_{-p,\mathbb{C}}, \epsilon > (1-r)^+$, 有

$$\|\tau_y\varphi\|_{p,q,r} \leq (1-2^{-\epsilon+(1-r)^+})^{-1/2}\|\varphi\|_{p,q+\epsilon,r}\|\mathcal{E}_y\|_{-p,-q,-r}. \tag{1.16}$$

这时，τ_y 可以延拓成为从 $(H_{p,q+\epsilon,r})_{\mathbb{C}}$ 到 $(H_{p,q,r})_{\mathbb{C}}$ 中的有界线性算子. 特别，若 $y \in E_{\mathbb{C}}$, 则 τ_y 可以延拓成为从 $(E)_{\mathbb{C}}^{-\beta}$ 到 $(E)_{\mathbb{C}}^{-\beta}$ 的连续线性算子，且有

$$\langle\langle \tau_y\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond \mathcal{E}_y\rangle\rangle, \forall \varphi \in (E)_{\mathbb{C}}^{-\beta}, G \in (E)_{\mathbb{C}}^{\beta}. \tag{1.14$'$}$$

(2) 设 $p \in \mathbb{Z}, q \in \mathbb{R}, \epsilon > 0, y \in H_{-p,\mathbb{C}}$, 使得 $|y|_{-p}^2 < 2^{-q}$, 则有

$$\|\tau_y\varphi\|_{p,q-\epsilon,1} \leq (1-2^{-\epsilon})^{-1/2}\|\varphi\|_{p,q,1}\|\mathcal{E}_y\|_{-p,-q,-1}$$

这时 τ_y 可以延拓为从 $(H_{p,q,1})_{\mathbb{C}}$ 到 $(H_{p,q-\epsilon,1})_{\mathbb{C}}$ 的有界线性算子.

(3) 设 $0 \leq \beta < 1$, $\varphi \in (E)_{\mathbb{C}}^{-\beta}$, $\eta \in E_{\mathbb{C}}$, 则由 (1.14)$'$ 有

$$S(\tau_\eta\varphi)(\xi) = S\varphi(\eta+\xi).$$

下面我们从推移算子出发定义 $(E)_{\mathbb{C}}^{\beta}$ 中的微分运算.

设 $\varphi \in (E)_{\mathbb{C}}^{\beta}, y \in E_{\mathbb{C}}^*$. 由 (1.14) 知：$\forall G \in (E)_{\mathbb{C}}^{-\beta}$

$$\begin{aligned}\lim_{t\downarrow 0}\langle\langle \frac{\tau_{ty}\varphi-\varphi}{t}, G\rangle\rangle &= \lim_{t\downarrow 0}\langle\langle \varphi, G \diamond \frac{\mathcal{E}_{ty}-1}{t}\rangle\rangle \\ &= \langle\langle \varphi, G \diamond I_1(y)\rangle\rangle.\end{aligned}$$

于是，我们可以通过如下关系式定义 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 的一个连续线性算子 D_y:

$$\langle\langle D_y\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond I_1(y)\rangle\rangle. \tag{1.17}$$

我们称 D_y 为 **Sobolev 微分 (算子)**. 由 (1.17) 容易看出：若 $\varphi \sim \{f_n\}$, 则 $D_y\varphi \sim \{h_n\}$, 其中

$$h_n = (n+1)\langle y, f_{n+1}\rangle, \ n \in \mathbb{N}_0. \tag{1.18}$$

由 (1.17) 及第四章 (3.17) 容易证明

定理 1.3 设 $p\in\mathbb{Z}$, $q,r\in\mathbb{R}$, $y\in H_{-p,\mathbb{C}}$. 则对 $\epsilon>(1-r)^-$, D_y 可以延拓成为从 $(H_{p,q+\epsilon,r})_{\mathbb{C}}$ 到 $(H_{p,q,r})_{\mathbb{C}}$ 中的有界线性算子, 并且我们有

$$\|D_y\varphi\|_{p,q,r}\leq C_{r,\epsilon}2^{-q/2}|y|_{-p}\|\varphi\|_{p,q+\epsilon,r}.\tag{1.19}$$

其中

$$C_{r,\epsilon}=(1-2^{-\epsilon+(1-r)^-})^{-1/2}.$$

特别, 对任意 $\varphi\in(E)_{\mathbb{C}}^{\beta}, y\longmapsto D_y\varphi$ 为 $E_{\mathbb{C}}^*$ 到 $(E)_{\mathbb{C}}^{\beta}$ 的连续映射, 对任意 $\varphi\in(E)_{\mathbb{C}}^{-\beta}, y\longmapsto D_y\varphi$ 为 $E_{\mathbb{C}}$ 到 $(E)_{\mathbb{C}}^{-\beta}$ 的连续映射; 对任意 $y\in E_{\mathbb{C}}^*, \varphi\longmapsto D_y\varphi$ 为 $(E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 的连续映射, 对任意 $y\in E_{\mathbb{C}}, \varphi\longmapsto D_y\varphi$ 为 $(E)_{\mathbb{C}}^{-\beta}$ 到 $(E)_{\mathbb{C}}^{-\beta}$ 的连续映射.

定理 1.4 (1) 设 $p\in\mathbb{Z}$, $q\in\mathbb{R}$, $r>-1$, $\epsilon>(1-r)^+$, $\varphi\in(H_{p,q+\epsilon,r})_{\mathbb{C}}$; $y\in H_{-p,\mathbb{C}}$, 则有

$$\lim_{t\downarrow 0}\|\frac{\tau_{ty}\varphi-\varphi}{t}-D_y\varphi\|_{p,q,r}=0.\tag{1.20}$$

(2) 设 $p\in\mathbb{Z}, q\in\mathbb{R}, \epsilon>2, \varphi\in(H_{p,q+\epsilon,r})_{\mathbb{C}}; y\in H_{-p,\mathbb{C}}$, 则有

$$\lim_{t\downarrow 0}\|\frac{\tau_{ty}\varphi-\varphi}{t}-D_y\varphi\|_{p,q,-1}=0.\tag{1.21}$$

(3) 设 $0\leq\beta<1$, $\varphi\in(E)_{\mathbb{C}}^{-\beta}$, $\eta\in E_{\mathbb{C}}$, 则有

$$SD_\eta\varphi(\xi)=\lim_{t\downarrow 0}\frac{1}{t}\{S\varphi(\xi+t\eta)-S\varphi(\xi)\}.\tag{1.22}$$

证明 (1) 由 (1.14) 和 (1.17) 得: $\forall G\in(E)_{\mathbb{C}}^{|r|}$,

$$\langle\langle\frac{\tau_{ty}\varphi-\varphi}{t}-D_y\varphi,G\rangle\rangle=\langle\langle\varphi,G\diamond(\frac{\mathcal{E}_{ty}-1}{t}-I_1(y))\rangle\rangle.$$

于是由第四章 (3.17) 得

$$\begin{aligned}&|\langle\langle\frac{\tau_{ty}\varphi-\varphi}{t}-D_y\varphi,G\rangle\rangle|\\&\leq C_{r,\epsilon}\|\varphi\|_{p,q+\epsilon,r}\|G\|_{-p,-q,-r}\|\frac{\mathcal{E}_{ty}-1}{t}-I_1(y)\|_{-p,-q,-r}.\end{aligned}$$

但由于 $r+1>0$,

$$\lim_{t\downarrow 0}\|\frac{\mathcal{E}_{ty}-1}{t}-I_1(y)\|^2_{-p,-q,-r}=\lim_{t\downarrow 0}\sum_{n=1}^{\infty}\frac{t^{n-1}2^{-nq}}{(n!)^{1+r}}|y|^{2n}_{-p}=0,$$

故得 (1.20).

(2) 当 t 足够小时有 $t^2|y|^2_{-p}<2^{-q}$, 从而 $\tau_{ty}\varphi$ 有定义. 其余证明与 (1) 的证明类似 (利用定理 1.2 的注 (2)).

(3) 由定理 1.2 的注 (3) 推得. ∎

系 1.5 设 $\varphi\in(E)^{\beta}_{\mathbb{C}}, y\in E^*$, 则 $\widetilde{\varphi}$ 沿 y 方向的 Gâteaux 微分处处存在, 且等于 $\widetilde{D_y\varphi}$.

证明 由上一定理知, $(\tau_{ty}\varphi-\varphi)/t$ 在 $(E)^{\beta}_{\mathbb{C}}$ 中收敛于 $D_y\varphi$. 于是 $\forall y\in E^*$,

$$\lim_{t\downarrow 0}\frac{\tilde{\varphi}(x+ty)-\tilde{\varphi}(x)}{t}=\lim_{t\downarrow 0}\langle\langle\frac{\tau_{ty}\varphi-\varphi}{t},\delta_x\rangle\rangle$$
$$=\langle\langle D_y\varphi,\delta_x\rangle\rangle=\widetilde{D_y\varphi}(x).$$ ∎

下一定理给出了广义泛函的 Taylor 展开.

定理 1.6 (1) 设 $p\in\mathbb{Z}, q\in\mathbb{R}, r>-1, \varphi\in(H_{p,q,r})_{\mathbb{C}}$. 则对 $y\in H_{-p,\mathbb{C}}$ 有

$$\tau_y\varphi=\sum_{k=0}^{\infty}\frac{1}{k!}D^k_y\varphi, \tag{1.23}$$

其中对任意 $\epsilon>(1-r)^+$, 级数在 $(H_{p,q-\epsilon,r})_{\mathbb{C}}$ 中收敛. 特别, 若 $\varphi\in(E)^{\beta}_{\mathbb{C}}$, 则对一切 $x,y\in E^*$ 有

$$\widetilde{\varphi}(x+y)=\sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{D^k_y\varphi}(x). \tag{1.24}$$

(2) 设 $p\in\mathbb{Z}, q\in\mathbb{R}, \varphi\in(H_{p,q,-1})_{\mathbb{C}}$. 如果 $y\in H_{-p,\mathbb{C}}$, 使得 $|y|^2_{-p}<2^{-q}$, 则 (1.23) 成立, 其中级数对任意 $\epsilon>2$ 在 $(H_{p,q-\epsilon,-1})_{\mathbb{C}}$ 中收敛.

证明 我们只证 (1),(2) 的证明类似. 由 (1.17) 有

$$\langle\langle D_y^k\varphi, G\rangle\rangle = \langle\langle \varphi, G\diamond I_k(y^{\otimes k})\rangle\rangle,\ \forall G\in(E)_{\mathbb{C}}^{|r|}. \tag{1.25}$$

于是 $\forall\epsilon>(1-r)^+$,

$$\|D_y^k\varphi\|_{p,q-\epsilon,r}\leq C_{r,\epsilon}\|\varphi\|_{p,q,r}\|I_k(y^{\otimes k})\|_{-p,-q,-r}.$$

由此推知 (1.24) 右边的级数在 $(H_{p,q-\epsilon,r})_{\mathbb{C}}$ 中收敛. 此外由 (1.25) 易知: $\forall G\in(E)_{\mathbb{C}}^{|r|}$,

$$\langle\langle \tau_y\varphi, G\rangle\rangle = \sum_{k=0}^{\infty}\frac{1}{k!}\langle\langle D_y^k\varphi, G\rangle\rangle.$$

于是有 (1.23).(1.24) 是 (1.23) 的直接推论. ∎

下一定理概括了微分算子的一些主要性质.

定理 1.7 设 $y,z\in E_{\mathbb{C}}^*,\xi,\eta\in E_{\mathbb{C}}$, $\varphi,\psi\in(E)_{\mathbb{C}}^{\beta},F,G\in(E)_{\mathbb{C}}^{-\beta}$, 则有

$$D_y(\varphi\psi)=\psi D_y\varphi+\varphi D_y\psi, \tag{1.26}$$
$$D_\xi(\varphi F)=FD_\xi\varphi+\varphi D_\xi F, \tag{1.27}$$
$$D_y(\varphi\diamond\psi)=\psi\diamond D_y\varphi+\varphi\diamond D_y\psi, \tag{1.28}$$
$$D_\xi(F\diamond G)=G\diamond D_\xi F+F\diamond D_\xi G, \tag{1.29}$$
$$(D_y+D_y^*)\varphi=I_1(y)\varphi, \tag{1.30}$$
$$(D_\xi+D_\xi^*)F=I_1(\xi)F, \tag{1.31}$$
$$D_yD_z\varphi=D_zD_y\varphi, \tag{1.32}$$
$$D_\xi D_\eta F=D_\eta D_\xi F, \tag{1.33}$$
$$D_y^*D_z^*F=D_z^*D_y^*F, \tag{1.34}$$
$$(D_yD_\xi^*-D_\xi^*D_y)\varphi=\langle y,\xi\rangle\varphi, \tag{1.35}$$
$$(D_\xi D_y^*-D_y^*D_\xi)F=\langle y,\xi\rangle F, \tag{1.36}$$
$$D_\xi^*(\varphi F)+FD_\xi\varphi=\varphi D_\xi^*F, \tag{1.37}$$
$$D_\xi^*(\varphi F)+\varphi D_\xi F=FD_\xi^*\varphi, \tag{1.38}$$
$$D_y^*(\varphi\psi)+\varphi D_y\psi=\psi D_y^*\varphi. \tag{1.39}$$

证明 令 $\xi_1, \xi_2 \in E_{\mathbb{C}}, m, n \in \mathbb{N}_0$. 容易验证 (1.26) 对 $\varphi = W_{\xi_1}^m$ 及 $\psi = W_{\xi_2}^n$ 成立, (1.28) 对 $\varphi = I_m(\xi_1^{\otimes m})$ 及 $\psi = I_n(\xi_2^{\otimes n})$ 成立. 于是由乘积 (或 Wick 积) 及微分运算的线性和连续性推知 (1.26) 及 (1.28) 对一切 $\varphi, \psi \in (E)_{\mathbb{C}}^{\beta}$ 成立. 在 (1.26)、(1.28) 中, 取 $\psi_n \in (E)_{\mathbb{C}}^{\beta}$, $\varphi_n \in (E)_{\mathbb{C}}^{\beta}$, 使得 $\psi_n \to F, \varphi_n \to G$, 即得 (1.27) 及 (1.29).

由第四章 (3.10) 的推广形式容易看出 (1.30) 对 $\varphi = I_n(f^{\otimes n})$ 成立, 其中 $f \in E_{\mathbb{C}}$. 于是基于同样理由, (1.30) 对一切 $\varphi \in (E)_{\mathbb{C}}^{\beta}$ 成立, 从而 (1.31) 也成立. (1.34) 显然成立, 从而 (1.32) 成立, 由此推知 (1.33) 也成立. 由 (1.28) 得

$$\begin{aligned} D_y D_\xi^* \varphi &= D_y(I_1(\xi) \diamond \varphi) = \varphi \diamond D_y I_1(\xi) + I_1(\xi) \diamond D_y \varphi \\ &= \langle y, \xi \rangle \varphi + D_\xi^* D_y \varphi, \end{aligned}$$

此即 (1.35). 由 (1.29) 类似可证 (1.36).

最后由 (1.26) 得

$$\begin{aligned} &\langle\langle D_\xi^*(\varphi F), \psi \rangle\rangle + \langle\langle F D_\xi \varphi, \psi \rangle\rangle \\ &= \langle\langle \varphi F, D_\xi \psi \rangle\rangle + \langle\langle \psi D_\xi \varphi, F \rangle\rangle \\ &= \langle\langle \varphi D_\xi \psi + \psi D_\xi \varphi, F \rangle\rangle = \langle\langle D_\xi(\varphi \psi), F \rangle\rangle \\ &= \langle\langle \varphi \psi, D_\xi^* F \rangle\rangle = \langle\langle \varphi D_\xi^* F, \psi \rangle\rangle, \end{aligned}$$

故 (1.37) 得证. 类似可证 (1.38) 及 (1.39). ∎

1.3 梯度算子与散度算子

在第三章我们引进了梯度算子和散度算子. 现在我们在白噪声框架下类似定义这两个算子. 由第四章定理 1.6 知, 白噪声分析框架下的检验泛函空间可连续稠密地嵌入到 Meyer- Watanabe 检验泛函空间中, 因此下面定义的梯度和散度算子是第二章中相应算子的连续延拓.

以下假定 $0 \le \beta < \infty$.

定义 1.8 设 $\varphi \in (E)_{\mathbb{C}}^{-\beta}$. 由定理 1.3 知存在唯一的 $D\varphi \in E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^{-\beta}$, 使得

$$\langle\langle D\varphi, \xi \otimes \psi \rangle\rangle = \langle\langle D_\xi \varphi, \psi \rangle\rangle, \ \xi \in E_{\mathbb{C}}, \psi \in (E)_{\mathbb{C}}^{\beta}. \tag{1.40}$$

我们称 $D\varphi$ 为 φ 的 梯度.

如果 $\varphi \in (E)_{\mathbb{C}}^{\beta}$, 则由定理 1.3 知， $D\varphi \in E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}^{\beta}$, 并且有

$$\langle\langle D\varphi, y \otimes \psi\rangle\rangle = \langle\langle D_y\varphi, \psi\rangle\rangle,\ y \in E_{\mathbb{C}}^*, \psi \in (E)_{\mathbb{C}}^{-\beta}. \tag{1.41}$$

定义 1.9 设 $F \in E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^{-\beta}$, 则存在唯一的 $D^*F \in (E)_{\mathbb{C}}^{-\beta}$, 使得

$$\langle\langle D^*F, \varphi\rangle\rangle = \langle\langle F, D\varphi\rangle\rangle,\ \varphi \in (E)_{\mathbb{C}}^{\beta}. \tag{1.42}$$

我们称 D^*F 为 F 的 **散度**.

如果 $F \in E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}^{\beta}$, 则由定理 1.3 知， $D^*F \in (E)_{\mathbb{C}}^{\beta}$, 并且有

$$\langle\langle D^*F, \varphi\rangle\rangle = \langle\langle F, D\varphi\rangle\rangle,\ \varphi \in (E)_{\mathbb{C}}^{-\beta}. \tag{1.43}$$

定理 1.10 D 为从 $(E)_{\mathbb{C}}^{-\beta}$ 到 $E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^{-\beta}$ 中的连续线性算子, 由 (1.43) 定义的 D^* 为 D 的对偶算子，它为从 $E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}^{\beta}$ 到 $(E)_{\mathbb{C}}^{\beta}$ 中的连续线性算子. D 到 $(E)_{\mathbb{C}}^{\beta}$ 上的限制为 $(E)_{\mathbb{C}}^{\beta}$ 到 $E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}^{\beta}$ 中的连续线性算子， (1.42) 定义的 D^* 是上述 D^* 的连续延拓，它是从 $E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^{-\beta}$ 到 $(E)_{\mathbb{C}}^{-\beta}$ 的连续线性算子.

我们称 D 为 **梯度算子**, 称 D^* 为 **散度算子**.

证明 直接从前面的定义及定理 1.3 推得. ∎

下一定理分别通过 D 及 D^* 给出了 D_y 及 D_y^* 的一个表示.

定理 1.11 设 $y \in E_{\mathbb{C}}^*, \xi \in E_{\mathbb{C}}, \varphi \in (E)_{\mathbb{C}}^{-\beta}, \psi \in (E)_{\mathbb{C}}^{\beta}$, 则有

$$D_y\psi = \langle D\psi, y\rangle, \quad D_\xi\varphi = \langle D\varphi, \xi\rangle, \tag{1.44}$$
$$D_y^*\varphi = D^*(y \otimes \varphi). \tag{1.45}$$

证明 我们有

$$\begin{aligned}\langle\langle D_y\psi, \varphi\rangle\rangle &= \langle\langle D\psi, y \otimes \varphi\rangle\rangle \\ &= \langle\langle\langle D\psi, y\rangle, \varphi\rangle\rangle, \quad \forall \varphi \in (E)_{\mathbb{C}}^{-\beta},\end{aligned}$$

故 (1.44) 第一式得证. 同理可证第二式.

另一方面，我们有

$$\langle\langle D^*(y\otimes\varphi),\psi\rangle\rangle=\langle\langle y\otimes\varphi,D\psi\rangle\rangle$$
$$=\langle\langle\varphi,D_y\psi\rangle\rangle=\langle\langle D_y^*\varphi,\psi\rangle\rangle,\ \forall\psi\in(E)_{\mathbb{C}}^{\beta},$$

故 (1.45) 得证. ∎

在白噪声框架下，我们也可以定义计数算子 N 如下：设 $\varphi\in(E)_{\mathbb{C}}^{-\beta},\varphi\sim\{f_n\}$, 令 $N\varphi\sim\{nf_n\}$, 则容易证明 N 为 $(E)_{\mathbb{C}}^{-\beta}$ 及 $(E)_{\mathbb{C}}^{\beta}$ 中的连续线性算子.

定理 1.12 我们有

$$N=D^*D. \tag{1.46}$$

证明 首先，由定理 1.10 易知： D^*D 为 $(E)_{\mathbb{C}}^{-\beta}$ 及 $(E)_{\mathbb{C}}^{\beta}$ 中的连续线性算子. 另外，由第二章命题 3.2 知， N 与 D^*D 在 $(E)_{\mathbb{C}}^{-\beta}$ 的一稠子集上一致，故 (1.46) 成立. ∎

下面我们在一个经典的白噪声分析框架下研究梯度和散度算子. 我们将采用第四章 1.3 节中的记号. 令 $\{e_i,i\geq 0\}$ 为由 A 的本征向量构成的 H 的一个标准正交基，其对应的本征值为 $\{\lambda_i,i\geq 0\}$. 依假定，存在 $p_0>0$, 使得 $\sum_{j=0}^{\infty}\lambda_j^{-2p_0}<\infty$.

定理 1.13 设 $\varphi\in(E)_{\mathbb{C}}$ $(\varphi\in(E)_{\mathbb{C}}^*)$, 则有

$$D\varphi=\sum_{j=0}^{\infty}e_j\otimes D_{e_j}\varphi, \tag{1.47}$$

其中级数在 $E_{\mathbb{C}}\otimes(E)_{\mathbb{C}}$ (相应地， $E_{\mathbb{C}}^*\otimes(E)_{\mathbb{C}}^*$) 中收敛.

证明 由于 $|e_j|_p=\lambda_j^p,p\in\mathbb{R}$. 故由 (1.19) 容易证明 (1.47) 中级数的收敛性. 此外，对 $\forall f,g\in E$, 我们有

$$\langle\langle\sum_{j=0}^{\infty}e_j\otimes D_{e_j}\varphi,f\otimes\mathcal{E}_g\rangle\rangle=\sum_{j=0}^{\infty}\langle f,e_j\rangle\langle\langle D_{e_j}\varphi,\mathcal{E}_g\rangle\rangle$$
$$=\sum_{j=0}^{\infty}\langle f,e_j\rangle\langle\langle D\varphi,e_j\otimes\mathcal{E}_g\rangle\rangle=\langle\langle D\varphi,f\otimes\mathcal{E}_g\rangle\rangle,$$

故有 (1.47). ∎

系 1.14 设 $\varphi \in (E)_{\mathbb{C}}, y \in E_{\mathbb{C}}^*$ ($\varphi \in (E)_{\mathbb{C}}^*, y \in E_{\mathbb{C}}$), 则有

$$D_y \varphi = \sum_{j=0}^{\infty} \langle y, e_j \rangle D_{e_j} \varphi, \tag{1.48}$$

其中级数在 $(E)_{\mathbb{C}}$ (相应地, $(E)_{\mathbb{C}}^*$) 中收敛.

证明 直接由 (1.47) 及 (1.44) 推得. ∎

定理 1.15 设 $\varphi \in (E)_{\mathbb{C}}, y \in E_{\mathbb{C}}$ ($\varphi \in (E)_{\mathbb{C}}^*, y \in E_{\mathbb{C}}^*$), 则有

$$D_y^* \varphi = \sum_{j=0}^{\infty} \langle y, e_j \rangle D_{e_j}^* \varphi, \tag{1.49}$$

其中级数在 $(E)_{\mathbb{C}}$ (相应地, $(E)_{\mathbb{C}}^*$) 中收敛.

证明 显然我们有

$$y \otimes \varphi = \sum_{j=0}^{\infty} \langle y, e_j \rangle e_j \otimes \varphi, \tag{1.50}$$

其中级数在 $E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}$ (相应地, $E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^*$) 中收敛. 于是由 (1.50) 及 (1.45) 推得 (1.49). ∎

§2. 广义泛函空间中的连续线性算子

设 $E \hookrightarrow H \hookrightarrow E^*$, $F \hookrightarrow K \hookrightarrow F^*$ 为两个 Gel'fand 三元组. 由第四章定理 1.5 下面的注, 我们对 $p \in \mathbb{Z}, q, \beta \in \mathbb{R}$, 可以定义 Hilbert 空间 $(H_{p,q,\beta})$ 及 $(K_{p,q,\beta})$. 本节将研究从 $(H_{p_1,q_1,\beta_1})_{\mathbb{C}}$ 到 $(K_{-p_2,-q_2,-\beta_2})_{\mathbb{C}}$ 中的有界线性算子或从 $(E)_{\mathbb{C}}^{\beta_1}$ 到 $(F)_{\mathbb{C}}^{\beta_2}$ 中的连续线性算子. 对 Hilbert 空间 X, Y, 我们用 $\mathcal{L}(X, Y)$ 表示从 X 到 Y 中的连续 (即有界) 线性算子的全体. 以下也用 $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{\beta_2})$ 表示从 $(E)_{\mathbb{C}}^{\beta_1}$ 到 $(F)_{\mathbb{C}}^{\beta_2}$ 中的连续线性算子的全体.

2.1 算子的象征与混沌分解

假定 $\beta_1,\beta_2<1$. 设 $A\in\mathcal{L}\big((H_{p_1,q_1,\beta_1})_{\mathbb{C}},(K_{-p_2,-q_2,-\beta_2})_{\mathbb{C}}\big)$. 令 $f\in E_{\mathbb{C}},g\in F_{\mathbb{C}}$. 由于假定了 $\beta_1,\beta_2<1$, 我们有 $\mathcal{E}_f\in(H_{p_1,q_1,\beta_1})_{\mathbb{C}}$, $\mathcal{E}_g\in(K_{p_2,q_2,\beta_2})_{\mathbb{C}}$, 于是可定义 $E_{\mathbb{C}}\times F_{\mathbb{C}}$ 上的如下函数:

$$\widehat{A}(f,g)=\langle\langle A\mathcal{E}_f,\mathcal{E}_g\rangle\rangle,\ f\in E_{\mathbb{C}},g\in F_{\mathbb{C}}. \tag{2.1}$$

我们称 $\widehat{A}$ 在 $E\times F$ 上的限制为 A 的 **象征 (symbol)**. 显然, 算子被它的象征唯一决定.

注 设 $0\le\beta_1,\beta_2<1$. 如果 $A\in\mathcal{L}((E)^{\beta_1},(F)^{\beta_2})$, 则 $\widehat{A}$ 可以唯一地连续延拓到 $E_{\mathbb{C}}\times F_{\mathbb{C}}^*$ 上. 如果 $A\in\mathcal{L}((E)^{-\beta_1},(F)^{-\beta_2})$, 则 $\widehat{A}$ 可以唯一地连续延拓到 $E_{\mathbb{C}}^*\times F_{\mathbb{C}}$ 上. 今后仍用 $\widehat{A}$ 表示这一延拓.

下一引理是第四章引理 2.8 的一种推广.

引理 2.1 设 $\beta_1,\beta_2<1,G$ 为 $E\times F$ 上的一复值函数, 满足下述条件:

(C1) $\forall f_1,g_1\in E,\ f_2,g_2\in F$, 映射 $(z,w)\longmapsto G(g_1+zf_1,g_2+wf_2)$ 在 $\mathbb{C}\times\mathbb{C}$ 上有整解析延拓 (仍记为 G);

(C2) 存在常数 $C,K_1,K_2>0,p_1,p_2\in\mathbb{Z}$, 使得

$$\begin{aligned}|G(zf,wg)|\le&C\exp\{K_1(|z||f|_{p_1})^{\frac{2}{1-\beta_1}}+K_2(|w||g|_{p_2})^{\frac{2}{1-\beta_2}}\},\\&f\in E,g\in F,z,w\in\mathbb{C}.\end{aligned} \tag{2.2}$$

如果 $p_1'>p_1,p_2'>p_2$, 使得 $\|I_{p_1p_1'}\|_{\mathrm{HS}}<\infty$, $\|I_{p_2p_2'}\|_{\mathrm{HS}}<\infty$, 且 q_1,q_2 满足

$$\begin{aligned}2^{q_1}>e^2\big(\tfrac{2K_1}{1-\beta_1}\big)^{1-\beta_1}\|I_{p_1p_1'}\|_{\mathrm{HS}}^2,\\2^{q_2}>e^2\big(\tfrac{2K_2}{1-\beta_2}\big)^{1-\beta_2}\|I_{p_2p_2'}\|_{\mathrm{HS}}^2,\end{aligned} \tag{2.3}$$

则存在 $A\in\mathcal{L}((H_{p_1',q_1,\beta_1})_{\mathbb{C}},(K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}})$, 使 $\widehat{A}(f,g)=G(f,g)$. 此外, 我们有

$$\|A\varphi\|_{-p_2',-q_2,-\beta_2}\le C'\|\varphi\|_{p_1',q_1,\beta_1}, \tag{2.4}$$

其中

$$
\begin{aligned}
C' =& C\Big[\big(1-2^{-q_1}e^2(\tfrac{2K_1}{1-\beta_1})^{1-\beta_1}\|I_{p_1p_1'}\|_{\mathrm{HS}}^2\big) \\
&\times\big(1-2^{-q_2}e^2(\tfrac{2K_2}{1-\beta_2})^{1-\beta_2}\|I_{p_2p_2'}\|_{\mathrm{HS}}^2\big)\Big]^{-1/2}.
\end{aligned} \tag{2.5}
$$

证明 首先，与第四章引理 2.7 类似可证，G 可唯一地延拓成为 $E_{\mathbb{C}}\times F_{\mathbb{C}}$ 上的二元整函数. 于是我们有如下的 Taylor 展开:

$$
G(zf,wg)=\sum_{l,m=0}^{\infty}G_{l,m}(g,f)w^l z^m,\quad f\in E,g\in F,w,z\in\mathbb{C}, \tag{2.6}
$$

其中 $G_{l,m}(g,f)$ 由 Cauchy 公式给出:

$$
G_{l,m}(g,f)=\big(\frac{1}{2\pi i}\big)^2\int_{|z|=R_1}\int_{|w|=R_2}\frac{G(zf,wg)}{z^{m+1}w^{l+1}}dz\,dw. \tag{2.7}
$$

由于 $G_{l,m}(g,f)$ 关于 f 及 g 分别是 E 和 F 上的齐次多项式，故由 Schwartz 核定理易知，存在 $a_{l,m}\in F_{\mathbb{C}}^{*\widehat{\otimes}l}\otimes E_{\mathbb{C}}^{*\widehat{\otimes}m}$, 使得

$$
\langle a_{l,m},g^{\otimes l}\otimes f^{\otimes m}\rangle=G_{l,m}(g,f),\quad f\in E,g\in F. \tag{2.8}
$$

与第四章引理 2.8 的证明完全类似，我们可以得到 $a_{l,m}$ 的如下范数估计:

$$
\begin{aligned}
|a_{l,m}|^2_{-p_2',-p_1'}\le& C^2(m!)^{\beta_1-1}(l!)^{\beta_2-1}\big[e^2(\tfrac{2K_1}{1-\beta_1})^{1-\beta_1}\big]^m \\
&\times\big[e^2(\tfrac{2K_2}{1-\beta_2})^{1-\beta_2}\big]^l\|I_{p_1p_1'}\|_{\mathrm{HS}}^{2m}\|I_{p_2p_2'}\|_{\mathrm{HS}}^{2l}.
\end{aligned} \tag{2.9}
$$

我们定义一算子 $A\in\mathcal{L}((H_{p_1',q_1,\beta_1})_{\mathbb{C}},(K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}})$ 如下: 设 $\varphi\in(H_{p_1',q_1,\beta_1})_{\mathbb{C}},\varphi\sim\{f_n\}$, 令

$$
h_l=\sum_{m=0}^{\infty}m!\langle a_{l,m},f_m\rangle. \tag{2.10}
$$

则有

$$|h_l|_{-p_2'}^2 \le \Big(\sum_{m=0}^{\infty} m!|a_{l,m}|_{-p_2',-p_1'}|f_m|_{p_1'}\Big)^2$$
$$\le \|\varphi\|_{p_1',q_1,\beta_1}^2 \sum_{m=0}^{\infty}(m!)^{1-\beta_1}2^{-mq_1}|a_{l,m}|_{-p_2',-p_1'}^2. \tag{2.11}$$

于是若令 $A\varphi \sim \{h_l\}$, 则由 (2.11) 及 (2.9) 立刻推得 (2.4). 此外, 由 (2.10),(2.8) 及 (2.6) 立刻看出 $\widehat{A} = G$. ∎

注 在引理的条件下, 对一切 $l, m \ge 0$, 令

$$A_{l,m}\varphi = m!I_l(\langle a_{l,m}, f_m\rangle), \tag{2.12}$$

其中 $\varphi \in (H_{p_1',q_1,\beta_1})_{\mathbb{C}}, \varphi \sim \{f_n\}$, 则 $\forall l, m \in \mathbb{N}_0$,

$$A_{l,m} \in \mathcal{L}\big((H_{p_1',q_1,\beta_1})_{\mathbb{C}}, (K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}}\big),$$

且有

$$A = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} A_{l,m}. \tag{2.13}$$

我们称 (2.13) 为算子 A 的 **混沌分解**. 我们有

$$\|A\|^2 = \sum_{l=0}^{\infty}\|\sum_{m=0}^{\infty} A_{l,m}\|^2,\ \sum_{m=0}^{\infty}\|A_{l,m}\| < \infty,$$

这里 $\|\cdot\|$ 为有界算子的范数. 此外, 设 $\varphi \in (H_{p_1',q_1,\beta_1})_{\mathbb{C}}, \varphi \sim \{f_n\}$, $\psi \in (K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}}, \psi \sim \{g_n\}$, 则

$$\langle\langle A\varphi, \psi\rangle\rangle = \sum_{l,m=0}^{\infty} l!m!\langle a_{l,m}, g_l \otimes f_m\rangle, \tag{2.14}$$

其中右边级数绝对收敛.

定义 2.2 设 $0 \leq \beta_1 < 1, 0 \leq \beta_2 < 1$,$G$ 为 $E \times F$ 上的一复值函数，如果它满足引理 2.1 的条件 (C1) 及 (C2), 则称 G 为 $\mathbf{U}_{\beta_1,\beta_2}$-**泛函**.

下一定理给出了算子的象征刻画.

定理 2.3 设 $0 \leq \beta_1 < 1, 0 \leq \beta_2 < 1$. 为要 $E \times F$ 上的一复值函数 G 是某个算子 $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$ 的象征，必须且只需 G 为 U_{β_1,β_2}- 泛函.

证明 设 $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$, 则 $A^* \in \mathcal{L}((F)_{\mathbb{C}}^{\beta_2}, (E)_{\mathbb{C}}^{-\beta_1})$. 由于

$$\widehat{A}(f,g) = \langle\langle A\mathcal{E}_f, \mathcal{E}_g\rangle\rangle = S\big(A^*\mathcal{E}_g\big)(f),$$

故容易推知 $\widehat{A}$ 为 U_{β_1,β_2}- 泛函. 反之，设 G 为 U_{β_1,β_2}- 泛函，则用第四章引理 2.7 的同样证明可证：

$$|G(\xi,\eta)| \leq C' \exp\{K'(|\xi|_p^{\frac{2}{1-\beta_1}} + |\eta|_p^{\frac{2}{1-\beta_2}})\}, \xi \in E_{\mathbb{C}}, \eta \in F_{\mathbb{C}},$$

其中 C', K' 为如下常数：任取 $0 < \rho < 1$,

$$C' = C(1-\rho)^{-(1+\frac{\beta_1+\beta_2}{2})}, \ K' = (2e^2)^{(\frac{1}{1-\beta_1}+\frac{1}{1-\beta_2})} K \rho^{-(\frac{1+\beta_1}{1-\beta_1}+\frac{1+\beta_2}{1-\beta_2})}.$$

特别，G 满足引理 2.1 的条件 (C2). 因此由引理 2.1 知，G 为某个算子 $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$ 的象征. ∎

定理 2.4 设 $0 \leq \beta_1 < 1, 0 \leq \beta_2 < 1$. 为要 $E \times F$ 上的一复值函数 G 是某个算子 $A \in ((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{\beta_2})$ 的象征，必须且只需它满足引理 2.1 的条件 (C1) 及如下条件：

(C3) $\forall p_2 \geq 0, \forall \epsilon > 0$ 存在 $C > 0, p_1 \geq 0$, 使得

$$|G(zf,wg)| \leq C\exp\{\epsilon[(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + (|w||g|_{-p_2})^{\frac{2}{1+\beta_2}}]\}$$
$$f \in E, \ g \in F, \ z,w \in \mathbb{C}. \tag{2.15}$$

证明 必要性容易验证，往证充分性. 对任给 $p_2' \geq 0, q_2 \geq 0$, 先取定 $p_2 > p_2'$ 使得 $\|I_{p_2'p_2}\|_{\mathrm{HS}} < \infty$, 再取 $\epsilon > 0$ 足够小，使得

$$2^{-q_2} > e^2\left(\tfrac{2\epsilon}{1+\beta_2}\right)^{1+\beta_2}\|I_{p_2'p_2}\|_{\mathrm{HS}}^2 .$$

对此 p_2 及 ϵ, 选取 $C > 0, p_1 \geq 0$. 使得 (2.15) 成立. 这时再取 $p_1' > p_1$ 及 $q_1 \geq 0$, 使得

$$2^{q_1} > e^2\left(\frac{2\epsilon}{1-\beta_1}\right)^{1-\beta_1}\|I_{p_1p_1'}\|_{\mathrm{HS}}^2 .$$

于是由引理 2.1 知，与 G 对应的算子 A 满足不等式

$$\|A\varphi\|_{p_2',q_2,\beta_2} \leq C'\|\varphi\|_{p_1',q_1,\beta_1},$$

其中

$$C' = C\Big[\left(1 - 2^{-q_1}e^2\left(\tfrac{2K_1}{1-\beta_1}\right)^{1-\beta_1}\|I_{p_1p_1'}\|_{\mathrm{HS}}^2\right) \\ \times \left(1 - 2^{q_2}e^2\left(\tfrac{2K_2}{1+\beta_2}\right)^{1+\beta_2}\|I_{p_2'p_2}\|_{\mathrm{HS}}^2\right)\Big]^{-1/2}.$$

这表明 $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{\beta_2})$. ∎

与广义泛函情形类似, 我们有定理 2.3 的如下两个重要推论.

定理 2.5 设 $0 \leq \beta_1 < 1, 0 \leq \beta_2 < 1$, $\{G_n, n \in \mathbb{N}\}$ 为 $E \times F$ 上一列 U_{β_1,β_2}- 泛函. 假定下列条件被满足:

(1) $\forall f \in E, g \in F, \{G_n(f,g), n \in \mathbb{N}\}$ 为 $\mathbb{C}$ 中的 Cauchy 列;

(2) 存在 $p_1, p_2 \geq 0, C, K > 0$ 使得

$$|G_n(zf, wg)| \leq C\exp K\left[(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + (|w||g|_{p_2})^{\frac{2}{1-\beta_2}}\right],\ z, w \in \mathbb{C}. \tag{2.16}$$

令 A_n 为以 G_n 为象征的算子, 则 $\forall \varphi \in (E)_{\mathbb{C}}^{\beta_1}$, $\{A_n\varphi, n \in \mathbb{N}\}$ 在 $(F)_{\mathbb{C}}^{-\beta_2}$ 中强收敛.

定理 2.6 设 $0 \leq \beta_1 < 1, 0 \leq \beta_2 < 1$, $(\Omega, \mathcal{F}, \nu)$ 为一测度空间, $\omega \longmapsto A_\omega$ 为 Ω 到 $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (E)_{\mathbb{C}}^{-\beta_2})$ 中的映射, $\widehat{A}_\omega$ 为 A_ω 的象征. 如果 A_ω 满足下列条件:

(1) $\forall f \in E, g \in F, \widehat{A}_\omega(f,g)$ 为 $(\Omega, \mathcal{F})$ 上的可测映射;

(2) 存在 $K > 0, p_1, p_2 \geq 0$ 及 Ω 上非负 ν 可积函数 $C(\omega)$, 使得对 ν-a.e.ω,

$$|\widehat{A}_\omega(zf, wg)| C(\omega)\exp\{K[(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + (|w||g|_{p_2})^{\frac{2}{1-\beta_2}}],\\ \forall f \in E,\ g \in F,\ z, w \in \mathbb{C}, \tag{2.17}$$

则存在 $p', q \geq 0$, 使得 $\forall \varphi \in (E)_{\mathbb{C}}^{\beta_1}$, $\omega \longmapsto A_\omega \varphi$ 在 $(K_{-p',-q,-\beta_2})_{\mathbb{C}}$ 中 Bochner 可积，且有

$$S\Big(\int_\Omega A_\omega \varphi d\nu(\omega)\Big)(g) = \int_\Omega S\big(\widehat{A}_\omega \varphi\big)(g) d\nu(\omega). \tag{2.18}$$

注 定理 2.4 也有两个类似推论.

现在转向研究 $\mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$. 设 $A \in \mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$，则 $A^* \in \mathcal{L}((F)_{\mathbb{C}}^1, (E)_{\mathbb{C}}^{-1})$. 显然，在 $E_{\mathbb{C}} \times F_{\mathbb{C}}$ 中存在 0 点某个邻域 U，使得下面的函数 $\widehat{A}$ 在 U 中有定义：

$$\widehat{A}(\xi, \eta) = \langle\langle A\mathcal{E}_\xi, \mathcal{E}_\eta \rangle\rangle, \quad (\xi, \eta) \in U. \tag{2.19}$$

对固定 ξ,$\widehat{A}(\xi, \cdot)$ 为 $A\mathcal{E}_\xi$ 的局部 S- 变换，从而 $\widehat{A}(\xi, \cdot) \in \mathrm{Hol}_0(F_{\mathbb{C}})$; 对固定 η,$\widehat{A}(\cdot, \eta)$ 为 $A^*\mathcal{E}_\eta$ 的局部 S- 变换，从而 $\widehat{A}(\cdot, \eta) \in \mathrm{Hol}_0(E_{\mathbb{C}})$. 因此，$\widehat{A}$ 在 U 中全纯. 我们称 $\widehat{A}$ 为 A 的 **局部象征**.

下一定理刻画了 $\mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$, 其证明与第四章定理 2.13 类似，故从略.

定理 2.7 设 $G \in \mathrm{Hol}_0(E_{\mathbb{C}} \times F_{\mathbb{C}})$. 则有唯一的 $A \in \mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$, 使得 A 的局部象征 $\widehat{A}$ 与 G 在 0 点某个邻域内一致.

借助于第四章定理 2.14 和定理 2.15, 我们得到定理 2.7 的两个重要推论.

定理 2.8 设 $A_n \in \mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$,$n \geq 1$. 如果存在 $E_{\mathbb{C}} \times F_{\mathbb{C}}$ 中 0 点某个邻域 U, 使得每个 $\widehat{A}_n$ 在 U 上有定义，$\{\widehat{A}_n, n \geq 1\}$ 在 U 上一致有界, 且在 U 上处处收敛, 则对一切 $\varphi \in (E)_{\mathbb{C}}^1$,$\{A_n \varphi, n \geq 1\}$ 在 $(F)_{\mathbb{C}}^{-1}$ 中强收敛.

定理 2.9 设 $\omega \longmapsto A_\omega$ 为测度空间 $(\Omega, \mathcal{F}, \nu)$ 到 $\mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$ 中的映射. 假定存在 $E_{\mathbb{C}} \times F_{\mathbb{C}}$ 中 0 点某个邻域 U, 使得每个 $\widehat{A}_\omega$ 在 U 上有定义且满足下列条件：

(1) $\forall (\xi, \eta) \in U, \omega \longmapsto \widehat{A}_\omega(\xi, \eta)$ 为可测映射；

(2) 存在非负 ν- 可积函数 $C(\omega)$, 使得对 a.e.ω,

$$|\widehat{A}_\omega(\xi, \eta)| \leq C(\omega), \ \forall (\xi, \eta) \in U,$$

则存在 $p \in \mathbb{N}_0, q \geq 0$, 使得 $\forall \varphi \in (E)^{1}_{\mathbb{C}}, \omega \longmapsto A_\omega \varphi$ 在 $(K_{-p,-q,-1})_{\mathbb{C}}$ 中 Bochner 可积, 且存在 $F_{\mathbb{C}}$ 中 0 点某个邻域 V_φ 使得

$$\langle\langle \int_\Omega A_\omega \varphi \, d\nu(\omega), \mathcal{E}_\eta \rangle\rangle = \int_\Omega \langle\langle A_\omega \varphi, \mathcal{E}_\eta \rangle\rangle d\nu(\omega), \ \eta \in V_\varphi.$$

现在研究一般情形 $\mathcal{L}((E)^{\beta_1}_{\mathbb{C}}, (F)^{-\beta_2}_{\mathbb{C}})$ 的刻画. 与广义泛函情形类似, 当 $\beta > 1$ 时, 由于不能定义一般广义泛函的 S- 变换或局部 S- 变换, 我们不能定义 $\mathcal{L}((E)^{\beta_1}_{\mathbb{C}}, (F)^{-\beta_2}_{\mathbb{C}})$ 中元素的象征或局部象征. 借助于第四章 §4 中关于广义泛函矩刻画的结果, 我们可以给出空间 $\mathcal{L}((E)^{\beta_1}_{\mathbb{C}}, (F)^{-\beta_2}_{\mathbb{C}})$ 的一个统一刻画.

下面用 $\mathcal{P}_E$ 及 $\mathcal{P}_F$ 表示 E^* 及 F^* 上多项式光滑泛函的全体.

定理 2.10 设 $0 \leq \beta_1, \beta_2 < \infty, G$ 为 $\mathcal{P}_E \times \mathcal{P}_F$ 上的一双线性型. 为要存在一算子 $A \in \mathcal{L}((E)^{\beta_1}_{\mathbb{C}}, (F)^{-\beta_2}_{\mathbb{C}})$, 使得

$$\langle\langle A\varphi, \psi \rangle\rangle = G(\varphi, \psi), \ \varphi \in \mathcal{P}_E, \psi \in \mathcal{P}_F, \tag{2.20}$$

必须且只需存在 $p_1, p_2 \geq 0, C > 0, K > 0$, 使得 $\forall j, k \in \mathbb{N}_0$, 有

$$| G(W^j_f, W^k_g) | \leq K C^{j+k} (j!)^{\frac{1+\beta_1}{2}} (k!)^{\frac{1+\beta_2}{2}} |f|^j_{p_1} |g|^k_{p_2}, \ f \in E, g \in F. \tag{2.21}$$

证明 留给读者作为练习.

2.2 广义算子的 S- 变换与 Wick 积

本节恒假定 $0 \leq \beta < 1$. 我们称 $\mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{-\beta}_{\mathbb{C}})$ 中的元为 **广义算子**. 设 $y \in E^*_{\mathbb{C}}$. 易知 $D_y \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{\beta}_{\mathbb{C}}), D^*_y \in \mathcal{L}((E)^{-\beta}_{\mathbb{C}}, (E)^{-\beta}_{\mathbb{C}})$. 在量子物理中, D_y 和 D^*_y 分别称为 **湮灭算子** 和 **增生算子**. 又设 $x \in E^*_{\mathbb{C}}$, 则 $D^*_y D_x$ 仍是广义算子, 但 $D_x D^*_y$ 一般无意义 (除非 $y \in E_{\mathbb{C}}$). 如果 $y \in E_{\mathbb{C}}$, 则由 (1.35) 知

$$D^*_y D_x = D_x D^*_y - \langle x, y \rangle.$$

当 $x, y \in E^*_{\mathbb{C}}$ 时, $D^*_y D_x$ 可以形式地看成上式右端两个无意义的项之差. 在量子物理中, 把 $D^*_y D_x$ 称为 $D_x D^*_y$ 的 Wick 重正化或

Wick 编序 (Wick ordering). 本节旨在将这种 Wick 编序推广为广义算子的 Wick 积运算.

上节我们定义了广义算子的象征. 下面为了今后记号的方便, 我们引进广义算子的 S- 变换, 它是广义泛函 S- 变换的自然推广. 为此, 先考虑广义泛函 $F \in (E)_{\mathbb{C}}^{-\beta}$, 它与检验泛函的乘积运算可以看成一个广义算子 (称为乘积算子). 令 $f, g \in E_{\mathbb{C}}$, 则

$$\begin{aligned} SF(f+g) &= \langle\langle F, \mathcal{E}_{f+g}\rangle\rangle = \langle\langle F, \mathcal{E}_f \mathcal{E}_g\rangle\rangle e^{-\langle f,g\rangle} \\ &= \langle\langle F\mathcal{E}_f, \mathcal{E}_g\rangle\rangle e^{-\langle f,g\rangle} = \widehat{F}(f,g) e^{-\langle f,g\rangle}. \end{aligned}$$

于是对一般的广义算子 $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{-\beta})$, 我们定义 A 的 **S-变换**为

$$\widetilde{A}(f,g) = \widehat{A}(f,g) e^{-\langle f,g\rangle}. \tag{2.22}$$

现在我们从广义算子的 S- 变换来看 Wick 编序. 我们有

$$\begin{aligned} \widehat{D_y^* D_x}(f,g) &= \langle\langle D_y^* D_x \mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\ &= \langle\langle D_x \mathcal{E}_f, D_y \mathcal{E}_g\rangle\rangle \\ &= \langle x, f\rangle \langle y, g\rangle \langle\langle \mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\ &= \langle x, f\rangle \langle y, g\rangle e^{\langle f,g\rangle}. \end{aligned}$$

于是 $\widetilde{D_y^* D_x}(f,g) = \langle x, f\rangle \langle y, g\rangle$. 另一方面, 容易看出

$$\widehat{D}_y^*(f,g) = \langle y, g\rangle e^{\langle f,g\rangle}, \quad \widehat{D}_x(f,g) = \langle x, f\rangle e^{\langle f,g\rangle},$$

于是 $\widetilde{D_y^*}(f,g) = \langle y, g\rangle$, $\widetilde{D_x}(f,g) = \langle x, f\rangle$. 因此最终有

$$\widetilde{D_y^* D_x} = \widetilde{D_y^*}\widetilde{D_x}. \tag{2.23}$$

下面我们再从广义算子的 S- 变换来看广义泛函的 Wick 积. 设 $F, G \in (E)_{\mathbb{C}}^{-\beta}$. 将 F, G 及 $F \diamond G$ 看成乘积算子, 它们的 S- 变换分别是:

$$\begin{aligned} \widetilde{F}(f,g) &= \langle\langle F\mathcal{E}_f, \mathcal{E}_g\rangle\rangle e^{-\langle f,g\rangle} = SF(f+g), \\ \widetilde{G}(f,g) &= SG(f+g), \\ \widetilde{F \diamond G}(f,g) &= S(F \diamond G)(f+g) = SF(f+g) SG(f+g). \end{aligned}$$

于是我们有

$$\widetilde{F \diamond G} = \widetilde{F}\widetilde{G}. \tag{2.24}$$

最后，令 $\mathcal{G}_{\beta,\beta}$ 及 $\mathcal{G}_{\beta,-\beta}$ 表示 $\mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{\beta})$ 及 $\mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{-\beta})$ 中元素 S- 变换的全体，则容易由定理 2.3 及 2.4 看出， $\mathcal{G}_{\beta,\beta}$ 及 $\mathcal{G}_{\beta,-\beta}$ 都对乘积封闭.

基于上面几个方面的观察，我们引进如下的

定义 2.11 设 $A,B \in \mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{-\beta})$. 以 $\widetilde{A}\widetilde{B}$ 为 S- 变换的算子称为 A 与 B 的 **Wick 积**, 记为 $A \diamond B$, 即有

$$\widetilde{A \diamond B} = \widetilde{A}\widetilde{B}. \tag{2.25}$$

由前面的讨论知，广义算子的 Wick 积既是 Wick 编序的推广，又是广义泛函 Wick 积的推广. 由定义知， Wick 积满足交换律和结合律. 此外， $\mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{\beta})$ 及 $\mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{-\beta})$ 按 Wick 积为代数，恒等算子是单位元.

下一引理中的 (ii) 表明，量子物理中算子乘积的 Wick 编序就是广义算子的 Wick 积.

引理 2.12 设 $A,B \in \mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{-\beta})$, $y \in E_{\mathbb{C}}^{*}$, 则有

(i) $(A \diamond B)^{*} = A^{*} \diamond B^{*}$;

(ii) $D_y^{*} \diamond A = D_y^{*}A, \quad D_y \diamond A = AD_y$.

证明 (i) 由 $\widetilde{K^{*}}(f,g) = \widetilde{K}(g,f)$ 推得, 其中 $K \in \mathcal{L}((E)_{\mathbb{C}}^{\beta},(E)_{\mathbb{C}}^{-\beta})$. 往证 (ii). 设 $f,g \in E$, 我们有

$$\begin{aligned}
\widetilde{D_y^{*}A}(f,g) &= e^{-\langle f,g\rangle}\langle\langle D_y^{*}A\mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\
&= e^{-\langle f,g\rangle}\langle\langle A\mathcal{E}_f, D_y\mathcal{E}_g\rangle\rangle \\
&= \langle y,g\rangle e^{-\langle f,g\rangle}\widehat{A}(f,g) = \widetilde{D_y^{*}}(f,g)\widetilde{A}(f,g).
\end{aligned}$$

由此推得 (ii) 的第一等式. 类似可证第二等式. ∎

定义 2.13 令

$$\mathcal{G}^{\beta} = \{S\varphi : \varphi \in (E)_{\mathbb{C}}^{\beta}\},\ \mathcal{G}^{-\beta} = \{S\psi : \psi \in (E)_{\mathbb{C}}^{-\beta}\}.$$

在 $\mathcal{G}^{-\beta}\times\mathcal{G}^{\beta}$ 上定义双线性型 $\langle\cdot,\cdot\rangle$ 如下:

$$\langle S\psi, S\varphi\rangle = \langle\langle\psi,\varphi\rangle\rangle,\ \varphi\in(E)^{\beta}_{\mathbb{C}}, \psi\in(E)^{-\beta}_{\mathbb{C}}. \tag{2.26}$$

引理 2.14 设 $F\in\mathcal{G}^{\beta}, G\in\mathcal{G}^{-\beta}$, 又设 $A\in\mathcal{L}((E)^{\beta}_{\mathbb{C}},(E)^{\beta}_{\mathbb{C}})$, $B\in\mathcal{L}((F)^{\beta}_{\mathbb{C}},(E)^{-\beta}_{\mathbb{C}})$, 则 $\forall f\in E_{\mathbb{C}}$,

$$F(f+\cdot)\in\mathcal{G}^{\beta}, G(f+\cdot)\in\mathcal{G}^{-\beta}; \widetilde{A}(f,\cdot)\in\mathcal{G}^{\beta}, \widetilde{B}(f,\cdot)\in\mathcal{G}^{-\beta}.$$

证明 直接由第四章定理 2.9 及定理 2.16 推得. ∎

引理 2.15 设 $F\in\mathcal{G}^{\beta}, G\in\mathcal{G}^{-\beta}, f\in E_{\mathbb{C}}$, 则有

$$G(f)=\langle G, e^{\langle f,\cdot\rangle}\rangle, \tag{2.27}$$

$$\langle G, Fe^{\langle f,\cdot\rangle}\rangle=\langle G(f+\cdot),F\rangle, \tag{2.28}$$

$$\langle e^{\langle f,\cdot\rangle}G, F\rangle=\langle G,F(f+\cdot)\rangle. \tag{2.29}$$

证明 设 $G=S\psi$, 则有

$$\begin{aligned}G(f)&=\langle\langle\psi,\mathcal{E}_f\rangle\rangle=\langle S\psi,S\mathcal{E}_f\rangle\\&=\langle G,e^{\langle f,\cdot\rangle}\rangle.\end{aligned}$$

(2.27) 得证. 为证 (2.28), 只需考虑 $F=e^{\langle g,\cdot\rangle}$ 这一特殊情形, 其中 $g\in E_{\mathbb{C}}$. 这时由 (2.27) 得

$$\langle G,e^{\langle g,\cdot\rangle}e^{\langle f,\cdot\rangle}\rangle=G(f+g)=\langle G(f+\cdot),e^{\langle g,\cdot\rangle}\rangle.$$

(2.28) 得证. 类似可证 (2.29). ∎

由 (2.26) 及 (2.28) 我们得到

定理 2.16 设 $\varphi\in(E)^{-\beta}_{\mathbb{C}}, \psi\in(E)^{\beta}_{\mathbb{C}}$, 则 $\forall f\in E_{\mathbb{C}}$ 有

$$S(\varphi\psi)(f)=\langle S\varphi(f+\cdot),S\psi(f+\cdot)\rangle. \tag{2.30}$$

证明 首先, $\forall g\in E_{\mathbb{C}}$ 有

$$\langle\langle\psi\mathcal{E}_f,\mathcal{E}_g\rangle\rangle=e^{\langle f,g\rangle}S\psi(f+g).$$

于是

$$\begin{aligned}
S(\varphi\psi)(f) &= \langle\langle \varphi\psi, \mathcal{E}_f\rangle\rangle \\
&= \langle\langle \varphi, \psi\mathcal{E}_f\rangle\rangle = \langle S\varphi, S(\psi\mathcal{E}_f)\rangle \\
&= \langle S\varphi, e^{\langle f,\cdot\rangle} S\psi(f+\cdot)\rangle \\
&= \langle S\varphi(f+\cdot), S\psi(f+\cdot)\rangle. \qquad \blacksquare
\end{aligned}$$

下一定理是上一定理的推广 (见下面的注).

定理 2.17 设 $A \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{-\beta}_{\mathbb{C}})$, $B \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{\beta}_{\mathbb{C}})$, 则 $\forall f, g \in E_{\mathbb{C}}$ 有

$$\widetilde{AB}(f,g) = \langle \widetilde{A}(f+\cdot, g), \widetilde{B}(f, g+\cdot)\rangle. \tag{2.31}$$

证明 我们有

$$\begin{aligned}
\widetilde{AB}(f,g) &= e^{-\langle f,g\rangle}\langle\langle AB\mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\
&= e^{-\langle f,g\rangle}\langle\langle A^*\mathcal{E}_g, B\mathcal{E}_f\rangle\rangle \\
&= e^{-\langle f,g\rangle}\langle S(A^*\mathcal{E}_g), S(B\mathcal{E}_f)\rangle \\
&= e^{-\langle f,g\rangle}\langle e^{\langle g,\cdot\rangle}\widetilde{A^*}(g,\cdot)\rangle, e^{\langle f,\cdot\rangle}\widetilde{B}(f,\cdot)\rangle \\
&= e^{-\langle f,g\rangle}\langle e^{\langle g,\cdot\rangle}\widetilde{A}(\cdot,g), e^{\langle f,\cdot\rangle}\widetilde{B}(f,\cdot)\rangle \\
&= e^{-\langle f,g\rangle}\langle e^{\langle g,f+\cdot\rangle}\widetilde{A}(f+\cdot,g), \widetilde{B}(f,\cdot)\rangle (\text{由}(2.28)) \\
&= \langle e^{\langle g,\cdot\rangle}\widetilde{A}(f+\cdot,g), \widetilde{B}(f,\cdot)\rangle \\
&= \langle \widetilde{A}(f+\cdot,g), \widetilde{B}(f,g+\cdot)\rangle (\text{由}(2.29)). \qquad \blacksquare
\end{aligned}$$

注 如果将定理 2.16 中的 φ, ψ 及 $\varphi\psi$ 看成乘积算子, 则由 (2.30) 得

$$\begin{aligned}
\widetilde{\varphi\psi}(f,g) &= S(\varphi\psi)(f+g) \\
&= \langle S\varphi(f+g+\cdot), S\psi(f+g+\cdot)\rangle \\
&= \langle \tilde{\varphi}(f+\cdot, g), \tilde{\psi}(f, g+\cdot)\rangle.
\end{aligned}$$

因此, 定理 2.17 确实推广了定理 2.16.

下一定理给出了 $\mathcal{L}((E)^{\beta}_{\mathbb{C}},(E)^{\beta}_{\mathbb{C}})$ 中一类算子的刻画.

定理 2.18 设 $B\in((E)^{\beta}_{\mathbb{C}},(E)^{\beta}_{\mathbb{C}})$, 则下列三断言等价:

(i) $BD_{\xi}=D_{\xi}B,\ \forall\xi\in E_{\mathbb{C}}$;

(ii) $\widetilde{B}(f,g)=\widetilde{B}(f,0),\ \forall f,g\in E_{\mathbb{C}}$;

(iii) $A\diamond B=AB,\ \forall A\in\mathcal{L}((E)^{\beta}_{\mathbb{C}},(E)^{-\beta}_{\mathbb{C}})$.

证明 (i)⇒(ii).$\forall f,g\in E_{\mathbb{C}}$ 我们有

$$\begin{aligned}\widetilde{BD_{\xi}}(f,g)&=e^{-\langle f,g\rangle}\widehat{BD_{\xi}}(f,g)=\langle f,\xi\rangle e^{-\langle f,g\rangle}\widehat{B}(f,g)\\&=\langle f,\xi\rangle\widetilde{B}(f,g),\end{aligned}\tag{2.32}$$

$$\begin{aligned}\widetilde{D_{\xi}B}(f,g)&=e^{-\langle f,g\rangle}\langle\langle D_{\xi}B\mathcal{E}_f,\mathcal{E}_g\rangle\rangle\\&=e^{-\langle f,g\rangle}\langle\langle B\mathcal{E}_f,D_{\xi}^{*}\mathcal{E}_g\rangle\rangle\\&=e^{-\langle f,g\rangle}\langle\langle B\mathcal{E}_f,I(\xi)\diamond\mathcal{E}_g\rangle\rangle\\&=e^{-\langle f,g\rangle}\lim_{\epsilon\downarrow0}\langle\langle B\mathcal{E}_f,\frac{1}{\epsilon}(\mathcal{E}_{g+\epsilon\xi}-\mathcal{E}_g)\rangle\rangle\\&=\lim_{\epsilon\downarrow0}\frac{1}{\epsilon}\big(\widetilde{B}(f,g+\epsilon\xi)-\widetilde{B}(f,g)\big)\\&\quad+\lim_{\epsilon\downarrow0}\frac{1}{\epsilon}(e^{-\langle f,g\rangle}-e^{-\langle f,g+\epsilon\xi\rangle})\langle\langle B\mathcal{E}_f,\mathcal{E}_{g-\epsilon\xi}\rangle\rangle\\&=\lim_{\epsilon\downarrow0}\frac{1}{\epsilon}\big(\widetilde{B}(f,g+\epsilon\xi)-\widetilde{B}(f,g)\big)+\langle f,\xi\rangle\widetilde{B}(f,g).\end{aligned}$$

若 (i) 成立, 则有

$$\lim_{\epsilon\downarrow0}\frac{1}{\epsilon}\big(\widetilde{B}(f,g+\epsilon\xi)-\widetilde{B}(f,g)\big)=0,\ \forall g,\xi\in E_{\mathbb{C}}.$$

于是 $\widetilde{B}(f,g)$ 不依赖于 g, 即有 (ii) 成立.

(ii)⇒(iii). 设 (ii) 成立, 则由 (2.31) 及 (ii) 知, $\forall f,g\in E_{\mathbb{C}}$,

$$\begin{aligned}\widetilde{AB}(f,g)&=\langle\widetilde{A}(f+\cdot,g),\widetilde{B}(f,0)\rangle\\&=\widetilde{B}(f,0)\langle\widetilde{A}(f+\cdot,g),1\rangle\\&=\widetilde{B}(f,g)\widetilde{A}(f,g)=\widetilde{A\diamond B}(f,g).\end{aligned}$$

(iii)⇒(i). 由 (2.32) 知，恒有 $BD_\xi = B \diamond D_\xi$, 故有 (iii) ⇒(i). ∎

作为定理 2.18 的对偶形式，我们有

定理 2.19 设 $A \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{-\beta}_{\mathbb{C}})$, 则下列三断言等价：

(i) $AD^*_\xi = D^*_\xi A$, $\forall \xi \in E_{\mathbb{C}}$;

(ii) $\widetilde{A}(f,g) = \widetilde{A}(0,g)$, $\forall f,g \in E_{\mathbb{C}}$;

(iii) $A \diamond B = AB$, $\forall B \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{\beta}_{\mathbb{C}})$.

证明 与上一定理证明类似，故从略. ∎

最后需要指出,本节所有结果都可用局部 S- 变换改述为 $\beta = 1$ 情形，我们建议读者给出它们的叙述和证明.

§ 3. 积分核算子与算子的积分核表示

本节将在一个经典的白噪声分析框架下研究从 $(E)_{\mathbb{C}}$ 到 $(E)^*_{\mathbb{C}}$ 中的连续线性算子. 我们假定 Gel'fand 三元组 $E \hookrightarrow H \hookrightarrow E^*$ 是由一可分 Hilbert 空间 H 及其上的一正自共轭算子 A 生成的 (见第四章 1.3 节),$\{|\cdot|_p, p \geq 0\}$ 为由 A 决定的标准范数列. 为了方便起见，我们设 $\|A^{-1}\|_{\mathrm{HS}} < \infty$(否则用 A^{p_0} 代替 A). 由第四章 1.3 节知，(E) 为 $\{(E)_p, p \in \mathbb{N}_0\}$ 的投影极限 (其中 $(E)_p = \mathcal{D}(\Gamma(A)^p)$),$(E)^*$ 为 $\{(E)_{-p}, p \in \mathbb{N}_0\}$ 的归纳极限. 从第二小节起，还进一步假定 $H = L^2(T, \mathcal{B}(T), \nu)$, 其中 T 为一 Hausdorff 空间， ν 为一 σ- 有限 Borel 测度.

今后，我们用 ρ 表示 $\|A^{-1}\|$, 用 δ 表示 $\|A^{-1}\|_{\mathrm{HS}}$. 依上述假定， $0 < \rho < 1$.

3.1 张量积的缩合

我们将推广第四章 3.1 中张量积缩合的定义. 令 $\{e_j, j \geq 0\}$ 为 H 的一个标准正交基，它由 A 的本征向量构成. 我们假定 $1 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots$ 为 A 的本征值，且 $Ae_j = \lambda_j e_j, \forall j \geq 0$. 显然 $\{e_j, j \geq 0\}$ 也构成 $H_{\mathbb{C}}$ 的基. 作为 $H_{\mathbb{C}}$ 中自共轭算子 A^{-1}, 我们仍

有

$$\|A^{-1}\| = \sup\{\lambda_j^{-1}, j \in \mathbb{N}_0\} = \lambda_0^{-1} = \rho,$$
$$\|A^{-1}\|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \lambda_j^{-2} = \delta.$$

设 $\alpha=(\alpha_1,\cdots,\alpha_n)\in\Lambda_n$, $e_\alpha = \otimes_{j=1}^n e_{\alpha_j}$, 则 $e_\alpha \in E^{\otimes n}$, $\{e_\alpha, \alpha\in\Lambda_n\}$ 既是 $H^{\otimes n}$ 的基，又是 $H_{\mathbb{C}}^{\otimes n}$ 的基.

引理 3.1 我们有

$$|e_\alpha|_p |e_\alpha|_{-p} = 1, \quad p\in\mathbb{R}, \tag{3.1}$$

$$|e_\alpha|_p \geq \rho^{-pn}, \quad p\geq 0, \alpha\in\Lambda_n, \tag{3.2}$$

$$|f|_p^2 = \sum_{\alpha\in\Lambda_n} |(f,e_\alpha)|^2 |e_\alpha|_p^2, \ f\in H_{p,\mathbb{C}}^{\otimes n}, \ p\in\mathbb{R}. \tag{3.3}$$

证明 令 $\lambda_\alpha^p = \prod_i \lambda_{\alpha_i}^p$. 由于 $|e_\alpha|_p = \lambda_\alpha^p$,(3.1) 和 (3.2) 显然成立. 又由于 $\{\lambda_\alpha^{-p} e_\alpha, \alpha\in\Lambda_n\}$ 为 $H_{p,\mathbb{C}}^{\otimes n}$ 的基，故易知 (3.3) 成立. ∎

设 $m,n\in\mathbb{N}, p,q\in\mathbb{R}$, 我们将用 $|\cdot|_{m,n;p,q}$ 表示 $H_{p,\mathbb{C}}^{\otimes m}\otimes H_{q,\mathbb{C}}^{\otimes n}$ 中的范数，即

$$|f|_{m,n;p,q} = |(A^p)^{\otimes m}\otimes(A^q)^{\otimes n} f|_0 .$$

与 (3.3) 类似可证

$$|f|_{m,n;p,q}^2 = \sum_{\alpha\in\Lambda_m,\sigma\in\Lambda_n} |(f,e_\alpha\otimes e_\sigma)|^2 |e_\alpha|_p^2 |e_\sigma|_q^2. \tag{3.4}$$

于是由 (3.4) 得

$$|f|_p = |f|_{m,n;p,p}, \quad f\in H_{p,\mathbb{C}}^{\otimes m+n}, \tag{3.5}$$

$$|f|_{m,n;p,q} \leq \rho^{mr+ns} |f|_{m,n;p+r,q+s}, \ p,q\in\mathbb{R}, \ r,s\geq 0. \tag{3.6}$$

引理 3.2 设 $f\in E_{\mathbb{C}}^{\otimes l+m}, g\in E_{\mathbb{C}}^{\otimes l+n}$, 我们按第四章 (3.4) 定义 $f\otimes_l g$, 则有

$$|f\otimes_l g|_{m,n;p,q}\le |f|_{m,l;p,r}|g|_{n,l;q,-r},\ \ p,q,r\in\mathbb{R},\tag{3.7}$$

$$|f\otimes_l g|_p\le \rho^{2pl}|f|_p|g|_p,\quad p\ge 0,\tag{3.8}$$

$$|f\otimes_l g|_{-p}\le \rho^{2pn}|f|_{-p}|g|_p,\quad p\ge 0,\tag{3.9}$$

$$|f\otimes_l g|_p\le \rho^{qn}|f|_{m,l;p,-(p+q)}|g|_{p+q},\ p\in\mathbb{R},\ q\ge 0.\tag{3.10}$$

证明 由于 $|e_\alpha|_{-r}|e_\alpha|_r=1$, 由 (3.4) 及第四章 (3.4) 得

$$\begin{aligned}|f\otimes_l g|^2_{m,n;p,q}&=\sum_{\sigma\in\Lambda_m,\delta\in\Lambda_n}\Big|\sum_{\alpha\in\Lambda_l}(f,e_\sigma\otimes e_\alpha)(g,e_\delta\otimes e_\alpha)\Big|^2|e_\sigma|_p^2|e_\delta|_q^2\\&\le\sum_{\alpha\in\Lambda_l,\sigma\in\Lambda_m}|(f,e_\sigma\otimes e_\alpha)|^2|e_\alpha|_r^2|e_\sigma|_p^2\\&\quad\times\sum_{\alpha\in\Lambda_l,\delta\in\Lambda_n}|(g,e_\delta\otimes e_\alpha)|^2|e_\alpha|_{-r}^2|e_\delta|_q^2,\end{aligned}$$

此即 (3.7). 在 (3.7) 中分别令 $r=q=p$ 及 $r=q=-p$ 并利用如下不等式

$$|g|_{n,l;p,-p}\le\rho^{2pl}|g|_p,\ |g|_{n,l;-p,p}\le\rho^{2pn}|g|_p,\ p\ge 0$$

即得 (3.8) 和 (3.9). 最后, 在 (3.7) 中令 $r=-(p+q)$ 得 (3.10). ∎

注 1 由 (3.8) 知, $\{f,g\}\longmapsto f\otimes_l g$ 为 $E_{\mathbb{C}}^{\otimes l+m}\times E_{\mathbb{C}}^{\otimes l+n}$ 到 $E_{\mathbb{C}}^{\otimes m+n}$ 中的连续双线性映射. 由 (3.9) 知, $\{f,g\}\longmapsto f\otimes_l g$ 可延拓成为 $E_{\mathbb{C}}^{*\otimes l+m}\times E_{\mathbb{C}}^{\otimes l+n}$ 到 $E_{\mathbb{C}}^{*\otimes m+n}$ 中的分别关于每个变元连续的双线性映射. 因此, (3.9) 和 (3.10) 中的 f 可以取为 $E_{\mathbb{C}}^{*\otimes l+m}$ 的元素.

注 2 由于对称化不使范数增大, 所以 (3.7)–(3.10) 对 $f\otimes_l g$ 的对称化 $f\widehat{\otimes}_l g$(作为 $E_{\mathbb{C}}^{\widehat{\otimes} m+n}$ 或 $E_{\mathbb{C}}^{*\widehat{\otimes} m+n}$ 的元素) 也成立.

引理 3.3 设 $F\in E_{\mathbb{C}}^{*\widehat{\otimes} k}, G\in E_{\mathbb{C}}^{*\widehat{\otimes} l}, h\in E_{\mathbb{C}}^{\widehat{\otimes} k+l+m}$, 则有

$$(F\widehat{\otimes}G)\widehat{\otimes}_{k+l}h=F\widehat{\otimes}_k(G\widehat{\otimes}_l h).\tag{3.11}$$

证明 首先，在 (3.10) 中令 $m=0$ 得

$$|f\otimes_l g|_p \le \rho^{qn}|f|_{-(p+q)}|g|_{p+q},\quad p\in \mathbb{R}, q\ge 0. \tag{3.12}$$

由此容易推知：对固定的 f 和 g,(3.11) 两边关于 h 连续. 但 (3.11) 对 $h=\xi^{\otimes k+l+m}(\xi\in E_{\mathbb{C}})$ 显然成立，故对一般的 h 也成立. ∎

引理 3.4 设 $F\in E_{\mathbb{C}}^{*\widehat{\otimes} l}, G\in E_{\mathbb{C}}^{*\widehat{\otimes} m}$. 则对 $f\in E_{\mathbb{C}}^{\widehat{\otimes} l+n}, g\in E_{\mathbb{C}}^{\widehat{\otimes} m+n}$ 有

$$\langle F\otimes_l f, G\otimes_m g\rangle = \langle F\otimes G, f\otimes_n g\rangle. \tag{3.13}$$

证明 由 (3.9) 及 (3.12) 容易看出 (3.13) 两边关于 f 和 g 都是连续双线性泛函. 此外，设 $\xi,\eta\in E_{\mathbb{C}}$, 对 $f=\xi^{\otimes l+n}$ 及 $g=\eta^{\otimes m+n}$, (3.13) 显然成立，从而 (3.13) 对一般的 f 及 g 成立. ∎

注 在引理中令 $m=0, G=1$, 则得

$$\langle F\otimes_l f, g\rangle = \langle F, f\otimes_n g\rangle. \tag{3.14}$$

3.2 积分核算子

从本小节开始，我们进一步假定

$$H=L^2(T,\mathcal{B}(T),\nu),$$

其中 T 为一 Hausdorff 空间， ν 为 $\mathcal{B}(T)$ 上的一 σ- 有限测度. 我们将把 H 中的元素与它的 ν- 等价类视为同一，并假定 H 为可分 Hilbert 空间. 例如，假定 Borel σ- 代数 $\mathcal{B}(T)$ 可分即可. 此外，我们还对拓扑空间 T 及 $L^2(T,\mathcal{B}(T),\nu)$ 上的正自共轭算子 A 作如下假定:

(H1) E 中每个元素在 T 上有连续版本，即对每个 $\xi\in E$, 存在 T 上连续函数 $\tilde{\xi}$, 使得对 ν-a.e.t, 有 $\xi(t)=\tilde{\xi}(t)$. 今后我们恒取 E 中元素的连续版本作为它所在等价类的代表元素;

(H2) 对一切 $t\in T$, 在 t 处的 Dirac δ- 函数 (或赋值映射) $\delta_t:\xi\longmapsto\xi(t)$ 为 E 上的连续线性泛函，即 $\delta_t\in E^*, \forall t\in T$;

(H3) $t \longmapsto \delta_t$ 为 T 到 E^* 中的连续映射.

今后我们用 ∂_t 及 ∂_t^* 分别记 D_{δ_t} 及 $D_{\delta_t}^*$, 称 ∂_t 为 **Hida 微分算子**, 并用 dt 简记 $\nu(dt)$. 此外, 用 $\|\cdot\|_p$ 简记 $\|\cdot\|_{p,0,0}$.

引理 3.5 对 $\varphi,\psi\in(E)_{\mathbb{C}}$, 令

$$\eta_{\varphi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m)=\langle\langle\partial_{s_1}^*\cdots\partial_{s_l}^*\partial_{t_1}\cdots\partial_{t_m}\varphi,\psi\rangle\rangle. \tag{3.15}$$

则 $\forall p>0$ 有

$$|\eta_{\varphi,\psi}|_p\le(1-\rho^{2p})^{-\frac{l+m+2}{2}}\sqrt{l!m!}\|\varphi\|_p\|\psi\|_p. \tag{3.16}$$

特别, $\eta_{\varphi,\psi}\in E_{\mathbb{C}}^{\widehat{\otimes}l}\otimes E_{\mathbb{C}}^{\widehat{\otimes}m}$.

证明 设 $\varphi\sim\{f_n\},\psi\sim\{g_n\}$. 由于

$$\eta_{\varphi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m)=\langle\langle\partial_{t_1}\cdots\partial_{t_m}\varphi,\partial_{s_1}\cdots\partial_{s_l}\psi\rangle\rangle,$$

故由 (1.18) 易知

$$\partial_{t_1}\cdots\partial_{t_m}\varphi\sim\left\{\frac{(n+m)!}{n!}\delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_m}\widehat{\otimes}_m f_{n+m}\right\},$$
$$\partial_{s_1}\cdots\partial_{s_l}\psi\sim\left\{\frac{(n+l)!}{n!}\delta_{s_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{s_l}\widehat{\otimes}_l g_{n+l}\right\}.$$

于是由 (3.13) 得

$$\begin{aligned}&\eta_{\varphi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m)\\&=\sum_{n=0}^{\infty}n!\frac{(n+m)!}{n!}\frac{(n+l)!}{n!}\langle\delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_m}\otimes_m f_{n+m},\delta_{s_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{s_l}\otimes_l g_{n+l}\rangle\\&=\sum_{n=0}^{\infty}\frac{(n+m)!(n+l)!}{n!}\langle\delta_{s_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{s_l}\otimes\delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_m},g_{n+l}\otimes_n f_{n+m}\rangle,\end{aligned}$$

即有

$$\eta_{\varphi,\psi}=\sum_{n=0}^{\infty}\frac{(n+m)!(n+l)!}{n!}g_{n+l}\otimes_n f_{n+m}. \tag{3.17}$$

由 (3.8)

$$|g_{n+l}\otimes_n f_{n+m}|_p\le \rho^{2pn}|f_{n+m}|_p|g_{n+l}|_p,$$

我们有

$$\begin{aligned}|\eta_{\varphi,\psi}|_p &\le \sum_{n=0}^{\infty}\frac{\sqrt{(n+m)!(n+l)!}}{n!}\rho^{2pn}\sqrt{(n+m)!}|f_{n+m}|_p\sqrt{(n+l)!}|g_{n+l}|_p\\ &\le C_{l,m,\rho,p}\|\varphi\|_p\|\psi\|_p,\end{aligned}\tag{3.18}$$

其中

$$C_{l,m,\rho,p}=\sup\nolimits_{n\ge 0}\frac{\sqrt{(n+m)!(n+l)!}}{n!}\rho^{2pn}.\tag{3.19}$$

由于

$$\sum_{n=0}^{\infty}\frac{(n+m)!}{n!}\rho^{2pn}=(1-\rho^{2p})^{-(m+1)}m!,$$

我们有

$$C_{l,m,\rho,p}\le(1-\rho^{2p})^{-\frac{l+m+2}{2}}\sqrt{l!m!}.\tag{3.20}$$

(3.16) 得证. ∎

注 由 (3.15) 定义的 $\eta_{\varphi,\psi}$ 与 (l,m) 有关，有时记为 $\eta_{\varphi,\psi}^{(l,m)}$. 如果从上下文可以明确 (l,m), 就简记为 $\eta_{\varphi,\psi}$.

在下一定义中我们引进今后常用的一些记号.

定义 3.6 (1) 设 $\kappa\in E_{\mathbb{C}}^{*\otimes n}$,$\sigma$ 为 $\{1,2,\cdots,n\}$ 的一个置换，则存在唯一的 $\kappa^{\sigma}\in E_{\mathbb{C}}^{*\otimes n}$, 使得

$$\langle\kappa^{\sigma},\xi_1\otimes\cdots\otimes\xi_n\rangle=\langle\kappa,\xi_{\sigma^{-1}(1)}\otimes\cdots\otimes\xi_{\sigma^{-1}(n)}\rangle,\xi_1,\cdots,\xi_n\in E_{\mathbb{C}},$$

设 $\kappa\in E_{\mathbb{C}}^{*\otimes l+m}$, 令

$$s_{l,m}(\kappa)=\frac{1}{l!m!}\sum_{\sigma\in\mathfrak{S}_l\times\mathfrak{S}_m}\kappa^{\sigma},$$

其中 $\mathfrak{S}_l$ 为 $\{1,2,\cdots,l\}$ 的置换全体. 则 $s_{l,m}(\kappa)$ 为 $E_{\mathbb{C}}^{*\widehat{\otimes}l}\otimes E_{\mathbb{C}}^{*\widehat{\otimes}m}$ 中唯一的元素，满足

$$\langle s_{l,m}(\kappa),\xi^{\otimes l}\otimes\eta^{\otimes m}\rangle=\langle\kappa,\xi^{\otimes l}\otimes\eta^{\otimes m}\rangle\ ,\xi,\eta\in E_{\mathbb{C}}.$$

我们称 $s_{l,m}(\kappa)$ 为 κ 的 $(\mathbf{l},\mathbf{m})$- 对称化.

(2) 设 $\kappa\in E_{\mathbb{C}}^{*\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m}$, 则存在唯一的 $t_{m,l}(\kappa)\in E_{\mathbb{C}}^{*\otimes m}\otimes E_{\mathbb{C}}^{*\otimes l}$, 使得

$$\langle t_{m,l}(\kappa),\eta\otimes\xi\rangle=\langle\kappa,\xi\otimes\eta\rangle,\ \eta\in E_{\mathbb{C}}^{\otimes m},\xi\in E_{\mathbb{C}}^{\otimes l}. \tag{3.21}$$

我们称 $t_{m,l}(\kappa)$ 为 κ 的 $(\mathbf{m},\mathbf{l})$- 换位.

定理 3.7 设 $\kappa\in E_{\mathbb{C}}^{*\otimes l+m}$, 则有唯一的 $\Xi_{l,m}(\kappa)\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}}^*)$, 使

$$\langle\langle\Xi_{l,m}(\kappa)\varphi,\psi\rangle\rangle=\langle\kappa,\eta_{\varphi,\psi}\rangle,\varphi,\psi\in(E)_{\mathbb{C}}, \tag{3.22}$$

其中 $\eta_{\varphi,\psi}$ 由 (3.15) 给出. 此外, 我们有 $\Xi_{l,m}(\kappa)=\Xi_{l,m}(s_{l,m}(\kappa))$,

$$\|\Xi_{l,m}(\kappa)\varphi\|_{-p}\le C_{l,m,\rho,p}|\kappa|_{-p}\|\varphi\|_p. \tag{3.23}$$

如果 $\varphi\in(E)_{\mathbb{C}},\varphi\sim\{f_n\}$, 则 $\Xi_{l,m}\varphi\sim\{h_n\}$, 其中

$$\begin{aligned}&h_n=0\ ,\ n\le l-1,\\&h_n=\frac{(m+n-l)!}{(n-l)!}s_{l,m}(\kappa)\widehat{\otimes}_m f_{n+m-l},\ n\ge l.\end{aligned} \tag{3.24}$$

若 $\kappa\in E_{\mathbb{C}}^{*\widehat{\otimes}l}\otimes E_{\mathbb{C}}^{*\widehat{\otimes}m}$, 则 $\Xi_{l,m}(\kappa)^*=\Xi_{m,l}(t_{m,l}(\kappa))$.

证明 由 (3.16) 易知 $\{\varphi,\psi\}\longmapsto\langle\kappa,\eta_{\varphi,\psi}\rangle$ 为 $(E)_{\mathbb{C}}\times(E)_{\mathbb{C}}$ 上的连续双线性泛函, 故由第一章定理 3.17 知, 存在 $\Xi_{l,m}(\kappa)\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}}^*)$ 使得 (3.22) 成立. (3.23) 由 (3.22) 及 (3.18) 推得. 由于 $\eta_{\varphi,\psi}\in E_{\mathbb{C}}^{\widehat{\otimes}l}\otimes E_{\mathbb{C}}^{\widehat{\otimes}m}$, 故 $\langle\kappa,\eta_{\varphi,\psi}\rangle=\langle s_{l,m}(\kappa),\eta_{\psi,\varphi}\rangle$, 从而有 $\Xi_{l,m}(\kappa)=\Xi_{l,m}(s_{l,m}(\kappa))$. 又由于 $\langle t_{m,l}(\kappa),\eta_{\psi,\varphi}\rangle=\langle\kappa,\eta_{\varphi,\psi}\rangle$, 故有 $\Xi_{l,m}(\kappa)^*=\Xi_{m,l}(t_{m,l}(\kappa))$. 最后证明 (3.24). 设 $\varphi\sim\{f_n\},\psi\sim\{g_n\}$, 则由 (3.17) 及 (3.13) 得

$$\begin{aligned}\langle\langle\Xi_{l,m}(\kappa)\varphi,\psi\rangle\rangle&=\sum_{n=0}^{\infty}\frac{(m+n)!(l+n)!}{n!}\langle\kappa,g_{l+n}\otimes_n f_{m+n}\rangle\\&=\sum_{n=0}^{\infty}\frac{(m+n)!(l+n)!}{n!}\langle s_{l,m}(\kappa)\otimes_m f_{n+m},g_{l+n}\rangle,\end{aligned} \tag{3.25}$$

由此推得 (3.24). ∎

注 1 设 $\kappa \in E_{\mathbb{C}}^{*\widehat{\otimes} l} \otimes E_{\mathbb{C}}^{*\widehat{\otimes} m}$, 则有

$$\begin{aligned}\langle\langle \Xi_{l,m}(\kappa)\mathcal{E}_f, \mathcal{E}_g\rangle\rangle &= \langle \kappa, \eta_{\mathcal{E}_f,\mathcal{E}_g}\rangle \\ &= \langle \kappa, g^{\otimes l} \otimes f^{\otimes m}\rangle e^{\langle f,g\rangle}. \end{aligned} \tag{3.26}$$

于是, $\Xi_{l,m}(\kappa)$ 为零算子当且仅当 $\kappa = 0$. 此外, 由 (3.26) 推知, 若另有 $w \in E_{\mathbb{C}}^{*\widehat{\otimes} j} \otimes E_{\mathbb{C}}^{*\widehat{\otimes} k}$, 则

$$\Xi_{l,m}(\kappa) \diamond \Xi_{k,j}(w) = \Xi_{l+k,m+j}(s_{l+k,m+j}(\kappa \otimes w)).$$

注 2 由 (3.22) 及 (3.15) 得: $\forall \varphi, \psi \in (E)_{\mathbb{C}}$,

$$\langle\langle \Xi_{l,m}(\kappa)\varphi, \psi\rangle\rangle = \langle \kappa, \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}\varphi, \psi\rangle\rangle \rangle.$$

因此, 我们可以把 $\Xi_{l,m}(\kappa)$ 及 $\Xi_{l,m}(\kappa)^*$ 形式地表示成

$$\begin{aligned}&\Xi_{l,m}(\kappa) \\ &= \int_{T^{l+m}} \kappa(s_1,\cdots s_l, t_1,\cdots,t_m)\partial_{s_1}^*\cdots\partial_{s_l}^*\partial_{t_1}\cdots\partial_{t_m} ds_1\cdots ds_l dt_1\cdots dt_m, \\ &\Xi_{l,m}(\kappa)^* \\ &= \int_{T^{l+m}} \kappa(s_1,\cdots s_l, t_1,\cdots,t_m)\partial_{t_1}^*\cdots\partial_{t_m}^*\partial_{s_1}\cdots\partial_{s_l} ds_1\cdots ds_l dt_1\cdots dt_m.\end{aligned}$$

我们称 $\Xi_{l,m}(\kappa)$ 为以 κ 为核的**积分核算子**.

下面我们将研究在什么条件下 $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$. 下一引理是解决这一问题的关键.

引理 3.8 设 $\kappa \in E_{\mathbb{C}}^{*\otimes l} \otimes E_{\mathbb{C}}^{*\otimes m}$ 与 $K \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{*\otimes l})$ 通过如下关系对应:

$$\langle Kf, g\rangle = \langle \kappa, g \otimes f\rangle = \langle \kappa \otimes_m f, g\rangle, g \in E_{\mathbb{C}}^{\otimes l}, f \in E_{\mathbb{C}}^{\otimes m}. \tag{3.27}$$

则下面条件等价:

(i) $\kappa \in E_{\mathbb{C}}^{\otimes l} \otimes E_{\mathbb{C}}^{*\otimes m}$;

(ii) $K \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{\otimes l})$;

(iii) $\forall p\geq 0,\exists C\geq 0,q\geq 0$,使得

$$|\langle\kappa,g\otimes f\rangle|\leq C|g|_{-p}|f|_{p+q}\ ,\forall g\in E_{\mathbb{C}}^{\otimes l},f\in E_{\mathbb{C}}^{\otimes m};\tag{3.28}$$

(iv) $\forall p\geq 0,\exists q\geq 0$, 使得 $|\kappa|_{l,m;p,-(p+q)}<\infty$.

证明 由于 $\mathcal{L}(E_{\mathbb{C}}^{\otimes m},E_{\mathbb{C}}^{\otimes l})\cong E_{\mathbb{C}}^{\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m}$(见 I.3.17), 故有 (i)$\Leftrightarrow$(ii).

(ii)$\Rightarrow$(iii). 由 K 的连续性, $\forall p\geq 0,\exists C\geq 0,q\geq 0$, 使得

$$|Kf|_p\leq C|f|_{p+q}.$$

于是由 (3.27) 推得 (3.28).

(iii)$\Rightarrow$(iv). 设 $p\geq 0$. 依假设, 存在 $C\geq 0,q\geq 1$, 使得

$$|\langle\kappa,g\otimes f\rangle|\leq C|g|_{-(p+1)}|f|_{p+q}\,,g\in E_{\mathbb{C}}^{\otimes l},f\in E_{\mathbb{C}}^{\otimes m}.$$

于是有

$$\begin{aligned}|\kappa|_{l,m;p,-(p+q+1)}^2&=\sum_{\alpha,\beta}|\langle\kappa,e_\alpha\otimes e_\beta\rangle|^2|e_\alpha|_p^2|e_\beta|_{-(p+q+1)}^2\\&\leq C^2\sum_{\alpha,\beta}|e_\alpha|_{-(p+1)}^2|e_\beta|_{p+q}^2|e_\alpha|_p^2|e_\beta|_{-(p+q+1)}^2\\&=C^2\sum_{\alpha,\beta}|e_\alpha|_{-1}^2|e_\beta|_{-1}^2\\&=C^2\delta^{2(l+m)}<\infty.\end{aligned}$$

(iv)$\Rightarrow$(ii). 由于 $Kf=\kappa\otimes_m f$, 故由 (3.7) 知, $\forall p\geq 0,\exists q\geq 0$, 使得

$$|Kf|_p\leq\rho^{qn}|\kappa|_{l,m;p,-(p+q)}|f|_{p+q}\quad,\ f\in E_{\mathbb{C}}^{\otimes m},$$

于是 $K\in\mathcal{L}(E_{\mathbb{C}}^{\otimes m},E_{C}^{\otimes l})$. ∎

定理 3.9 设 $\kappa\in E_{\mathbb{C}}^{*\otimes l+m}$, 则 $\Xi_{l,m}(\kappa)\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}})$ 当且仅当 $\kappa\in E_{\mathbb{C}}^{\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m}$. 此外, $\forall q>0$ 有

$$\|\Xi_{l,m}(\kappa)\varphi\|_p\leq C_{l,m,\sqrt{\rho},q}|s_{l,m}(\kappa)|_{l,m;p,-(p+q)}\|\varphi\|_{p+q}.\tag{3.29}$$

其中 $C_{l,m,\sqrt{\rho},q}$ 由 (3.19) 定义.

证明 设 $\varphi\in(E)_{\mathbb{C}},\varphi\sim\{f_n\}$, 则 $\Xi_{l,m}(\kappa)\varphi\sim\{h_n\}$, 其中 h_n 由 (3.24) 给出, 故对 $p\geq 0$, 由 (3.10) 有

$$
\begin{aligned}
&\|\Xi_{l,m}(\kappa)\varphi\|_p^2=\sum_{n=0}^{\infty}(l+n)!\Big(\frac{(n+m)!}{n!}\Big)^2|s_{l,m}(\kappa)\widehat{\otimes}_m f_{n+m}|_p^2\\
&\leq\sum_{n=0}^{\infty}(l+n)!\Big(\frac{(n+m)!}{n!}\Big)^2\rho^{2qn}|s_{l,m}(\kappa)|_{l,m;p,-(p+q)}^2|f_{n+m}|_{p+q}^2\\
&=|s_{l,m}(\kappa)|_{l,m;p-(p+q)}^2\sum_{n=0}^{\infty}(n+m)!|f_{n-m}|_{p+q}^2\frac{(l+n)!(n+m)!}{n!m!}\rho^{2qn}\\
&\leq C_{l,m,\sqrt{\rho},q}^2|s_{l,m}(\kappa)|_{l,m;p,-(p+q)}^2\|\varphi\|_{p+q}^2.
\end{aligned}
$$

于是 (3.29) 成立. 充分性得证.

往证必要性. 设 $\Xi_{l,m}(\kappa)\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}})$. 则 $\forall p\geq 0$, 存在 $C\geq 0,q\geq 0$, 使得

$$
\|\Xi_{l,m}(\kappa)\varphi\|_p\leq C\|\varphi\|_{p+q},\quad \varphi\in(E)_{\mathbb{C}}.
$$

特别令 $\varphi=I_m(f),\psi=I_l(g)$, 其中 $f\in E_{\mathbb{C}}^{\widehat{\otimes}m}$, $g\in E_{\mathbb{C}}^{\widehat{\otimes}l}$, 则由 (3.15) 易知 $\eta_{\varphi,\psi}^{(l,m)}=l!m!g\otimes f$. 故有

$$
\begin{aligned}
|\langle\kappa,g\otimes f\rangle|&=\frac{1}{l!m!}|\langle\langle\Xi_{l,m}(\kappa)\varphi,\psi\rangle\rangle|\leq\frac{C}{l!m!}\|\varphi\|_{p+q}\|\psi\|_{-p}\\
&=\frac{C}{\sqrt{l!m!}}|g|_{-p}|f|_{p+q}.
\end{aligned}
$$

于是由引理 3.8 知 $\kappa\in E_{\mathbb{C}}^{\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m}$. ∎

系 3.10 设 $\kappa\in E_{\mathbb{C}}^{*\otimes l}\otimes E_{\mathbb{C}}^{\otimes m}$, 则 $\Xi_{l,m}(\kappa)$ 可以延拓成为 $(E)_{\mathbb{C}}^*$ 中的连续线性算子.

证明 我们有 $t_{m,l}(\kappa)\in E_{\mathbb{C}}^{\otimes m}\otimes E_{\mathbb{C}}^{*\otimes l}$. 从而由定理 3.9 知

$$
\Xi_{l,m}(\kappa)^*=\Xi_{m,l}(t_{m,l}(\kappa))\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}}).
$$

由此立得欲证结论. ∎

下面我们给出几个积分核算子的例子.

例 1 设 $y \in E_{\mathbb{C}}^*$, 则 $D_y = \Xi_{0,1}(y), D_y^* = \Xi_{1,0}(y)$. 更一般地, 设 $y_1, \cdots, y_n \in E_{\mathbb{C}}^*$, 则 $D_{y_1} \cdots D_{y_m} = \Xi_{0,m}(\otimes_{j=1}^m y_j)$, $D_{y_1}^* \cdots D_{y_m}^* = \Xi_{m,0}(\otimes_{j=1}^m y_j)$.

例 2 令 $\kappa = \tau_1$ 为如下定义的 $E \otimes E^*$ 中的元素:

$$\langle \tau_1, g \otimes f \rangle = \langle g, f \rangle,\ g \in E, f \in E^*$$

(注意: τ_1 在 $E^* \otimes E^*$ 中的对称化即为由 (IV.1.2) 定义的 τ), 则通过 (3.27) 与 τ_1 对应的算子 K 是恒等算子, 且由定理 3.9 知, $\Xi_{1,1}(\tau_1) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$.

设 $f, g \in E_{\mathbb{C}}$, 则

$$\begin{aligned}\langle\langle \Xi_{1,1}(\tau_1)\mathcal{E}_f, \mathcal{E}_g \rangle\rangle &= \langle \tau_1, \eta_{\mathcal{E}_f, \mathcal{E}_g}^{(1,1)} \rangle \\ &= \langle \tau_1, g \otimes f \rangle e^{\langle f, g \rangle} = \langle g, f \rangle e^{\langle f, g \rangle}.\end{aligned}$$

另一方面, 设 N 为计数算子, 则

$$\begin{aligned}\langle\langle N\mathcal{E}_f, \mathcal{E}_g \rangle\rangle &= \langle\langle \sum_{n=1}^{\infty} \frac{1}{(n-1)!} I_n(f^{\otimes n}), \sum_{n=1}^{\infty} \frac{1}{n!} I_n(g^{\otimes n}) \rangle\rangle \\ &= \sum_{n=1}^{\infty} \frac{\langle f, g \rangle^n}{(n-1)!} = \langle f, g \rangle e^{\langle f, g \rangle}.\end{aligned}$$

因此我们有

$$N = \Xi_{1,1}(\tau_1),$$

即 N 有如下形式表达式:

$$N = \int_{T^2} \tau_1(s,t) \partial_s^* \partial_t ds dt = \int_T \partial_t^* \partial_t dt. \tag{3.30}$$

例 3 考虑 $\Xi_{0,2}(\tau)$. 由定理 3.9 知 $\Xi_{0,2}(\tau) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$. 我们称 $\Xi_{0,2}(\tau)$ 为 **Gross 意义下的 Laplace 算子**, 记为 Δ_G, 其形式表达为

$$\Delta_G = \int_{T^2} \tau(s,t) \partial_s \partial_t ds dt = \int_T \partial_t^2 dt. \tag{3.31}$$

$\triangle_G$ 的对偶算子 $\triangle_G^* = \Xi_{2,0}(\tau)$, 其形式表达为

$$\triangle_G^* = \int_{T^2} \tau(s,t)\partial_s^*\partial_t^* ds dt = \int_T \partial_t^{*2} dt.$$

事实上, $\forall\varphi \in (E)_{\mathbb{C}}, \forall F \in (E)_{\mathbb{C}}^*$, 我们有 (通过 S- 变换验证)

$$\triangle_G \varphi = \int_T \partial_t^2 \varphi dt, \tag{3.32}$$

$$\triangle_G^* F = \int_T \partial_t^{*2} F dt = I_2(\tau) \diamond F. \tag{3.33}$$

(3.32) 中的积分是 Bochner 意义下的积分. 此外, 若 $\varphi \sim \{f_n\}$, 则 $\triangle_G \varphi \sim \{h_n\}$, 其中

$$h_n = (n+2)(n+1)\tau \otimes_2 f_{n+2}, \quad n \geq 0.$$

3.3 广义算子的积分核表示

由 §2 知, 广义算子 $A \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}}^*)$ 有如下混沌分解:

$$A\varphi = \sum_{l,m=0}^{\infty} I_{l,m}(a_{l,m})\varphi, \ \varphi \in (E)_{\mathbb{C}}, \tag{3.34}$$

其中 $I_{l,m}(a_{l,m})$ 如 (2.12) 定义, 即有

$$\widehat{I_{l,m}(a_{l,m})}(f,g) = \langle a_{l,m}, g^{\otimes l} \otimes f^{\otimes m}\rangle, \tag{3.35}$$

序列 $\{a_{l,m}\}$ 由 A 如下确定:

$$\langle\langle AI_m(f^{\otimes m}), I_l(g^{\otimes l})\rangle\rangle = l!m!\langle a_{l,m}, g^{\otimes l} \otimes f^{\otimes m}\rangle, \ f,g \in E. \tag{3.36}$$

类似于积分核算子, 下面我们借助于算子 ∂_t 及 ∂_t^* 给出 $I_{l,m}(a_{l,m})$ 的形式表示. 为此, 设 $\varphi, \psi \in (E)_{\mathbb{C}}, \varphi \sim \{f_n\}, \psi \sim \{g_n\}$, 则有

$$\begin{aligned}
&\langle\langle I_{l,m}(a_{l,m})\varphi, \psi\rangle\rangle \\
&= \langle\langle I_{l,m}(a_{l,m}) I_m(f_m), I_l(g_l)\rangle\rangle \\
&= l!m!\langle a_{l,m}, g_l \otimes f_m\rangle \\
&= l!m! \int_{T^{l+m}} a_{l,m}(\mathbf{s},\mathbf{t}) g_l(\mathbf{s}) f_m(\mathbf{t}) ds_1, \cdots ds_l dt_1, \cdots dt_m.
\end{aligned} \tag{3.37}$$

其中 $\mathbf{s}=(s_1,\cdots,s_l),\mathbf{t}=(t_1,\cdots,t_m)$. 另一方面，由 (1.18) 易知

$$\langle\langle\partial_{t_1}\cdots\partial_{t_m}\varphi,1\rangle\rangle=m!f_m(t_1,\cdots,t_m),$$

于是

$$l!m!g_l(\mathbf{s})f_m(\mathbf{t})=\langle\langle\langle\langle\partial_{t_1}\cdots\partial_{t_m}\varphi,1\rangle\rangle,\partial_{s_1}\cdots\partial_{s_l}\psi\rangle\rangle.$$

如果我们用 P_0 表示到 0- 阶混沌的投影算子 (即 $\forall F\in(E)_{\mathbb{C}}^*,P_0F=\langle\langle F,1\rangle\rangle$), 则上式可改写成

$$g_l(s_1,\cdots,s_l)f_m(t_1,\cdots,t_m)=\langle\langle\partial_{s_1}^*\cdots\partial_{s_l}^*P_0\partial_{t_1}\cdots\partial_{t_m}\varphi,\psi\rangle\rangle. \tag{3.38}$$

因此由 (3.37) 及 (3.38) 我们得到算子 $I_{l,m}(a_{l,m})$ 的如下形式表示:

$$\begin{aligned}&I_{l,m}(a_{l,m})\\&=\int_{T^{l+m}}a_{l,m}(\mathbf{s},\mathbf{t})\partial_{s_1}^*\cdots\partial_{s_l}^*P_0\partial_{t_1}\cdots\partial_{t_m}ds_1\cdots ds_ldt_1\cdots dt_m.\end{aligned} \tag{3.39}$$

下面我们利用积分核算子给出广义算子的另一种分解 (称为**积分核表示**).

定理 3.11 设 $A\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}}^*)$. 则存在 $\kappa_{l,m}\in E_{\mathbb{C}}^{*\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m},l,m\geq 0$, 使得

$$A\varphi=\sum_{l,m=0}^{\infty}\Xi_{l,m}(\kappa_{l,m})\varphi,\ \varphi\in(E)_{\mathbb{C}}, \tag{3.40}$$

其中级数在 $(E)_{\mathbb{C}}^*$ 中强收敛. 若 $A\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}})$, 则相应的 $\kappa_{l,m}\in E_{\mathbb{C}}^{\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m}$, 且 (3.40) 中级数在 $(E)_{\mathbb{C}}$ 中收敛. 算子 A 的 S- 变换为

$$\widetilde{A}(f,g)=\sum_{l,m=0}^{\infty}\langle\kappa_{l,m},g^{\otimes l}\otimes f^{\otimes m}\rangle,\ f,g\in E_{\mathbb{C}}. \tag{3.41}$$

此外，我们可取

$$\kappa_{l,m}=\sum_{n=0}^{l\wedge m}\frac{(-1)^n}{n!}a_{l-n,m-n}\otimes\tau_n, \tag{3.42}$$

其中 $\{a_{l,m}\}$ 由 (3.36) 给出， $\tau_n \in E^{\hat{\otimes} n} \otimes E^{*\hat{\otimes} n}$ 由下式确定：

$$\langle \tau_n, g^{\otimes n} \otimes f^{\otimes n} \rangle = \langle f, g \rangle^n,\ g \in E, f \in E^*. \tag{3.43}$$

我们称 $\{\kappa_{l,m}\}$ 为算子 A 的 **积分核序列**.

证明 显然 $\widetilde{A}$ 为 $U_{0,0}$- 泛函，故存在 $B \in \mathcal{L}((E)_{\mathbb{C}}, (E)^*_{\mathbb{C}})$ 及 $\{\kappa_{l,m} \in E^{*\otimes l}_{\mathbb{C}} \otimes E^{*\otimes m}_{\mathbb{C}}, l, m \in \mathbb{N}_0\}$, 使得 $\widehat{B}(f,g) = \widetilde{A}(f,g), \forall f, g \in E_{\mathbb{C}}$, 且使 (3.41) 成立. 由 (2.9) 知, 存在 $p_1 \geq 0$, 当 p 充分大， $\kappa_{l,m}$ 有如下的范数估计：

$$|\kappa_{l,m}|^2_{-p} \leq C^2 (l!m!)^{-1} (2e^2 K)^{l+m} \|I_{p_1 p}\|^{2(l+m)}_{\mathrm{HS}}. \tag{3.44}$$

于是由 (3.23) 及 (3.20) 得

$$\|\Xi_{l,m}(\kappa_{l,m})\varphi\|_{-p} \leq C(1-\rho^{2p})^{-\frac{l+m+2}{2}} (2e^2 K)^{\frac{l+m}{2}} \|I_{p_1 p}\|^{l+m}_{\mathrm{HS}} \|\varphi\|_p.$$

由于

$$\begin{aligned} \|I_{p_1, p_1+k+1}\|^2_{\mathrm{HS}} &= \|A^{-(k+1)}\|^2_{\mathrm{HS}} \\ &= \sum_{j=0}^{\infty} \lambda_j^{-2(k+1)} \leq \rho^{2k} \|A^{-1}\|^2_{\mathrm{HS}}, \end{aligned}$$

取 $p > p_1$ 足够大可使 $\|I_{p_1 p}\|^2_{\mathrm{HS}} < (2e^2 K)^{-1}(1-\rho^{2p})$. 这时有

$$\sum_{l,m=0}^{\infty} \|\Xi_{l,m}(\kappa_{l,m})\varphi\|_{-p} < \infty.$$

这表明 (3.40) 中级数在 $(E)^*_{\mathbb{C}}$ 中强收敛，且由 (3.26) 及 (3.41) 知 (3.40) 成立. 若 $A \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$, 则由 (3.41) 易知， $\kappa_{l,m} \in E^{\otimes l}_{\mathbb{C}} \otimes E^{*\otimes m}_{\mathbb{C}}$, 且类似可证 (3.40) 中级数在 $(E)_{\mathbb{C}}$ 中收敛.

最后， (3.42) 容易由如下等式推得：

$$\begin{aligned} &\sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, g^{\otimes l} \otimes f^{\otimes m} \rangle \\ &= \sum_{k,j=0}^{\infty} \langle a_{k,j}, g^{\otimes k} \otimes f^{\otimes j} \rangle \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle f, g \rangle^n. \end{aligned}$$ ∎

注 1 由于未要求 $\kappa_{l,m}$ 为 (l,m)- 对称的，故序列 $\{\kappa_{l,m}\}$ 不唯一，但 $\kappa_{l,m}$ 的 (l,m)- 对称化唯一.

注 2 设 A' 和 A'' 为广义算子，其积分核序列分别是 $\{\kappa'_{l,m}\}$ 和 $\{\kappa''_{l,m}\}$, 则算子 $A'\diamond A''$ 的积分核序列是 $\{\kappa_{l,m}\}$:

$$\kappa_{l,m}=\sum_{i+k=l,j+n=m}\kappa'_{i,j}\otimes\kappa''_{k,n}. \tag{3.45}$$

需要指出，本节所有结果都可用局部 S- 变换改述为 $\beta=1$ 情形.

下面给出广义算子积分核表示的若干例子，它们都可以通过算子象征的 Taylor 展开来证明. 请读者自行补足之.

例 1 设 P_0 为到 0- 阶混沌的投影算子，我们有

$$P_0=\sum_n(-1)^n(n!)^{-1}\Xi_{n,n}(\tau_n).$$

例 2 设 $y\in E^*$, 则推移算子 τ_y 有如下表示:

$$\tau_y=\sum_{n=0}^{\infty}\frac{1}{n!}\Xi_{0,n}(y^{\otimes n}).$$

作为推论，我们得到 $\tau_y\varphi$ 的 Taylor 展开 (见 (1.23)):

$$\tau_y\varphi=\sum_{n=0}^{\infty}\frac{1}{n!}D_y^n\varphi,\quad \varphi\in(E)_{\mathbb{C}},$$

其中级数在 $(E)_{\mathbb{C}}$ 中收敛.

例 3 令 $\varphi\in(E)^*_{\mathbb{C}},\varphi\sim\{f_n\}$. 作为乘积算子，它的积分核表示为

$$\varphi=\sum_{l,m=0}^{\infty}\binom{l+m}{m}\Xi_{l,m}(f_{l+m}).$$

此外，$\varphi\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}})$, 当且仅当 $\varphi\in(E)_{\mathbb{C}}$.

例 4 设 $\lambda \in \mathbb{C}$, 则刻度变换算子 σ_λ 有如下表示:

$$\sigma_\lambda = \sum_{l,m=0}^{\infty} \frac{(\lambda-1)^l(\lambda^2-1)^m}{l!m!2^m} \Xi_{l,l+2m}(\tau_l \otimes \tau^{\otimes m}).$$

其中 τ_l 如 (3.43) 定义.

例 5 设 $\lambda \in \mathbb{C}$, 则

$$\Gamma(\lambda) = \sum_{m=0}^{\infty} \frac{(\lambda-1)^m}{m!} \Xi_{m,m}(\tau_m).$$

§4. 在量子物理中的若干应用

本节介绍广义泛函空间中的算子理论在量子物理中的若干应用. 我们用广义算子来定义量子随机积分, 该积分推广了 Hudson-Parthasarathy 意义下的量子随机积分; 用广义算子及 Wick 运算给出了 Klein- Gordon 场的一种严格的数学解释. 此外, 我们在白噪声框架下研究了无穷维经典 Dirichlet 型. 关于广义算子理论在无穷维调和分析及量子概率中的应用, 读者可参看 Obata[4,5,6,8].

4.1 量子随机积分

令测度空间 $(T, \mathcal{B}(T), \nu)$ 满足 3.2 节中的假定, 我们将采用 3.2 节中的记号. 设 $\{K_t, t \in T\}$ 为一 $\mathcal{L}((E)_{\mathbb{C}}, (E)^*_{\mathbb{C}})$- 值可测过程, M 为 T 的一 Borel 子集, 我们可以定义如下形式的积分:

$$\int_M K_t W_{l,m}(dt) \equiv \int_M K_t \diamond \partial_t^{*l} \partial_t^m dt, \tag{4.1}$$

只要上式右端的 Bochner 积分在 $\mathcal{L}((E)_{\mathbb{C}}, (E)^*_{\mathbb{C}})$ 中存在. 这里左边的积分是一个形式记号, 称 $W_{l,m}(dt) = \partial_t^{*l}\partial_t^m dt$ 为 **量子白噪声测度**. 由 (4.1) 定义的积分的 S- 变换为

$$G(\xi, \eta) = \int_M \widetilde{K}_t(\xi, \eta)\eta(t)^l \xi(t)^m dt, \ \xi, \eta \in E. \tag{4.2}$$

特别, 当 $T = \mathbb{R}_+,\nu$ 为 Lebesque 测度, $\{K_t\}$ 为 $L^2(E^*,\mu)$ 上的算子值适应过程时, 以上定义的积分本质上即为 Hudson-Parthasarathy 意义下的量子随机积分. 这时令

$$A_t^* = \int_0^t \partial_s^* ds,\ A_t = \int_0^t \partial_s ds,\ N_t = \int_0^t \partial_s^* \partial_s ds, \tag{4.3}$$

分别称不定积分 A_t^* 、 A_t 及 N_t 为量子增生过程、湮灭过程及计数过程. 与经典的 Brown 运动及 Poisson 过程对应的量子 Brown 运动和量子 Poisson 过程分别是

$$Q_t = A_t^* + A_t, \quad P_t^\lambda = N_t + \sqrt{\lambda} Q_t + \lambda t, \tag{4.4}$$

其中 $\lambda > 0$. 事实上, 每个 Q_t 及 P_t^λ 为 $L^2(E^*,\mu)$ 上的自共轭算子, 且 $\{Q_t 1\}$ 为标准 Brown 运动, $\{P_t^\lambda 1\}$ 为强度 λ 的 Poisson 过程 (见 Meyer[3]).

由 (4.1) 定义的积分从两方面推广了 Hudson-Parthasarathy 意义下的量子随机积分. 其一是被积过程 $\{K_t\}$ 不限于 $L^2(E^*,\mu)$ 上的算子值过程, 它可以是广义算子值过程; 其二是不要求被积过程为适应过程. 此外, 这种量子随机积分将参数空间 $\mathbb{R}_+$ 推广到了较一般的测度空间. 上述推广对研究随机场和量子场是有用的.

下面解释如何借助算子乘积的 Wick 编序导出量子随机积分的 Itô 公式. 前面说过, $Q_t = A_t^* + A_t$ 为量子 Brown 运动. 利用对易关系 $[A_t, A_t^*] = t$ 容易证明: $\forall n \in \mathbb{N}_0$

$$Q_t^n = \sum_{2j+k+l=n} \frac{n!}{2^j j!\, k!\, l!} t^j A_t^{*k} A_t^l. \tag{4.5}$$

令 $f(x) = x^n$, 将 $f(Q_t) = Q_t^n$ 视为变量 t, A_t^* 及 A_t 的函数, 则按

微分的链式法则得

$$
\begin{aligned}
df(Q_t) = & \sum_{2j+k+l=n} \frac{n!}{2^j j!\,k!\,l!}\left[t^j k A_t^{*k-1} A_t^l dA_t^*\right. \\
& \left. + t^j l A_t^{*k} A_t^{l-1} dA_t + j t^{j-1} A_t^{*k} A_t^l dt\right] \\
= & \, n \sum_{2j+k+l=n-1} \frac{(n-1)!}{2^j j!\,k!\,l!} t^j A_t^{*k} A_t^l d(A_t^* + A_t) \\
& + \frac{n(n-1)}{2} \sum_{2j+k+l=n-2} \frac{(n-2)!}{2^j j!\,k!\,l!} t^j A_t^{*k} A_t^l dt.
\end{aligned}
$$

从而有

$$
df(Q_t) = f'(Q_t)dQ_t + \frac{1}{2} f''(Q_t)dt. \tag{4.6}
$$

于是我们证明了 f 为多项式情形的**量子 Itô 公式**.

在前面的推导中，我们先将 Q_t^n 表示成算子 A_t^* 与 A_t 按 Wick 序的积的某种函数，然后将 A_t^* 和 A_t 看成普通变量，对 Q_t^n 按普通微分的链式法则进行微分. 由于 A_t 与 A_t^* 是非交换的，所以在导出 (4.5) 式时用到了对易关系 $[A_t, A_t^*] = t$, 这导致了 Itô 公式中二阶项的出现.

有了量子随机积分我们就可以讨论量子随机微分方程. 求解量子随机微分方程通常可以用 S- 变换将它转化为求解泛函的积分方程. 下面举两个例子来说明这一点.

例 1 考虑量子随机微分方程

$$
dX_t = X_t dA_t,
$$

此即如下的积分方程

$$
X_t = X_0 + \int_0^t X_s \diamond \partial_s ds.
$$

在方程两边取 S- 变换得

$$
\widetilde{X}_t(\xi,\eta) = \widetilde{X}_0(\xi,\eta) + \int_0^t \widetilde{X}_s(\xi,\eta)\xi(s)ds.
$$

因此有

$$\widetilde{X}_t(\xi,\eta)=\widetilde{X}_0(\xi,\eta)\exp\left\{\int_0^t\xi(s)ds\right\},$$

从而

$$X_t=X_0\diamond e^{A_t^*},$$

其中 X_0 为广义算子.

例 2 考虑量子随机积分方程

$$X_t\equiv X_0+\int_0^t X_s\diamond A_s^{*l}A_s^m ds.$$

取 S- 变换得

$$\widetilde{X}_t(\xi,\eta)=\widetilde{X}_0(\xi,\eta)+\int_0^t\widetilde{X}_s(\xi,\eta)\eta(s)^l\xi(s)^m ds.$$

从而有

$$\widetilde{X}_t(\xi,\eta)=\widetilde{X}_0(\xi,\eta)\exp\left\{\int_0^t\eta(s)^l\xi(s)^m ds\right\}.$$

若用 $\exp^\diamond K$ 表示 K 的 "Wick 指数 ": $\sum_{n=0}^\infty(n!)^{-1}K^{\diamond n}$, 则有

$$X_t=X_0\diamond\exp^\diamond(A_t^{*l}A_t^m).$$

容易证明 $\exp^\diamond(A_t^{*l}A_t^m)$ 仍是广义算子, 所以方程在 $\mathcal{L}((E),(E)^*)$ 中有唯一解.

4.2 Klein-Gordon 场

设 $\square=\nabla_t^2-\nabla_x^2=\frac{\partial^2}{\partial t^2}-\sum_{j=1}^3\frac{\partial^2}{\partial x_j^2}$ 为 Minkowski 空间 $R\times R^3$ 上的波算子, 物理上的自由场 $\{\phi(t,x):(t,x)\in\mathbb{R}\times\mathbb{R}^3\}$ 作为一定意义下的算子值分布满足 Klein-Gordon 方程 ($m>0$ 为常数)

$$(\square+m^2)\phi=0 \tag{4.7}$$

以及等时对易关系

$$[\phi(t,x),\phi(t,y)]=0,\quad [\dot\phi(t,x),\dot\phi(t,y)]=0,$$
$$[\phi(t,x),\dot\phi(t,y)]=i\delta(x-y),$$

其中 $\dot\phi(t,x)=\frac{\partial}{\partial t}\phi(t,x)$. 按物理上的形式推导，有

$$\phi(t,x)=\int_{R^3}\left[f_k^*(t,x)\partial_k^*+f_k(t,x)\partial_k\right]dk,$$

其中 ∂_k^* 与 ∂_k 为 Fock 空间 $\Gamma(L^2(R^3))$ 上点态增生算子与湮灭算子，而

$$f_k^*(t,x)=\frac{1}{\sqrt{(2\pi)^3 2\omega_k}}e^{i\omega_k t-ikx},$$
$$f_k(t,x)=\frac{1}{\sqrt{(2\pi)^3 2\omega_k}}e^{-i\omega_k t+ikx}$$

为经典场方程

$$(\Box+m^2)f=0 \tag{4.8}$$

的一对共轭解，其中 $\omega_k=\sqrt{k^2+m^2}$. Klein-Gordon 场 $\phi(t,x)$ 的能量算子与动量算子分别是

$$H(t)=\frac{1}{2}\int_{R^3}[(\nabla_t\phi)^2+(\nabla_x\phi)^2+m^2\phi^2]dx, \tag{4.9}$$
$$P(t)=-\int_{R^3}\nabla_t\phi\cdot\nabla_x\phi dx. \tag{4.10}$$

从数学上看，以上推导过程尚需明确化，特别是：

(i) ∂_k^* 并非 Fock 空间上算子 (其定义域为 $\{0\}$), 而只能解释为广义算子或算子值分布；

(ii) 算子值分布的乘积，例如 $(\nabla_t\phi)^2,(\nabla_x\phi)^2$ 及 ϕ^2 不是普通的乘积，只能解释为 Wick 积.

下面我们将在广义算子的框架下将以上运算严格化. 令

$$H = L^2(\mathbb{R}^3),\ E = \mathcal{S}(\mathbb{R}^3),\ E^* = \mathcal{S}^*(\mathbb{R}^3).$$

考虑经典的白噪声分析框架. 以下用 $\mathcal{L}$ 简记空间 $\mathcal{L}((E)_{\mathbb{C}}, (E)^*_{\mathbb{C}})$. 直接将 Klein-Gordon 方程 (4.7) 视为关于抽象函数 $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathcal{L}$ 的波动方程, 则容易验证 $\mathcal{L}$- 值函数

$$\phi(t,x) = \int_{\mathbb{R}^3} \left[f_k^*(t,x)\partial_k^* + f_k(t,x)\partial_k\right] dk$$

满足方程 (4.7).$\phi(t,x)$ 的 S- 变换为

$$\widetilde{\phi(t,x)}(\xi,\eta) = \int_{\mathbb{R}^3} \left[f_k^*(t,x)\eta(k) + f_k(t,x)\xi(k)\right] dk,\ \xi,\eta \in \mathcal{S}(\mathbb{R}^3),$$

它满足经典波动方程 (4.8).

利用 Wick 积, 重正化了的能量算子与动量算子分别是

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left[(\nabla_t\phi)^{\diamond 2} + (\nabla_x\phi)^{\diamond 2} + m^2\phi^{\diamond 2}\right] dx,$$
$$P(t) = -\int_{\mathbb{R}^3} \nabla_t\phi \diamond \nabla_x\phi dx,$$

它们的 S- 变换分别是

$$\widetilde{H(t)}(\xi,\eta) = \frac{1}{2}\int_{\mathbb{R}^3} \left[(\nabla_t\tilde{\phi})^2 + (\nabla_x\tilde{\phi})^2 + m\tilde{\phi}^2\right] dx,$$
$$\widetilde{P(t)}(\xi,\eta) = -\int_{\mathbb{R}^3} \nabla_t\tilde{\phi} \cdot \nabla_x\tilde{\phi}\, dx.$$

利用上式直接计算得: $\forall \xi,\eta \in \mathcal{S}(\mathbb{R}^3), \frac{d}{dt}\widetilde{H(t)} = 0, \frac{d}{dt}\widetilde{P(t)} = 0$, 从而 $H(t) = H(0), P(t) = P(0)$, 此即能量守恒与动量守恒. 经过直接但稍长的运算, 可以推得如下结果 (详见 Huang-Luo[2] 及骆顺龙 [1]):

$$H(0) = \int_{\mathbb{R}^3} \omega_k \partial_k^* \partial_k dk,$$
$$P(0) = \int_{\mathbb{R}^3} k \partial_k^* \partial_k dk.$$

因此，从广义算子出发，经过 Wick 运算，我们实际上得知 $H(t)$ 和 $P(t)$ 的每个分量都属于 $\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}})$ 及 $\mathcal{L}((E)_{\mathbb{C}}^*,(E)_{\mathbb{C}}^*)$，并且它们均为 Fock 空间 $\Gamma(H)$ 上本性自共轭算子. 此外，还可以证明下面的 Heisenberg 方程成立：

$$\nabla_t\phi(t,x)=i[H(t),\phi(t,x)], \tag{4.11}$$

其中右边的交换子为广义算子.

4.3 无穷维经典 Dirichlet 型

Dirichlet 型是经典的 Dirichlet 积分的推广. 1959 年 Beurling 和 Deny 首次引进 Dirichlet 空间概念，建立了位势论的 L^2- 框架. 到 70 年代，Fukushima 和 Silverstein 等人将它进一步发展成了一个系统的 Dirichlet 型理论. 这一理论是联系位势论和 Markov 过程的一个桥梁. 由于它在非相对论量子力学及 Euclid 场论中有重要应用，近年来这一理论发展迅速，其中最重要的理论突破是由马志明和 Albeverio 建立的拟正则狄氏型框架，该框架特别适用于无穷维分析. 有兴趣的读者可参看马志明和 Röckner 的专著 [1].

下面介绍白噪声分析在 Dirichlet 型中的一个应用，内容取自 Hida-Kuo-Potthoff-Streit[1].

设 X 为一可分距离空间，m 为 $\mathcal{R}(X)$ 上一 σ- 有限测度，则 $H=L^2(X,m)$ 为一可分 Hilbert 空间. 设 $(\mathcal{E},\mathcal{D})$ 为 $L^2(X,m)$ 上的一对称稠定正双线性型. 令 $\mathcal{E}_1(u,v)=\mathcal{E}(u,v)+(u,v), u,v\in\mathcal{D}$. 若 $\mathcal{D}$ 在 $\mathcal{E}_1$- 范数下完备，则称 $(\mathcal{E},\mathcal{D})$ 为闭的. 如果下述条件成立：

$$u\in\mathcal{D}\Rightarrow u^+\wedge 1\in\mathcal{D},\ \mathcal{E}(u^+\wedge 1,u^+\wedge 1)\le\mathcal{E}(u,u),$$

则称 $(\mathcal{E},\mathcal{D})$ 有压缩性 (或马氏性). 我们把 H 上具有马氏性的对称闭正双线性型 $(\mathcal{E},\mathcal{D})$ 称为 H 上的 Dirichlet 型.

在非相对论量子力学中，一个具有 d- 个自由度的系统的动力学由 $L^2(\mathbb{R}^d,dx)$ 上一正自共轭算子 H(称为 Hamilton 算子或能量算子) 决定：$H=-\frac{1}{2}\Delta+V$, 其中 Δ 为 Laplace 算子，V 为位

势函数. 假定算子 H 关于 $C_0^\infty(I\!R^d)$ 为本性自共轭且下有界，则 H 的最小本征值为单重的，相应的本征函数 φ 可取为严格正，且 $\int_{I\!R^d}\varphi^2(x)dx=1$. 称 φ 为基态 (或真空态). 为简单起见，不妨设 H 的最小本征值为 0. 令 $\nu(dx)=\varphi^2(x)dx, Wf(x)=f(x)/\varphi(x)$, 则 W 为 $L^2(I\!R^d,dx)$ 到 $L^2(I\!R^d,d\nu)$ 上的酉变换. 令 $H_\nu=WHW^{-1}$, 则 H_ν 为 $L^2(I\!R^d,\nu)$ 上的正自共轭算子. 称 H_ν 为 H 的基态表示. 如果 φ 比较光滑，则由分部积分公式知， $H_\nu=-\frac{1}{2}\Delta+b\nabla$, 其中 $b=-\nabla\log\varphi$, 但当 V 是奇异位势时， H_ν 没有上述表达式. 这时令

$$\mathcal{E}_\nu(u,v)=\frac{1}{2}\int_{I\!R^d}\nabla u\cdot\nabla v\ d\nu,\ u,v\in C_0^\infty(I\!R^d), \tag{4.12}$$

则 $(\mathcal{E}_\nu, C_0^\infty(I\!R^d))$ 的最小闭扩张为 Dirichlet 型, 称为**经典 Dirichlet 型**, 它联系于一扩散过程. 用这种基态表示，我们可以讨论具有奇异位势的 Hamilton 算子. 在量子场论中，与之相对应的是无穷维经典 Dirichlet 型:

$$\mathcal{E}(\varphi,\psi)=\int_{E^*}D\varphi\cdot D\psi\ d\nu,\ \varphi,\psi\in(E), \tag{4.13}$$

其中 $E=\mathcal{S}(I\!R^d), E^*=\mathcal{S}^*(I\!R^d)$, ν 为 $\mathcal{S}^*(I\!R^d)$ 上的一有限测度,

$$D\varphi\cdot D\psi=\sum_{j=1}^\infty(D_{e_j}\varphi)(D_{e_j}\psi). \tag{4.14}$$

这里及以下，我们采用本章 1.3 节及第四章 1.3 节中的记号. 特别，令 A 表示 $L^2(I\!R)$ 中的谐振子，设 $\{e_j, j\ge 1\}$ 为由 $A^{\otimes d}$ 的本征函数构成的 $L^2(I\!R^d)$ 的基. 一个重要的问题是：对测度 ν 加什么条件，由 (4.13) 定义的双线性型可以扩张成为一 Dirichlet 型？下面我们将讨论这一问题.

首先，假定 ν 是与一正的 Hida 广义泛函 Φ 联系的测度. 我们将证明 $D\varphi\cdot D\psi\in(E)$, 于是 (4.13) 有明确定义，且有

$$\mathcal{E}(\varphi,\phi)=\langle\langle D\varphi\cdot D\psi,\Phi\rangle\rangle,\ \varphi,\psi\in(E). \tag{4.15}$$

今后我们用 $\mathcal{E}_\Phi$ 表示由 (4.15) 定义的 (E) 上的双线性型.

在下一引理中 $\|\cdot\|_p$ 不是 L^p- 范数，而是 $(E)_p$ 中的范数 (见第四章 1.3 节).

引理 4.1 设 $\varphi,\psi\in(E)$, 则 $D\varphi\cdot D\psi\in(E)$. 此外，存在常数 $C>0$, 使得 $\forall p\in\mathbb{N}_0$, 有

$$\|\,|D\varphi|^2-|D\psi|^2\,\|_p\le C\|\varphi-\psi\|_{p+2}(\|\varphi\|_{p+2}+\|\psi\|_{p+2}),\qquad(4.16)$$

其中 $|D\varphi|^2=D\varphi\cdot D\varphi$. 特别，$\varphi\mapsto|D\varphi|^2$ 为 (E) 到 (E) 中的连续映射.

证明 设 $F\in(E),F\sim\{f_n\}$. 由第四章 (1.25) 知,

$$\sum_{n=0}^{\infty}n!\|A^{-1}\|^{-2n}|f_n|_p^2\le\sum_{n=0}^{\infty}n!|f_n|_{p+1}^2.$$

注意到 $\|A^{-1}\|<1$, 于是由第四章 (4.11) 知，存在 $K>0$, 使得

$$\|(D_{e_j}\varphi)^2\|_p\le K\|D_{e_j}\varphi\|_{p+1}^2.$$

由此进一步利用 (1.19) 易知，存在常数 $K'>0$, 使得

$$\begin{aligned}\|(D_{e_j}\varphi)^2\|_p&\le K'|e_j|_{-(p+1)}^2\|\varphi\|_{p+2}^2\\&\le K'|e_j|_{-1}^2\|\varphi\|_{p+2}^2.\end{aligned}$$

于是我们有

$$\|\,|D\varphi|^2\,\|_p\le K'\|A^{-1}\|_{\mathrm{HS}}^2\|\varphi\|_{p+2}^2.$$

这表明 $|D\varphi|^2\in(E)$.

为证不等式 (4.16), 我们将 (4.16) 左端改写成:

$$\|\,|D\varphi|^2-|D\psi|^2\,\|_p=\|\sum_{k=1}^{\infty}D_{e_k}(\varphi-\psi)D_{e_k}(\varphi+\psi)\|_p,$$

按照上述证明不难推得 (4.16). ∎

由于 ν 是 $(\mathcal{S}^*(\mathbb{R}^d),\mathcal{B})$ 上的一 Radon 测度，从而多项式光滑泛函 $\mathcal{P}$ 在 $L^2(\nu)$ 中稠. 但 $\mathcal{P}\subset(E)$, 故由 (4.15) 定义的双线性型在 $L^2(\nu)$ 中是稠定的. 容易看出，它还是正的和对称的. 下面我们将证明，对 Φ 作某种假定后，$\mathcal{E}_\Phi$ 还是可闭的.

令 $l^2(\nu)\equiv L^2(\mathbb{R}^d)\otimes L^2(\nu)$, 则梯度算子 D 为 $L^2(\nu)$ 到 $l^2(\nu)$ 中的稠定线性算子. 令 D' 为 D 的共轭算子. 如能证明 D 是可闭的，则 $D'D$ 将是本性自共轭的 (见第一章定理 1.7), 这将蕴含 $\mathcal{E}_\Phi$ 是可闭的. 因此，由第一章定理 1.5 知，为证 $\mathcal{E}_\Phi$ 可闭，只需证明：作为 $L^2(\nu)$ 到 $l^2(\nu)$ 的稠定线性算子 D, 它的共轭算子 D' 是稠定的.

令 $\mathcal{C}$ 表示由 $\{\sin W_\xi,\cos W_\xi,\xi\in E\}$ 生成的代数，其中 $W_\xi(x)=\langle x,\xi\rangle, x\in\mathcal{S}^*(\mathbb{R}^d)$. 容易证明，由 $\{e^{iW_\xi},\xi\in E\}$ 生成的 (复域上的) 代数在 $(E)_{\mathbb{C}}$ 中稠，故 $\mathcal{C}$ 在 (E) 中稠. 由此可以推知 $(\mathcal{E},\mathcal{C})$ 的可闭性与 $(\mathcal{E},(E))$ 的可闭性是等价的，且二者闭包相同. 下面我们将研究 $(\mathcal{E},\mathcal{C})$ 的可闭性. 为此只要研究以 $\mathcal{C}$ 为定义域的从 $L^2(\nu)$ 到 $l^2(\nu)$ 的算子 D 的可闭性.

在下一引理的叙述中，我们采用第二章 3.4 节中引进的 Sobolev 范数 $\|\cdot\|^{\sim}_{k,p}$, 即 $\|F\|^{\sim}_{k,p}=\|(I+N)^{k/2}\|_{L^p(\mu)}$, 其中 $k\in\mathbb{N},p>1$.

引理 4.2 设 Φ 为 $\mathcal{S}^*(\mathbb{R}^d)$ 上的可测函数，满足 $\Phi>0,\mu$-a.e., 且对某 $p>1,\Phi^{1/2}\in\mathbb{D}_1^{2p}$. 令 $d\nu=\Phi d\mu$, 则定义域为 $\mathcal{C}$ 的算子 D 是 $L^2(\nu)$ 到 $l^2(\nu)$ 中的稠定可闭算子.

证明 首先，由第二章 (3.41) 知，存在常数 $C>0$, 使得

$$\|\Phi\|^{\sim}_{1,p}=\|\Phi^{1/2}\Phi^{1/2}\|^{\sim}_{1,p}\le C\|\Phi^{1/2}\|^{\sim 2}_{1,2p},$$

于是由假定知 $\Phi\in\mathbb{D}_1^p$. 因此，Φ 属于 D 的 (L^p)- 定义域，并且有 $D\Phi=2\Phi^{1/2}D\Phi^{1/2}$. 下面我们用 $l_0^2(\nu)$ 表示由如下形式的 F 构成的 $l^2(\nu)$ 的子空间：

$$F=\sum_k e_k\otimes F_k,$$

其中 $F_k \in \mathcal{C}$, 并且只有有限多个 F_k 不为零. 显然 $l_0^2(\nu)$ 在 $l^2(\nu)$ 中稠. 由于 $\forall \varphi \in \mathcal{C}$,

$$\begin{aligned}
\int (\Phi^{-1}(D_{e_k}\Phi)\varphi)^2 d\nu &= \int \Phi^{-1}(D_{e_k}\Phi)^2 \varphi d\mu \\
&\le \|\varphi\|_\infty^2 \int (\Phi^{-1/2} D_{e_k}\Phi)^2 d\mu \\
&= 4\|\varphi\|_\infty^2 \int (D_{e_k}\Phi^{1/2})^2 d\mu \\
&\le 4\|\varphi\|_\infty^2 \|\Phi^{1/2}\|_{1,2}^{\sim 2}.
\end{aligned}$$

于是如下定义的线性算子 W_Φ 是 $l^2(\nu)$ 到 $L^2(\nu)$ 中的稠定算子:

$$W_\Phi F = \sum_k \Phi^{-1}(D_{e_k}\Phi)F_k. \tag{4.17}$$

令 $\varphi \in \mathcal{C}, F \in l_0^2(\nu)$, 则

$$\begin{aligned}
&\left(\varphi, (D^* - W_\Phi)F\right)_{L^2(\nu)} \\
&= \int \left(D(\varphi\Phi), F\right)_{L^2(\mathbb{R}^d)} d\mu - \int \varphi(D\Phi, F)_{L^2(\mathbb{R}^d)} d\mu \\
&= \int (D\varphi, F)_{L^2(\mathbb{R}^d)} \Phi d\mu = (D\varphi, F)_{L^2(\nu)}.
\end{aligned}$$

这表明

$$D'F = (D^* - W_\Phi)F, \quad F \in l_0^2(\nu).$$

特别, D' 为稠定的. ∎

下一定理是本节的主要结果.

定理 4.3 在引理 4.2 的条件下, $(\mathcal{E}, \mathcal{C})$ 是可闭的, 且其闭包 $(\mathcal{E}, \overline{\mathcal{C}})$ 为 $L^2(\nu)$ 上的 Dirichlet 型, 其中 $\overline{\mathcal{C}}$ 为 $\mathcal{C}$ 关于 $\mathcal{E}_1$- 范数的完备化.

证明 由引理 4.2 知, $(\mathcal{E}, \overline{\mathcal{C}})$ 是 $L^2(\nu)$ 上的对称闭正双线性型. 为证 $(\mathcal{E}, \overline{\mathcal{C}})$ 是 Dirichlet 型, 只要验证它有马氏性 (即压缩性).

为此，设 $\varphi \in \mathcal{C}$. 令 $g \in C_b^1(\mathbb{R})$, 假定 $\sup_x |\varphi(x)| \le R$. 由数学分析中的 Weierstrass 定理知，存在一列多项式 $\{g_n, n \in \mathbb{N}\}$, 使得 g_n' 在 $[-R, R]$ 上一致收敛于 g'. 令

$$g_n(t) = \int_{-R}^{t} g_n'(s)ds + g(-R),$$

则 $\{g_n, n \in \mathbb{N}\}$ 为在 $[-R, R]$ 上一致收敛于 g 的多项式序列. 显然 $g_n \circ \varphi$ 在 $L^2(\nu)$ 中收敛于 $g \circ \varphi$. 下面的估计表明，当 $m, n \to \infty$ 时， $a_{m,n} \equiv \mathcal{E}(g_n \circ \varphi - g_m \circ \varphi, g_n \circ \varphi - g_m \circ \varphi) \to 0$:

$$\begin{aligned}
a_{n,m} &= \int D(g_n \circ \varphi - g_m \circ \varphi)|^2 d\nu \\
&= \int g_n' \circ \varphi - g_m' \circ \varphi| |D\varphi|^2 d\nu \\
&\le \sup_{|t| \le R} |g_n'(t) - g_m'(t)| \mathcal{E}(\varphi, \varphi).
\end{aligned}$$

因此，$(\mathcal{E}, \mathcal{C})$ 的可闭性蕴含 $g \circ \varphi \in \overline{\mathcal{C}}$.

下面证明 $\varphi \in \overline{\mathcal{C}} \Rightarrow g \circ \varphi \in \overline{\mathcal{C}}$. 在 $\mathcal{C}$ 中取一序列 $\{\varphi_n, n \in \mathbb{N}\}$ 使得当 $n \to \infty$ 时 $\mathcal{E}_1(\varphi - \varphi_n, \varphi - \varphi_n) \to 0$. 特别在 $L^2(\nu)$ 中有 $\varphi_n \to \varphi$, 从而也有 $g \circ \varphi_n \to g \circ \varphi$. 此外，我们有如下估计：

$$\begin{aligned}
&\mathcal{E}(g \circ \varphi_n - g \circ \varphi_m, g \circ \varphi_n - g \circ \varphi_m) \\
&= \int |g' \circ \varphi_n D\varphi_n - g' \circ \varphi_m D\varphi_m|^2 \, d\nu \\
&\le 2 \int |g' \circ \varphi_n - g' \circ \varphi_m|^2 |D\varphi_n|^2 \, d\nu \\
&\quad + \int |g' \circ \varphi_m|^2 |D(\varphi_n - \varphi_m)|^2 \, d\nu \\
&\le 2 \int |g' \circ \varphi_n - g' \circ \varphi_m|^2 |D\varphi_n|^2 \, d\nu \\
&\quad + 2 \sup_{t \in \mathbb{R}} |g'(t)|^2 \int D(\varphi_n - \varphi_m)|^2 d\nu.
\end{aligned}$$

易见，当 $n,m\to\infty$ 时，最后不等式右端两项趋于零，从而 $g\circ\varphi_n$ 按 $\mathcal{E}_1$- 范数收敛于 $g\circ\varphi$. 因此， $g\circ\varphi\in\overline{\mathcal{C}}$.

最后证明 $(\mathcal{E},\overline{\mathcal{C}})$ 的压缩性. 根据 Dirichlet 型理论中的熟知结果 (例如见 Ma-Röckner[1]), 这等价于要验证 $(\mathcal{E},\overline{\mathcal{C}})$ 具有如下性质: $\forall\epsilon>0$, 存在 $C_\epsilon:\mathbb{R}\to[-\epsilon,1+\epsilon]$, 使得

(a) $C_\epsilon(t)=t,t\in[0,1];0\le C_\epsilon(t)-C_\epsilon(s)\le t-s,t,s\in\mathbb{R},s\le t$;

(b) $\forall\varphi\in\overline{\mathcal{C}}$, $C_\epsilon\circ\varphi\in\overline{\mathcal{C}}$, 且 $\mathcal{E}(C_\epsilon\circ\varphi,C_\epsilon\circ\varphi)\le\mathcal{E}(\varphi,\varphi)$.

满足 (a) 的 C_ϵ 容易构造， (b) 的第一个条件已被证明是满足的，第二个条件实际上对满足 (a) 的 C_ϵ 恒成立 (注意 $|C'_\epsilon|\le1$):

$$\begin{aligned}\mathcal{E}(C_\epsilon\circ\varphi,C_\epsilon\circ\varphi)&=\int|DC_\epsilon\circ\varphi|^2\,d\nu\\&=\int(C'_\epsilon\circ\varphi)^2|D\varphi|^2\,d\nu\\&\le\int|D\varphi|^2\,d\nu=\mathcal{E}(\varphi,\varphi).\end{aligned}$$

定理证毕. ∎

下一定理给出了 $(\mathcal{E},\mathcal{C})$ 可闭性的另一个充分条件.

定理 4.4 设 $\Phi\in(E)^*_+$. 如果 $D\Phi=B\cdot\Phi$, 其中 $B\in E^*\otimes(E)$, $B\cdot\Phi$ 为如下定义的 $E^*\otimes(E)^*$ 中的元素:

$$\langle\langle B\cdot\Phi,\eta\otimes\varphi\rangle\rangle=\langle\langle\Phi,\langle B,\eta\rangle\varphi\rangle\rangle,\ \eta\in E,\varphi\in(E),$$

则 $(\mathcal{E}_\Phi,\mathcal{C})$ 可闭，且 $(\mathcal{E}_\Phi,\overline{\mathcal{C}})$ 为 $L^2(\nu)$ 上的 Dirichlet 型.

证明 由假定， $\forall\eta\in E,\varphi\in(E)$, 我们有

$$\begin{aligned}\langle\langle D\Phi,\eta\otimes\varphi\rangle\rangle&=\langle\langle B\cdot\Phi,\eta\otimes\varphi\rangle\rangle\\&=\langle\langle\Phi,\langle B,\eta\rangle\varphi\rangle\rangle,\end{aligned}\tag{4.18}$$

其中 $\langle\cdot,\cdot\rangle$ 为 $\mathcal{S}^*(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ 上的典则双线性型， $\langle B,\eta\rangle\in(E)$. 由 (4.18) 容易推知： $\forall F\in l^2_0(\nu),\varphi\in(E)$, 有

$$\langle\langle D\Phi,\varphi\cdot F\rangle\rangle=\langle\langle\Phi,\langle B,F\rangle\varphi\rangle\rangle,\tag{4.19}$$

其中

$$\langle B, F\rangle \equiv \sum_k \langle B, e_k\rangle \cdot F_k.$$

因此， $\forall \varphi \in \mathcal{C}, F \in l_0^2(\nu)$,

$$\begin{aligned}(\varphi, D^* F)_{L^2(\nu)} &= \langle\langle \Phi, \varphi D^* F\rangle\rangle \\ &= \langle\langle D(\Phi\varphi), F\rangle\rangle \\ &= \langle\langle \Phi, (D\varphi, F)_{L^2(\mathbb{R}^d)}\rangle\rangle + \langle\langle D\Phi, \varphi \cdot F\rangle\rangle \\ &= (D\varphi, F)_{l^2(\nu)} + \langle\langle \Phi, \langle B, F\rangle \varphi\rangle\rangle \\ &= (D\varphi, F)_{l^2(\nu)} + (\varphi, \langle B, F\rangle)_{l^2(\nu)}.\end{aligned}$$

这表明

$$D'F = D^*F - \langle B, F\rangle,\ F \in l_0^2(\nu),$$

从而 D' 是稠定的．故 $(\mathcal{E}, \mathcal{C})$ 可闭，且 $(\mathcal{E}, \overline{\mathcal{C}})$ 为 $L^2(\nu)$ 上的 Dirichlet 型． ∎

作为定理 4.4 的一个应用，我们考虑 $(\mathcal{S}^*(\mathbb{R}^d), \mathcal{B})$ 上的一 Gauss 测度 ν，假定其特征泛函为

$$\widehat{\nu}(\xi) = \int e^{i\langle x, \xi\rangle} \nu(dx) = \exp\{-\tfrac{1}{2}(\xi, K\xi)\},$$

其中 K 为 $L^2(\mathbb{R}^d)$ 上可逆正自共轭算子，且 K 为从 $\mathcal{S}(\mathbb{R}^d)$ 到 $\mathcal{S}(\mathbb{R}^d)$ 的连续算子．显然 ν 对应于一 Hida 广义泛函，记为 Φ(即 $\Phi = \frac{d\nu}{d\mu}$)．通过 S- 变换，利用 (1.22) 可以证明： $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$D_f \Phi = \langle \cdot, (K^{-1} - I) f\rangle \Phi.$$

令

$$B = \sum_k e_k \otimes \langle \cdot, (K^{-1} - I) e_k\rangle,$$

则 $B \in \mathcal{S}^*(\mathbb{R}^d) \otimes (E)$，且有 $D\Phi = B \cdot \Phi$．故由定理 4.4 知，通过 (4.13) ν 联系 $L^2(\nu)$ 上一 Dirichlet 型．

附录 A Hermite 多项式与 Hermite 函数

实 Hermite 多项式 定义为

$$H_n(u) \equiv (-1)^n e^{u^2/2} \frac{d^n}{du^n} e^{-u^2/2}, \quad u \in \mathbb{R}, n \in \mathbb{N}_0, \tag{A.1}$$

它们是函数 $\exp\{tu - t^2/2\}$ 按 t 展开的幂级数的系数:

$$\exp\{tu - t^2/2\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u), \quad t, u \in \mathbb{R}. \tag{A.2}$$

利用此展开式容易证明:

定理 A.1 Hermite 多项式可以表示为

$$H_n(u) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k u^{n-2k}}{2^k k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{A.3}$$

反之

$$u^n = n! \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(u)}{2^k k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{A.4}$$

$\{H_n, n \in \mathbb{N}\}$ 满足微分方程:

$$H_n'(u) = nH_{n-1}(u), \qquad n \geq 1, \tag{A.5}$$

$$H_n''(u) - uH_n'(u) + nH_n(u) = 0, \quad n \geq 0 \tag{A.6}$$

和递推公式

$$H_0(u) \equiv 1, \qquad H_1(u) = u,$$

$$H_{n+1}(u) = uH_n(u) - nH_{n-1}(u), \qquad n \geq 1, \tag{A.7}$$

以及乘法公式:

$$H_m(u)H_n(u)=\sum_{k=0}^{m\wedge n}k!\binom{m}{k}\binom{n}{k}H_{m+n-2k}(u). \qquad (A.8)$$

对 $\lambda\in\mathbb{R}$ 有

$$H_n(\lambda u)=n!\sum_{k=0}^{[n/2]}\frac{(\lambda^2-1)^k\lambda^{n-2k}}{2^k k!(n-2k)!}H_{n-2k}(u). \qquad (A.9)$$

证明 由 e^{tu} 及 $e^{-t^2/2}$ 关于 t 的幂级数代入 (A.2) 式，比较等式两边 t^n 的系数，即得 (A.3) 和 (A.4) 式；将 (A.2) 式两边关于 u 微分，比较两边幂级数的系数，即得 (A.5) 和 (A.6) 式；再由 (A.2) 式可知

$$\begin{aligned}\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}H_m(u)H_n(u)&=\exp\left\{(s+t)u-\frac{(s+t)^2}{2}+st\right\}\\&=\sum_{j=0}^{\infty}\frac{(s+t)^j}{j!}H_j(u)\sum_{k=0}^{\infty}\frac{s^k t^k}{k!}\\&=\sum_{j,k=0}^{\infty}\frac{H_j(u)}{j!k!}\sum_{l=0}^{j}\binom{j}{l}s^{l+k}t^{j-l+k},\end{aligned}$$

在后一式中令 $l+k=m$，$j-l+k=n$ 得

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}\sum_{k=0}^{m\wedge n}k!\binom{m}{k}\binom{n}{k}H_{m+n-2k}.$$

比较 $s^m t^n$ 的系数即得乘法公式 (A.8). 特别，(A.8) 中当 $m=1$ 时就是递推公式 (A.7). 最后，由 (A.2) 可知

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(\lambda u)&=\exp\left\{\lambda tu-\frac{t^2}{2}\right\}\\&=\exp\left\{\lambda tu-\frac{\lambda^2t^2}{2}+\frac{(\lambda^2-1)t^2}{2}\right\}\\&=\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}H_j(u)\sum_{k=0}^{\infty}\frac{(\lambda^2-1)^k t^{2k}}{2^k k!},\end{aligned}$$

在最后一式中令 $j+2k=n$, 即化为

$$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/2]} \frac{(\lambda^2-1)^k \lambda^{n-2k}}{2^k k!(n-2k)!} H_{n-2k}(u),$$

比较 t^n 的系数, 即得 (A.9) 式. ∎

考虑 $\mathbb{R}$ 上的 Gauss 测度

$$\gamma(du) \equiv (2\pi)^{-1/2} \exp\{-u^2/2\} du$$

及 Hilbert 空间 $L^2(\mathbb{R},\gamma)$, 我们有

定理 A.2 Hermite 多项式构成 $L^2(\mathbb{R},\gamma)$ 中的正交系:

$$\int_{\mathbb{R}} H_m(u) H_n(u) \gamma(du) = n! \delta_{mn}, \qquad m, n \in \mathbb{N}_0. \tag{A.10}$$

记 $i=\sqrt{-1}$, 则

$$H_n(u) = \int_{\mathbb{R}} (u \pm iv)^n \gamma(dv), \qquad n \in \mathbb{N}_0, \tag{A.11}$$

且

$$H_n(u+v) = \sum_{k=0}^{n} \binom{n}{k} u^k H_{n-k}(v), \qquad n \in \mathbb{N}_0. \tag{A.12}$$

当 $t^2<1$ 时有

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u) H_n(v) = \frac{1}{\sqrt{1-t^2}} \exp\left\{-\frac{t^2u^2 - 2tuv + t^2v^2}{2(1-t^2)}\right\}. \tag{A.13}$$

证明 由 (A.2) 式

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m! n!} \int_{\mathbb{R}} H_m(u) H_n(u) \gamma(du) \\
&= \int_{\mathbb{R}} \exp\left\{(s+t)u - \frac{s^2+t^2}{2}\right\} \gamma(du) \\
&= \exp\left\{-\frac{s^2+t^2}{2} + \frac{(s+t)^2}{2}\right\} = e^{st} \\
&= \sum_{n=0}^{\infty} \frac{(st)^n}{n!}.
\end{aligned}$$

比较 $s^m t^n$ 的系数即得 (A.10) 式. 利用围道积分可得

$$\int_{\mathbb{R}} \exp\{t(u \pm iv)\}\gamma(dv) = \exp\left\{tu - \frac{t^2}{2}\right\}.$$

将上式两边展开成 t 的幂级数 (右边利用 (A.2) 式) 并比较 t^n 的系数即得 (A.11) 式. 由 (A.11) 式可知

$$\begin{aligned} H_n(u+v) &= \int_{\mathbb{R}} (u+v+iy)^n \gamma(dy) \\ &= \sum_{k=0}^{n} \binom{n}{k} u^n \int_{\mathbb{R}} (v+iy)^{n-k} \gamma(dy). \end{aligned}$$

于是 (A.12) 式得证. 再由 (A.11) 式可知

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u) H_n(v) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{t(u+ix)(v+iy)\}\gamma(dx)\gamma(dy). \end{aligned}$$

计算右边积分即得 (A.13) 式. ∎

由 (A.4) 及乘法公式 (A.8) 可知 Hermite 多项式构成多项式环的线性基底, 由正交性 (A.10) 及多项式在 $L^2(\mathbb{R}, \gamma)$ 中的稠密性可知 $\{(n!)^{-1/2} H_n\}$ 构成 $L^2(\mathbb{R}, \gamma)$ 的正交基. 现在考虑 Hilbert 空间 $L^2(\mathbb{R}) = L^2(\mathbb{R}, du)$, 其中 du 为 Lebesgue 测度. 对 $f \in L^2(\mathbb{R})$, 令

$$Jf(u) \equiv \pi^{1/4} e^{u^2/4} f(u/\sqrt{2}), \tag{A.14}$$

则

$$\|Jf\|^2_{L^2(\mathbb{R},\gamma)} = \|f\|^2_{L^2(\mathbb{R})},$$

且

$$J^{-1} f(u) = \pi^{-1/4} e^{-u^2/2} f(\sqrt{2}u). \tag{A.15}$$

从而 $J: L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R},\gamma)$ 为 Hilbert 空间同构. 令

$$\begin{aligned} h_n(u) &\equiv (n!)^{-1/2} J^{-1} H_n(u) \\ &= (n!)^{-1/2}\pi^{-1/4}e^{-u^2/2}H_n(\sqrt{2}u), \end{aligned} \tag{A.16}$$

则 $\{h_n, n\in \mathbb{N}_0\}$ 为 $L^2(\mathbb{R})$ 的正交基，称为 **Hermite 函数**. 由定义及 Hermite 多项式性质容易推出:

$$h_n'(u) + uh_n(u) = \sqrt{2n}h_{n-1}(u), \qquad n \geq 1. \tag{A.17}$$

此外，我们经常要用到以下一些有用的估计，证明可参看 Hille-Phillips[1] 或 G.Szegö [1].

定理 A.3 对固定 $u \in \mathbb{R}$ 有

$$h_n(u) = O(n^{-1/4}), \tag{A.18}$$

$$\int_0^u h_n(v)dv = O(n^{-3/4}), \tag{A.19}$$

而

$$\|h_n\|_{L^\infty} \equiv \sup_{u\in\mathbb{R}} |h_n(u)| = O(n^{-1/12}), \tag{A.20}$$

$$\|h_n\|_{L^1} \equiv \int_{\mathbb{R}} |h_n(u)|du = O(n^{1/4}). \tag{A.21}$$

因 $H_n(u) = (n!)^{1/2}\pi^{1/4}e^{u^2/4}h_n(u/\sqrt{2})$, 故由 (A.20) 可知

$$| H_n(u) | \leq c(n!)^{1/2}e^{u^2/4}. \tag{A.22}$$

更精确地，在上一不等式中可令 $c = 1.2$, 此时 (A.22) 称为 Cramér 估计 (见 Erdélyi[1], p.208).

附录 B 局部凸空间及其对偶

我们扼要介绍本书用到的拓扑线性空间的某些基本概念，详细内容请参看 Bourbaki[1], Schaefer[1] 或 Treves[1].

1. 半范、范数与 H 范

设 X 为域 $\mathbb{K}$(实数域 $\mathbb{R}$ 或复数域 $\mathbb{C}$) 上的线性空间. X 上的非负实值函数 p 若具有性质：

$$\text{(N.1)} \qquad p(x-y) \le p(x)+p(y), \qquad x,y \in X;$$

$$\text{(N.2)} \qquad p(\lambda x) = |\lambda| p(x), \qquad x \in X, \lambda \in \mathbb{K};$$

称为 X 上的 **半范**. 若还具有性质：

$$\text{(N.3)} \qquad x \neq 0 \Rightarrow p(x) > 0,$$

则称为 **范数**. 若此外还具有“平行四边形”性质：

$$\text{(N.4)} \quad p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2,\ x,y \in X,$$

则称为 **Hilbert 范**(简称 **H 范**). 具有性质 (N.1), (N.2) 及 (N.4) 的函数 p 称为 **H 半范**.

设 p 为 X 上的半范. 令

$$N_p \equiv p^{-1}(0) = \{x \in X : p(x) = 0\}.$$

由性质 (N.1) 及 (N.2) 可知，N_p 是 X 的线性子空间 (若 p 为范数，则 $N_p = \{0\}$). 令

$$\widehat{X}_p \equiv X/N_p$$

为商空间，即由 $p(x-y)=0$ 建立的等价关系 $x \sim y$ 中所有等价类 $\widehat{x}$ 构成的线性空间，其商映射记为

$$Q_p : X \longrightarrow \widehat{X}_p \qquad \text{(B.1)}$$

即 $\widehat{x}=Q_p x$ 为含 x 的等价类. 在 $\widehat{X}_p$ 上定义函数

$$\widehat{p}(Q_p x)=p(x).$$

易见, $\widehat{p}$ 为 $\widehat{X}_p$ 上的范数, $(\widehat{X}_p,\widehat{p})$ 为赋范空间, 将其完备化得一 Banach 空间 $(\overline{X}_p,\bar{p})$. 若此 Banach 空间是可分的, 我们称此半范 p 为 **可分半范**.

若 p 为 X 上的 H 半范, 则由此而得的 Banach 空间实际上是 Hilbert 空间.

设 p 和 q 为 X 上的两个半范, 若 $\exists c>0$ 使

$$p(x)\leq cq(x), \qquad \forall x\in X,$$

则称 p 囿于 q, 记为 $p\prec q$. 此时 $N_q\subset N_p$, q 等价必为 p 等价, 因而

$$I_{pq}\equiv Q_p Q_q^{-1}\ :\ \widehat{X}_q\longrightarrow\widehat{X}_p \tag{B.2}$$

可开拓为 $\overline{X}_q$ 到 $\overline{X}_p$ 的连续线性算子.

设 p 和 q 为 X 上的两个可分 H 半范, 若存在 $\overline{X}_q$ 中的正交基 $\{e_n\}$ 使

$$\sum_{n=1}^{\infty}\bar{p}(I_{pq}e_n)^2<\infty, \tag{B.3}$$

则称 p **HS 囿于** q, 记为 $p\prec_{\mathrm{HS}} q$. 容易证明 (B.3) 中的和不依赖于正交基 $\{e_n\}$ 的选取, 它恰好是算子 I_{pq} 的 Hilbert-Schmidt 范数的平方.

2. 局部凸拓扑线性空间、有界集

线性空间 X 中赋以拓扑结构 $\mathcal{T}$, 使加法及数乘运算连续, 则 $(X,\mathcal{T})$ 称为 **拓扑线性空间** 或 **拓扑向量空间**.

在拓扑线性空间中, 由于加法的连续性, 任何一点的邻域都可以由零点邻域系 $\mathcal{N}(0)$ 平移而得到, 因此其拓扑由 $\mathcal{N}(0)$ 所唯一确定.

设 V 为 X 的子集，若 $\forall x \in X, \exists\lambda_0$ 使 $\lambda x \in V$ 对一切满足 $|\lambda| \leq \lambda_0$ 之 $\lambda \in \mathbb{K}$ 成立，则称 V 为 **吸收集**. 由数乘运算的连续性易知，零点的每一邻域都是吸收集.

在 Banach 空间 $(X, \|\cdot\|)$ 中，可取不同半径的开球 $\{x, \|x\| < \epsilon\}$ $(\epsilon > 0)$ 作为 $\mathcal{N}(0)$ 的基. 但在无穷维分析中，只考虑由单个范数生成的拓扑线性空间是不够的，有必要考虑由某一族半范 $\Gamma = \{p_\alpha; \alpha \in A\}$ 生成的拓扑线性空间 $(X, \mathcal{T})$, 其中 A 是任意指标集. 这时 $\mathcal{N}(0)$ 的基可以由以下集合构成：

$$\{x:\ p_{\alpha_j}(x) < \epsilon_j,\ j = 1, 2, \cdots, n\}, \tag{B.4}$$

其中 $n \in \mathbb{N}, \epsilon_j > 0, \alpha_j \in A\,(j = 1, 2, \cdots, n)$. 为使 $(X, \mathcal{T})$ 为 Hausdorff 空间，代替 (N.3) 的是，总体 Γ 满足

$$x \neq 0 \Rightarrow \exists\alpha \in A,\ p_\alpha(x) > 0. \tag{B.5}$$

由一族满足性质 (B.5) 的半范 Γ 生成的拓扑线性空间称为 **局部凸空间** (locally convex space, 简记为 LCS). 这一定义等价于几何的定义，即存在凸邻域基的 Hausdorff 拓扑线性空间为局部凸空间.

设 Γ_1 和 Γ_2 为 X 上两族半范，若由 Γ_1 生成之拓扑弱于由 Γ_2 生成之拓扑，则称 Γ_1**弱于**Γ_2, 记为 $\Gamma_1 \prec \Gamma_2$; 若 $\Gamma_1 \prec \Gamma_2$ 且 $\Gamma_2 \prec \Gamma_1$, 则称 Γ_1**等价** 于 Γ_2, 记为 $\Gamma_1 \sim \Gamma_2$. 等价的半范族生成相同的拓扑.

局部凸空间可距离化的充要条件是其拓扑可由可数半范族生成. 完备的可距离化的局部凸空间称为 **Fréchet 空间**. 每一个 Banach 空间都是 Fréchet 空间.

赋范空间中的有界集可以用范数来定义，而一般拓扑线性空间中则不然. 拓扑线性空间 X 中的子集 B, 若被零点每一邻域吸收，即

$$\forall U \in \mathcal{N}(0),\ \exists\lambda \in \mathbb{K} \text{ 使 } B \subset \lambda U, \tag{B.6}$$

则称为 **有界集**. 若 X 为局部凸空间，其拓扑由半范族 Γ 生成，则子集 B 为有界集的充要条件是:

$$\sup_{x \in B} p(x) < \infty, \qquad \forall p \in \Gamma. \tag{B.7}$$

局部凸空间可赋范的充要条件是零点具有有界邻域. 每点具有紧邻域的拓扑空间称为 **局部紧空间**, 局部凸空间当且仅当有限维时为局部紧.

3. 投影拓扑与拓扑投影极限

设 X 为一线性空间, $\{X_\alpha, \mathcal{T}_\alpha; \alpha \in A\}$ 为一族局部凸空间, 其拓扑 $\mathcal{T}_\alpha$ 由半范族 Γ_α 生成. 设

$$f_\alpha \ : X \longrightarrow X_\alpha, \quad \alpha \in A$$

为一族线性映射, 且 $\bigcap_\alpha f_\alpha^{-1}(0) = \{0\}$. X 中使所有 f_α 为连续的 **最弱** 局部凸拓扑 $\mathcal{T}$, 称为关于 $(X_\alpha, \mathcal{T}_\alpha, f_\alpha; \alpha \in A)$ 的 **投影拓扑**, 它可由半范族 $\{p_\alpha \circ f_\alpha; p_\alpha \in \Gamma_\alpha, \alpha \in A\}$ 生成. 任意局部凸空间 Y 到 X 的线性映射 T 为 $\mathcal{T}$- 连续的充要条件是: $\forall \alpha \in A$, $f_\alpha \circ T$ 为 $\mathcal{T}_\alpha$ 连续. X 中子集 B 为有界的充要条件是: $\forall \alpha \in A$, $f_\alpha(B)$ 在 X_α 中有界.

例 1 设 X 为局部凸空间 $(Y, \mathcal{T})$ 的线性子空间, $f : X \longrightarrow Y$ 为自然嵌入映射, 则 X 中关于 $(Y, \mathcal{T}, f)$ 的投影拓扑就是 $\mathcal{T}$ 在子空间 X 中的 **导出拓扑**, 记为 $\mathcal{T}|_X$.

例 2 设 $X = \prod_{\alpha \in A} X_\alpha$ 为乘积空间, f_α 为 X 到 X_α 的坐标投影, 则 X 中关于 $(X_\alpha, \mathcal{T}_\alpha, f_\alpha; \alpha \in A)$ 的投影拓扑就是 **乘积拓扑**, 记为 $\prod_{\alpha \in A} \mathcal{T}_\alpha$.

例 3 设 $\{X_n, \mathcal{T}_n; n \in \mathbb{N}\}$ 为一列局部凸空间且

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$$

当 $m \geq n$ 时, $\mathcal{T}_n|_{X_m} \prec \mathcal{T}_m$. 又设

$$X = \bigcap_n X_n,$$

f_n 为 X 到 X_n 的自然嵌入映射. 则 X 中关于 $(X_n, \mathcal{T}_n, f_n; n \in \mathbb{N})$ 的投影拓扑 $\mathcal{T}$ 称为此局部凸空间投影序列的 **投影极限拓扑**. 赋

予此拓扑的空间 X 称为序列 $(X_n, \mathcal{T}_n)$ 的 (**拓扑**) **投影极限**, 记为 $X = \varprojlim X_n$(这种极限还可以推广到足标集为任意定向半序集情形).X 上的线性泛函 f 为 $\mathcal{T}$- 连续的充要条件为: $\exists n \in \mathbb{N}$, 使 f 可延拓为 X_n 上的 $\mathcal{T}_n$- 连续线性泛函. 若对一切 $m \geq n$, $\mathcal{T}_n|_{X_m} = \mathcal{T}_m$, 则称此投影极限是 **严格的**.

4. 归纳拓扑与拓扑归纳极限

仍设 X 为一线性空间, $\{X_\alpha, \mathcal{T}_\alpha; \alpha \in A\}$ 为一族局部凸空间. 设

$$g_\alpha \ : \ X_\alpha \longrightarrow X \,, \quad \alpha \in A$$

为一族线性映射, 且 $X = \mathrm{span}\{\bigcup_\alpha g_\alpha(X_\alpha)\}$. X 中使所有 g_α 为连续的 **最强** 局部凸拓扑 $\mathcal{T}$, 称为关于 $(X_\alpha, \mathcal{T}_\alpha, g_\alpha; \alpha \in A)$ 的 **归纳拓扑**.X 到任意局部凸空间 Y 的线性映射 T 为 $\mathcal{T}$- 连续的充要条件是: $\forall \alpha \in A, T \circ g_\alpha$ 为 $\mathcal{T}_\alpha$- 连续.

例 1 设 M 是局部凸空间 $(Y, \mathcal{T})$ 的闭线性子空间, $X = Y/M$ 为商空间, $g : Y \longrightarrow X$ 为商映射, 则 X 中关于 $(Y, \mathcal{T}, g)$ 的归纳拓扑就是 **商拓扑**.

例 2 设 $\{X_\alpha, \mathcal{T}_\alpha; \alpha \in A\}$ 为一族局部凸空间. $X = \bigoplus_{\alpha \in A} X_\alpha$ 为其线性空间直和 (即乘积空间 $\prod_{\alpha \in A} X_\alpha$ 中只含有限个坐标分量不为零的元素所构成的子空间), g_α 为 X_α 到 X 中的自然嵌入映射, 则 X 中关于 $(X_\alpha, \mathcal{T}_\alpha, g_\alpha; \alpha \in A)$ 的归纳拓扑就是 **直和拓扑**, 记为 $\bigoplus_{\alpha \in A} \mathcal{T}_\alpha$.

例 3 设 $\{X_n, \mathcal{T}_n; n \in \mathbb{N}\}$ 为一列局部凸空间. 且

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

当 $m \geq n$ 时, $\mathcal{T}_m|_{X_n} \prec \mathcal{T}_n$. 又设

$$X = \bigcup_n X_n,$$

g_n 为 X_n 到 X 的自然嵌入映射, 则 X 中关于 $(X_n, \mathcal{T}_n, g_n; n \in \mathbb{N})$ 的归纳拓扑 $\mathcal{T}$ 称为此局部凸空间归纳序列的 **归纳极限拓扑**. 赋

予此拓扑的空间 X 称为序列 $(X_n,\mathcal{T}_n)$ 的 (**拓扑**) **归纳极限**, 记为 $X=\varinjlim X_n$ (它同样可推广到任意定向半序足标集情形). 若对一切 $m\geq n$, $\mathcal{T}_m|_{X_n}=\mathcal{T}_n$, 则称此归纳极限是 **严格的**.

在归纳极限拓扑中, X 中的子集 B 为有界集的充要条件是: $\exists n_0\in\mathbb{N}$ 使 $B\subset X_{n_0}$ 且 B 为 X_{n_0} 中有界集. 序列 $\{x_n\}$ 收敛于 x 的充要条件是: $\exists n_0\in\mathbb{N}$, 使 $\{x_n\}\subset X_{n_0}$ 且在 X_{n_0} 中 $x_n\to x$.

注意一般归纳极限拓扑未必为 Hausdorff 拓扑 , 但若极限是严格的, 则为 Hausdorff 拓扑.

5. 对偶空间和弱拓扑

设 X,Y 为两个线性空间, 若在 $X\times Y$ 上存在双线性泛函 $\langle x,y\rangle$, 满足分离性质:

$$\begin{aligned}&\langle x,y\rangle=0\ ,\ \forall y\in Y\Rightarrow x=0,\\&\langle x,y\rangle=0\ ,\ \forall x\in X\Rightarrow y=0.\end{aligned}\tag{B.8}$$

则 X 和 Y 通过此泛函建立了一种对偶关系, 简称 $\langle X,Y\rangle$ 为 (线性空间) **对偶**. 例如, 线性空间 X 上的线性泛函全体构成线性空间 X', 称为 X 的 **代数对偶** 空间. 对 $x\in X$ 及 $f\in X'$, 定义双线性泛函

$$\langle x,f\rangle\equiv f(x),$$

则 $\langle X,X'\rangle$ 为一对偶.

若 $(X,\mathcal{T})$ 为局部凸空间, X 上的 $\mathcal{T}$- 连续线性泛函全体构成线性空间 X^*, 称为 $(X,\mathcal{T})$ 的 **拓扑对偶** 空间. 显然, X^* 是 X' 的线性子空间, 且 $\langle X,X^*\rangle$ 仍为一对偶.

设 $\langle X,Y\rangle$ 为对偶. X 中由半范族:

$$p_y(x)\equiv|\langle x,y\rangle|\ ,\quad y\in Y\tag{B.9}$$

生成之拓扑称为关于此对偶的 **弱拓扑**, 记为 $\sigma(X,Y)$, 它是使线性泛函 $f_y(x)=\langle x,y\rangle, y\in Y$ 为连续之最弱局部凸拓扑. 由对称性, 在 Y 中也可定义弱拓扑 $\sigma(Y,X)$.

6. 相容性和 Mackey 拓扑

设 $\langle X,Y\rangle$ 为对偶， X 中某局部凸拓扑 $\mathcal{T}$ 称为与对偶**相容**，如果它使得 X 上 $\mathcal{T}$- 连续线性泛函全体构成的线性空间 X^* 和 Y 重合 (Y 中每一元素 y 可看作 X 上的线性泛函 $\langle\cdot,y\rangle$, 因而 Y 可以看作 X 的代数对偶 X' 的线性子空间). 显然，弱拓扑 $\sigma(X,Y)$ 是与对偶相容的最弱拓扑.

为了引进更强的拓扑，可以考虑如下半范族：

$$p_S(x)\equiv\sup_{y\in S}|\langle x,y\rangle|,\ S\in\mathfrak{S},\tag{B.10}$$

其中 $\mathfrak{S}$ 为 Y 中某一覆盖空间 Y 的非空子集类. 由 (B.7) 可以看出，当且仅当 p_S 有限，亦即 S 为 $\sigma(Y,X)$- 有界集时， p_S 为半范. 由此半范族生成 X 中的拓扑，即在 $\mathfrak{S}$ 中每一集合 S 上一致收敛拓扑，称为 **$\mathfrak{S}$- 拓扑**. 特别，当 $\mathfrak{S}$ 为 Y 中有限子集全体时，就得到点点收敛拓扑，即弱拓扑 $\sigma(X,Y)$.

为了得到最强的相容拓扑，我们注意到 p_S(关于某拓扑) 连续的充要条件是： $\forall\epsilon>0$, $\exists U\in\mathcal{N}(0)$ 使 $\forall x\in U$ 有 $p_S(x)<\epsilon$, 也就是说， S 为 X 上线性泛函的**等度连续集合**. X 上任一相容拓扑都是 Y 中等度连续集合上一致收敛拓扑. 由 Alaoglu-Bourbaki 定理 (例如参看 Taylor & Lay[1], p.166), 等度连续集合是相对 $\sigma(Y,X)$- 紧的. 若取 $\mathfrak{S}$ 为 Y 中绝对凸[1] 弱紧子集全体，此 $\mathfrak{S}$- 拓扑称为 **Mackey 拓扑**, 记为 $\tau(X,Y)$. 我们有以下重要的

Mackey-Arens 定理 (参看 Schaefer[1], p.131) 设 $\langle X,Y\rangle$ 为对偶，则 X 上的局部凸拓扑 $\mathcal{T}$ 和此对偶相容的充要条件是：

$$\sigma(X,Y)\prec\mathcal{T}\prec\tau(X,Y).\tag{B.11}$$

一切相容拓扑有相同的有界集族和闭凸集族. 根据这个定理，凸集 (特别是线性子空间) 关于任一相容拓扑的闭包是相同的.

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[1] 线性空间 X 中集合 V, 若满足： $x,y\in V,\ \lambda,\mu\in\mathbb{K},|\lambda|+|\mu|\le1\Rightarrow\lambda x+\mu y\in V$, 称为绝对凸集.

设 $(X,\mathcal{T})$ 为局部凸空间. 考虑对偶 $\langle X,X^*\rangle$, 显然有

$$\sigma(X,X^*)\prec\mathcal{T}\prec\tau(X,X^*).$$

于是, 一切弱有界集必有界.

7. 强拓扑和自反性

设 $(X,\mathcal{T})$ 为局部凸空间, X^* 为其拓扑对偶, $\mathfrak{S}$ 为 X^* 中 $\sigma(X^*,X)$- 有界集全体所构成的子集类. 此时 X 中的 $\mathfrak{S}$ 拓扑即为在 $\sigma(X^*,X)$- 有界集上一致收敛拓扑, 称为 **强拓扑**, 记为 $\beta(X,X^*)$. 为了区别原空间 $(X,\mathcal{T})$ 和赋以强拓扑的局部凸空间 $(X,\beta(X,X^*))$, 我们常将后者简记为 X_β(类似地还有 X_σ,X_τ 等). 若 $(X,\mathcal{T})$ 和 X_β 一致, 称为 **桶式**(barreled)[2] 空间. 若 $(X,\mathcal{T})$ 和 X_τ 一致, 称为 **Mackey 空间**. 所有 Fréchet 空间都是桶式空间. 所有桶式空间都是 Mackey 空间. 以 $(X_\beta)^*$ 表示 X 上强连续线性泛函所构成的线性空间 (类似地还有 $(X_\sigma)^*$,$(X_\tau)^*$ 等), 由 Mackey-Arens 定理可知, 作为线性空间,

$$(X_\sigma)^*=X^*=(X_\tau)^*\subset(X_\beta)^*. \tag{B.12}$$

对称地, 在 X^* 上有弱 * 拓扑 $\sigma(X^*,X)$, Mackey 拓扑 $\tau(X^*,X)$ 及强拓扑 $\beta(X^*,X)$. 为了区别 X^* 赋以不同拓扑所得的不同局部凸空间, 可以分别记为 X^*_σ,X^*_τ 及 X^*_β (注意 $X^*_\beta\neq(X_\beta)^*$). X^*_β 称为 X 的 **强对偶空间**.

X^* 上强连续线性泛函全体构成线性空间 $X^{**}\equiv(X^*_\beta)^*$ 称为二次对偶. 一般地 $X\subset X^{**}$, 若 $X=X^{**}$ 且 $\mathcal{T}=\beta(X,X^*)$, 即 $(X,\mathcal{T})$ 与二次强对偶空间 $X^{**}_\beta\equiv(X^*_\beta)^*_\beta$ 一致, 称为 **自反空间**.

X 为自反空间的充要条件是: X 为桶式空间且其中一切有界集为相对弱紧. 特别, Banach 空间为自反的充要条件是单位球弱紧.

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[2] 闭绝对凸吸收集称为桶 (barrel). 在桶式空间中, 一切桶均为零点邻域.

8. 对偶映射

设 X,Y 为局部凸空间, X^*,Y^* 分别为其拓扑对偶，以 $\mathcal{L}(X,Y)$ 表示 X 到 Y 中线性连续映射所构成的线性空间. 若 $T\in\mathcal{L}(X,Y)$, 则对任一 $f\in Y^*$ 有 $f\circ T\in X^*$, 线性映射: $f\longmapsto f\circ T$ 记为

$$T^* : Y^*\longrightarrow X^*,$$

称为 T 之**对偶映射**. 同时考虑对偶 $\langle X,X^*\rangle$ 及 $\langle Y,Y^*\rangle$,T 和 T^* 的关系可归结为

$$\langle Tx,f\rangle=\langle x,T^*f\rangle,\quad x\in X,f\in Y^*. \tag{B.13}$$

由上式可知, T 和 T^* 都是 σ- 连续的, 即

$$T\in\mathcal{L}(X_\sigma,Y_\sigma)\ ,\quad T^*\in\mathcal{L}(Y^*_\sigma,X^*_\sigma). \tag{B.14}$$

当且仅当 T 的值域 $\mathcal{R}(T)$ 在 Y 中稠密 (因为线性子空间在相容拓扑中有相同闭包, 故等价于弱稠密) 时 T^* 为单射.

若 X,Y 都是 Mackey 空间, 则连续性和 σ- 连续性等价. 特别, 当 X,Y 都是 Banach 空间时, 若 X 连续、稠密地嵌入 Y, 则其强对偶空间 Y^* 连续、稠密地嵌入 X^*.

考虑一列 Hilbert 空间

$$X_1\supset X_2\supset\cdots\supset X_n\supset X_{n+1}\supset\cdots.$$

若 $\forall n\in\mathbb{N},X_{n+1}$ 连续、稠密地嵌入 X_n, $X=\varprojlim X_n$ 为其拓扑投影极限, 则对偶地有

$$X_1^*\subset X_2^*\subset\cdots\subset X_n^*\subset X_{n+1}^*\subset\cdots,$$

且 $\forall n\in\mathbb{N},X_n^*$ 连续、稠密地嵌入 X^*_{n+1}, $X^*=\varinjlim X_n^*$ 为其拓扑归纳极限.

由于可列 Hilbert 空间 X(参看第一章 §3) 是自反空间, 而归纳拓扑极限是使一切嵌入映射为连续之最强局部凸拓扑, 所以在

其对偶空间 X^* 中，归纳极限拓扑、强拓扑和 Mackey 拓扑相互等价.

9. 均匀凸空间和 Banach-Saks 定理

设 X 为赋范空间，若 $\forall \epsilon \in (0,2), \exists \delta > 0$ 使 $\forall x, y \in X$,

$$\|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon \Longrightarrow \|\tfrac{1}{2}(x+y)\| \leq 1-\delta, \tag{B.15}$$

则 X 称为 **均匀凸空间**.

例 1 (Hilbert 空间) 由平行四边形性质知, 当 $\|x\| \leq 1, \|y\| \leq 1$ 及 $\|x-y\| \geq \epsilon$ 时有 $\|x+y\|^2 \leq 4-\epsilon^2$. 取 $\delta = 1-(1-\epsilon^2/4)^{1/2}$ 即得 Hilbert 空间的均匀凸性.

例 2 (L^p 空间) 设 $1 < p < \infty$, 由 Clarkson 不等式:

$$\|\tfrac{1}{2}(f+g)\|_p^p + \|\tfrac{1}{2}(f-g)\|_p^p \leq \tfrac{1}{2}(\|f\|_p^p + \|g\|_p^p) \quad (2 \leq p < \infty) \tag{B.16}$$

$$\|\tfrac{1}{2}(f+g)\|_p^q + \|\tfrac{1}{2}(f-g)\|_p^q \leq (\tfrac{1}{2}(\|f\|_p^p + \|g\|_p^p))^{q-1} \quad (1 < p \leq 2) \tag{B.17}$$

(其中 q 为 p 的共轭指数) 可知, 当 $p \geq 2$ 时

$$\|\tfrac{1}{2}(f+g)\|_p \leq (1-(\epsilon/2)^p)^{1/p},$$

取 $\delta = \epsilon^p/2^p p + o(\epsilon^p)$; 当 $p \leq 2$ 时

$$\|\tfrac{1}{2}(f+g)\|_p \leq (1-(\epsilon/2)^q)^{1/q},$$

取 $\delta = (p-1)\epsilon^2/8 + o(\epsilon^2)$, 得证 L^p 空间的均匀凸性.

均匀凸 Banach 空间为自反 Banach 空间. 我们知道, 自反 Banach 空间中有界集为相对 **弱** 紧, 但在均匀凸 Banach 空间中有以下的所谓 **Banach-Saks** 性质:

Banach-Saks-Kakutani 定理 设 $\{x_n\}$ 为均匀凸空间 X 中的有界序列, 则存在子序列 $\{x_{n_j}\}$, 使算术平均序列 $S_k \equiv \frac{1}{k}\sum_{j=1}^{k} x_{n_j}$ 在 X 中 **强收敛**.

读者可参看 Diestel[1]. 更进一步可以证明: 若 A 为均匀凸 Banach 空间 X 的闭凸子集, 则 $\forall x \in X$ 存在唯一元素 $P_A(x) \in A$ 使

$$\|P_A(x) - x\| = \inf_{y \in A} \|y - x\|.$$

评　　注

第 一 章

§1. 1.1—1.4 的内容主要取材于 Kato[1] 的第五、六章. 1.5 的内容可参看 Kuo[1].

§2. 本节中 Hilbert 空间张量积的定义取自 Reed-Simon[1]. Fock 空间的概念源于 Fock[1]. 二次量子化的严格数学理论最早见于 Cook[1], 更详尽的讨论可参看 Segal[2] 和 Simon[1].

§3. 赋可列范空间概念由 Gel′fand 首先引进, 见 Gel′fand-Shilov[1]. Grothendieck[1] 研究使 Schwartz 核定理成立的一类广泛的局部凸空间时首先定义了核空间, 其一般理论参看 Schaefer[1],Treves[1] 和 Itô[2], Gel′fand-Vilenkin[1] 中有关于可列 Hilbert 核空间的详细讨论. 一般局部凸空间的投影张量积即为 Treves[1] 中所定义的 π- 张量积, 此外还有一种 ϵ- 张量积, 但在核空间情形两种张量积是相同的. 在 Hilbert 空间情形, 它不同于 Hilbert 空间张量积.

§4. 4.1 的主要内容取材于 Kuo[1] 和 Skorohod[2]. 4.2—4.3 的内容基于 Kuo[1], 其中 Fernique 定理的证明取自 Da Prato-Zubczyk[1]. 有关 Minlos 定理和 Gross 定理的推广形式见 Yan[2].

第 二 章

随机变分学奠基性工作是 Malliavin[1]. 此后许多学者从不同途径建立了严格的数学理论. Stroock[1,2] 及 Kusuoka-Stroock[1,3] 系统地发展了无穷维的对称扩散半群理论. Bismut[1] 应用更为直接的 Girsanov 变换方法, 得到了 Wiener 空间上的分部积分公式. Shigekawa[1], S. Watanabe[1], Ikeda-Watanabe[1] 及 Meyer[2] 等发展了 Wiener 泛函的 Sobolev 空间方法, 形成了一个统一的无穷维 Sobolev 空间理论. 作为系统阐述随机变分学理论的论文或专著, 读者可参看 S. Watanabe[1], Ikeda-Watanabe[3], Norris[1], Ocone [2], Huang[4], Malliavin[5], Ustunel[4] 及 Nualart[1].

§1. 抽象 Wiener 空间的概念是由 Gross[1] 引进的. 因为其中微分结构完全取决于 Cameron-Martin 空间 H, Itô[3,4] 试图建立只依赖于 H 而不依赖于其它附加结构的随机变分学, 由此引导到以 Segal[1] 的

Gauss 概率空间作为基本框架. Malliavin[5], Nualart[1] 以及本书都是在这个基本框架下来阐述随机变分学的基本理论的.

不可约 Gauss 概率空间, 数值模型和内蕴性质等概念取自 Malliavin[5]. 平方可积泛函的混沌分解最早由 Wiener[2] 获得. 本节主要结果均属于 Itô[1], 部分证明参照 Nualart[1].

§2. Ornstein-Uhlenbeck 半群的超压缩性首先由 Nelson[1] 证明. 此处简单的证明取自 Neveu[1]. Cameron-Martin[1] 首先得到 Wiener 测度的拟不变性质 (定理 2.5), 在一般 Gauss 概率空间中的定理的表述和证明属于 Malliavin[5]. Wiener 泛函的 Sobolev 空间理论在 S. Watanabe[1] 及 Sugita[1,2] 中有较详细的论述.

§3. L^p 乘子定理 (定理 3.8) 属于 Meyer[2], 此处证明取自 Shigekawa[4]. Meyer 不等式 (定理 3.15) 在 $p=2$ 的特殊情形已由 M. Krée-P. Krée[1] 获得. Meyer[2] 利用 Littlewood-Payley 不等式予以证明. 这里基于 Hilbert 变换在 L^p 中的有界性的简洁证明来自 Pisier[1]. 广义泛函首先由 S. Watanabe[1] 引进. Malliavin 随机变分学中的公式总汇可参看 Nualart -Zakai[4]. 命题 3.23 取自 Sugita[3].

§4. Malliavin[1] 利用调和分析引理及分部积分公式首次给出了 Wiener 泛函分布密度存在性及光滑性的基本结果. 该技巧在 Stroock[1], Bismut[1] 及 Shigekawa[1] 中得到了发展. Bouleau-Hirsch[1,2] 利用 Dirichlet 型方法减弱了分布密度存在的条件 (定理 4.7).S. Watanabe[1] 定义了广义函数和 Winer 泛函的复合, 证明了泛函分布密度光滑性的重要结果 (定理 4.9).Donsker δ- 函数的例子取自 Ikeda-Watanabe [3]. 基于白噪声分析的讨论参看 Kuo[3] 及本书第四章例 2.24 及例 2.25. 连续过程极大值分布密度的例子取自 Nualart-Vives[1]. 在 Kusuoka-Stroock[4] 中, 还可以找到一些目前还不能用分析方法获得的结果.

第 三 章

§1. Skorohod[1] 定义了对 Brown 运动的非适应随机积分. Gaveau-Trauber[1] 证明了 Skorohod 积分和散度算子 δ 的等价性. 本节基于混沌分解的讨论主要依照 Nualart-Pardoux[1], Nualart-Zakai[1,2] 和 Yan[1]. 定理 1.7 最早由 Clark[1] 获得 (还可参看 Haussmann[1]), 但需假定 F 为 Fréchet 可微. Ocone[1] 将其推广到了 $F \in \mathbb{D}_1^2$ 的情

形. Karatzas-Ocone-Li[1] 证明了公式对 $F \in I\!D_1^1$ 仍成立. 此处极简单的证明取自 Yan[1](还可参看 Nualart-Zakai[2]). 对 Clark 公式的一般形式讨论可参看 Wu[3]. 定理 1.9 的证明详见 Ikeda-Watanabe[1] 和 Stroock[3], 这里我们利用 Picard 迭代及引理 1.4 给出了一个简单证明. Hörmander 定理的概率证明最早由 Malliavin[1] 给出, 但证明较复杂 (参看 Ikeda-Watanabe[1] 或 Huang[4]), 这里的简单证明取自 Norris[1], 其中关键引理 1.11 来自 Stroock[3]. 另一种证明方法可参看 Bismut[1]. 在 Kusuoka-Stroock[3] 中还讨论了对 Hörmander 条件的改进.

§2. 本节大部分结果取自 Malliavin[5] 和 Sugita[3], 但对证明作了一些改进. Malliavin[2] 首先引进 (k,p)- 容度和疏集的概念, 开创了拟必然分析研究领域. 容度不变性问题首先由 Itô 提出, Albeverio 、Fukushima 等 [1] 给出了回答. 此处定理 2.15 的证明取自 Malliavin[5]. 在抽象 Wiener 空间框架下, Sugita[3] 证明了 Meyer-Watanabe 正广义泛函是测度. 在白噪声框架下类似的定理参看 Kondratiev-Samoylenko[1], Yokoi[1] 和本书第四章定理 4.9. 它们之间关系的讨论可参看 Huang[5]. 关于随机过程轨道拟必然性质的讨论, 可参看 Fukushima[1], Takeda[1], Yoshida[1], Denis[1], Ren[1,3,5] 以及 Ren[4] 中所列文献.

§3. 关于 Skorohod 积分和 Stratonovich 积分的 Riemann 和逼近主要取材于 Nualart- Pardoux[1]. 非适应随机积分和 Itô 公式还有许多不同的途径, 例如参看 Hitsuda[1], Sevljakov[1], Ogawa[1], Sekiguchi-Shiota[1], Kuo-Russek[1], Asch-Potthoff[1] 和 Ustunel[2].

关于非适应随机微分方程和 Girsanov 变换主要取材于 Kusuoka[1] 和 Buckdahn [2,3,4]. 读者还可参看 Ramer[1], Buckdahn[1], Enchev[1], Enchev-Stroock[1], Ustunel-Zakai[3,5] 以及 Y. N. Zhang[1]. 其它类型的方程和研究方法可参看 Ocone-Pardoux[1], Buckdahn-Nualart[1] 及 Pardoux[1] 中所列文献.

Malliavin 分析的应用在此不能一一列举. 例如, 关于 Atiyah-Singer 指标定理的概率证明参看 Bismut[5] 和 Watanabe[3], 在滤波问题中的应用可参看 Bismut-Michel[1], 关于热核渐近性质的研究可参看 Watanabe[2] 和 Ikeda[1], 关于随机振荡积分的研究参看 Gaveau-Moulinier[1], 关于 Wiener 空间上随机变量独立性和其梯度正交性关系的研究可参看 Ustunel-Zakai[1,2]. 此外, 关于带跳跃过程的随机变分问题的研究可参

看 Bismut[3], Bichteler-Gravereaux-Jacod[1] 及 Wu[1,2]. 其它应用以及理论的进一步发展可参看 Malliavin[4] 及其中所引参考文献.

第四章

§1. 有关 Wiener-Itô-Segal 同构的经典文献是 Wiener[2],Itô[1] 和 Segal[2]. Wick 张量积 : $x^{\otimes n}$: 的取名源于量子物理中的 Wick 编序, 后者得名于 Wick[1]. 在一些文章中把 Wick 张量积称为 Wick 编序, 似乎欠妥. Kubo-Takenaka[1] 首次用二次量子化方法构造了经典的白噪声分析框架. Meyer-Yan[2] 和 Kondratiev-Leukert-Potthoff-Streit-Westerkamp[1] 进一步讨论了一般 Gel'fand 三元组的二次量子化. 用推广了的二次量子化方法构造白噪声分析框架是 Kondratiev-Streit[1] 首先提出的. 这一方法也适用于非 Gauss 分析 (见 Kondratiev- Streit-Westerkamp-Yan[1]). 近年来, 由于实际问题需要或理论上的兴趣, 提出了若干新的白噪声分析框架 (例如见 Meyer-Yan[3], Potthoff-Timpel[1], Huang-Song[1], Imkeller-Yan[2]).

§2. 在经典的白噪声分析框架下, 广义泛函空间的刻画定理及其两个重要推论首先由 Potthoff-Streit[1] 给出, 检验泛函空间的刻画则在 Kuo-Potthoff-Streit[1] 中给出. Yan[7] 包含上述结果的一个精细化. T. S. Zhang[1] 给出了泛函空间的另一种刻画. Kondratiev-Leukert-Potthoff 等 [1] 进一步给出了一般 Gel'fand 三元组的二次量子化情形下的泛函空间刻画. 2.1 及 2.3 中关于一般框架下的相应结果属于 Kondratiev-Streit[1]. 2.2 中的结果则属于 Kondratiev-Leukert-Streit[1]. 例 2.32 由 Kubo-Takenaka[1] 给出. 例 2.4 中 Donsker δ- 泛函由 Kuo[3] 给出. 例 2.26 来自 Kondratiev-Streit[1]. H. Watanabe[1] 首先用白噪声分析研究了多维 Brown 运动的自交局部时, 这里的例 2.27 取自 He-Yang-Yao-Wang[1].

§3. 泛函的乘积公式 (3.10) 是经典的 (参见 Meyer[3] 关于该公式的评注). 引理 3.1 和定理 3.2 是 Potthoff-Yan[1] 中相应于 $\beta=0$ 情形的结果的推广. 泛函的 Wick 积源于量子场论中的 "S- 积", 它是由 Wick[1] 引入的 (也见 Simon[1]). 1965 年, Hida- Ikeda[1] 首次在概率论中引进 Wick 积. Meyer-Yan[1] 用 S- 变换定义了白噪声分析框架下的广义泛函的 Wick 积. 关于 Wick 积在随机分析中的广泛应用可参看 Holden et al.[1]. 用白噪声分析研究 Feynman 积分始于 Hida-Streit[1].

Meyer-Yan[1], Hu-Meyer[1] 以及 De Faria-Potthoff-Streit[1] 继续了这一研究. 3.3 采用 Meyer-Yan[1] 的构架介绍了 Khandekar-Streit[1] 的结果.

§4. 广义泛函空间的矩刻画首先由 Kondratiev-Streit[1] 给出，这里用 Yan[6] 中的重正化算子简化了原证明. 在经典的白噪声分析框架下，正广义泛函的测度表示由 Yokoi[1] 首先给出，一般框架下的相应结果 (定理 4.9) 属于 Kondratiev-Streit[1].

第五章

§1. 广义泛函的分析运算在 Potthoff-Yan[1] 中有系统研究， Kuo-Potthoff-Yan[1] 及 Yan[10] 给出了进一步的结果. 本节内容是上述文献中的结果到一般框架下的推广.

§2. Fock 空间中算子象征的概念源于 Berezin[2] 及 Krée-Rączka[1]. 在白噪声分析框架下，广义算子的象征刻画定理是 Obata[3] 首先给出的，它是 Potthoff-Streit[1] 的广义泛函刻画定理的自然推广. 由于 Obata 的证明用到积分核算子，因此比较繁，而且不适用于一般框架. Obata[8] 给出了一般框架下广义算子的象征刻画定理的叙述. 这里给出的简单证明来自 Luo-Yan[1]. 广义算子的混沌分解源于 Berezin[2]，在白噪声分析框架下， Huang[6] 首先给出这一结果. 广义算子的 Wick 积源于量子场论中的增生和湮灭算子乘积的 Wick 编序，它的严格数学定义由 Huang-Luo[1] 给出. 2.2 中的结果取自 Luo-Yan [1].

§3. 积分核算子概念源于 Berezin[2]. 在白噪声分析框架下， Kubo-Takenaka[1] 首先用 Hida 微分算子及其对偶算子表示一类算子. Hida-Obata-Saitô[1] 系统地研究了积分核算子. 广义算子的积分核表示 (定理 3.11) 属于 Obata[4]，在那里被称为 Fock 展开. 这里借助于算子的混沌分解给出了该结果的简化证明. 公式 (3.39) 属于 Huang[6]. 本节主要参考了 Obata[4].

§4. Huang[1] 首先将白噪声分析用于量子概率，提出量子白噪声测度概念. 本节 4.1–4.2 的内容取自 Huang-Luo[1] 及骆顺龙 [1]，4.3 的内容来自 Hida-Kuo-Potthoff-Streit [1]. 有关白噪声分析在无穷维 Dirichlet 型中的进一步应用可参见 Albeverio-Hida 等 [1,2], Hida-Potthoff-Streit[1] 及 Razafimanantena[1]. 关于广义算子理论在无穷维调和分析及量子概率中的应用，可参看 Obata[4,5,6,8].

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符 号 说 明

$\equiv$ (定义为); $\Longrightarrow$ (蕴含); $\Longleftrightarrow$ (命题等价);
$\prec$ (拓扑弱于); $\sim$ (等价); ▮ (证明结束);
$\cong$ (同构); $\hookrightarrow$ (连续、稠密嵌入);
$\uparrow$ (单调上升趋于极限); $\uparrow\uparrow$ (严格单调上升趋于极限);
$\downarrow$ (单调下降趋于极限); $\downarrow\downarrow$ (严格单调下降趋于极限);
$x\longmapsto f(x)$ (映射 f); 1_A (集合 A 的示性函数);
$f|_A$ (映射 f 在集合 A 上的限制);
$\overline{\mathcal{F}}^\mu$ (σ- 代数 $\mathcal{F}$ 关于测度 μ 的完备化);
$a\wedge b\equiv\min(a,b)$; $a\vee b\equiv\max(a,b)$;
$x^+\equiv x\vee 0$; $x^-\equiv-(x\wedge 0)$; $\binom{n}{k}\equiv\frac{n!}{k!(n-k)!}$;
$|\cdot|_X$, $\|\cdot\|_X$ (Banach 空间 X 中的范数);
$(\cdot,\cdot)_H$ (Hilbert 空间 H 中的内积);
X' (线性空间 X 的代数对偶, B5);
X^* (拓扑线性空间 X 的拓扑对偶, B5);
$\langle\cdot,\cdot\rangle$, $\langle\langle\cdot,\cdot\rangle\rangle$ (典则双线性型, B5);
$\partial_j\equiv\partial/\partial x_j$ $(x=(x_1,\cdots,x_m)\in\mathbb{R}^m)$;
$\partial_\alpha\equiv\partial_1^{\alpha_1}\cdots\partial_m^{\alpha_m}$ $\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbb{N}_0^m$);
$x^\alpha\equiv x_1^{\alpha_1}\cdots x_m^{\alpha_m}$; $|\alpha|\equiv\sum_j\alpha_j$; $\alpha!\equiv\prod_j(\alpha_j!)$;
$\otimes$ (张量积, I2.1, I2.3, I3.3); $\widehat{\otimes}$ (对称张量积, I2.1);
$\widetilde{\otimes}$ (投影张量积, I2.3);
$:x^{\otimes n}:$ (Wick 张量积, IV1.1);
$\otimes_r$ (张量积的缩合, II1.3, IV3.1, V3.1);
$\oplus$ (直和 I1.1, I2.2); $\widehat{A}$ (算子 A 的象征, V2.1);
A^τ (矩阵 A 的转置); $\mathcal{A}^\alpha(E_{\mathbb{C}}^*)$ (IV2.3);
$\widetilde{A}$ (算子 A 的闭包, I1.1-1.3;算子 A 的 S- 变换, V2.2);
a.a. (几乎所有); a.e. (几乎处处); a.s. (几乎必然);
$\mathcal{B}(T)$ (拓扑空间 T 的 Borel 子集 σ- 代数);
$\mathcal{B}(X,Y)$ (I3.3); $\beta(X,Y)$ (B7); X_β (B7);
$\mathbb{C}$ (复数域); $X_{\mathbb{C}}\equiv X+iX$($X$的复化, $IV2$);
C^k, C^∞ (k 次、无限次连续可微函数);
C_b^k, C_b^∞ (C^k, C^∞ 中具有各阶有界偏导数函数);
C_0^k, C_0^∞ (C^k, C^∞ 中具有紧支集函数);

q.s., q.e. (III2.1); $Q \equiv (I-\mathcal{L})^{-1/2}$ (II3.3);
Q_p (B1); $\mathcal{R}(A)$ (I1.1); $\rho(A)$ (I1.4);
$\mathbb{R}$ (实数域); $\mathbb{R}_+$ (非负实数); Re (实部);
$\mathbb{R}^d$ (d 维实数空间); $\mathbb{R}^\infty$ (实数序列空间);
$(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$ (数值模型, II1.2);
$\mathcal{S}(\mathbb{R}^m)$, $\mathcal{S}^*(\mathbb{R}^m)$ (I3.2); $\mathcal{S}_M(E)$ (II1.1);
$\mathfrak{S}, \mathfrak{S}_l$ (置换群, I2.1, V3.2); $s_{l,m}(\kappa)$ (V3.2);
$\sigma(\mathcal{G})$ (由 $\mathcal{G}$ 生成的 σ- 代数);
$\sigma(f_\alpha . \alpha \in \Gamma)$ (由 $\{f_\alpha \in \Gamma\}$ 生成的 σ- 代数);
$\sigma(A), \sigma_p(A)$ (I1.4); $\sigma(X,Y)$ (B5);
$\Sigma = (\sigma_{ij})$ (II4.1);
span (线性包); spec (谱集); supp (支集);
τ (IV1.1); τ_n (V3.3); $\tau(X,Y)$ (B6);
Tr (I1.5); T_t (II2.1, II3.1); $t_{m,l}(\kappa)$ (V3.2);
X_σ (B7); X_τ (B7);
$\mathcal{W}(A)$ (I1.1); $W^{k,p}(\mathbb{R}^m)$ (Sobolev 空间);
$\Xi_{l,m}(\kappa)$ (V3.2); $\widehat{X}_p$ (B1); $\mathbb{Z}$ (整数集).

Introduction to Infinite Dimensional Stochastic Analysis

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Contents

Preface

The infinite dimensional analysis as a branch of mathematical sciences was formed in the late 19th and early 20th centuries. Motivated by problems in mathematical physics, the first steps in this field were taken by V. Volterra, R. Gâteaux, P. Lévy and M. Fréchet, among others (see the preface to Lévy[2]). Nevertheless, the most fruitful direction in this field is the infinite dimensional integration theory initiated by N. Wiener and A. N. Kolmogorov which is closely related to the developments of the theory of stochastic processes. It was Wiener who constructed for the first time in 1923 a probability measure on the space of all continuous functions (i.e. the Wiener measure) which provided an ideal mathematical model for Brownian motion. Then some important properties of Wiener integrals, especially the quasi-invariance of Gaussian measures, were discovered by R. Cameron and W. Martin[1, 2, 3]. In 1931, Kolmogorov[1] deduced a second order partial differential equation for transition probabilities of Markov processes with continuous trajectories (i.e. diffusion processes) and thus revealed the deep connection between theories of differential equations and stochastic processes. The stochastic analysis created by K. Itô (also independently by Gihman [1]) in the forties is essentially an infinitesimal analysis for trajectories of stochastic processes. By virtue of Itô's stochastic differential equations one can construct diffusion processes via direct probabilistic methods and treat them as functionals of Brownian paths (i.e. the Wiener functionals). This affords a possibility of using probabilistic methods to investigate deterministic differential equations and many other pure analytical problems. On the other hand, during the same decade of this century, the famous work on functional integration approach to mathematical physics derived by R. Feynman and M. Kac as well as rapid developments in quantum field theory gave new impulsions to the analysis in infinite dimension.

The classical notions of functions and derivatives in finite dimensional analysis were long felt to be restrictive in mathematical physics. Notions of generalized functions and derivatives were first introduced by S. L. Sobolev[1] in 1936 to meet the needs of solving equations in mathematical physics. The theory of Sobolev spaces has played an important role in modern treatment of partial differential operators. L. Schwartz systematically developed this idea and established theory

of distributions. Many singular objects in classical physics such as Dirac delta functions thus obtained mathematically rigorous meanings. However, up to now there still exist a lot of intuitive notions and heuristic calculations in physics which remain meaningless from the mathematical viewpoint. To put them on a sound mathematical foundation is quite important for the development of theoretical physics and is a real challenge to mathematicians and physicists.

The same situation also occurred in infinite dimensional analysis. Since many important functionals (e.g. diffusion processes regarded as Wiener functionals) are not differentiable in Fréchet sense, it is essential to generalize the notions of functionals and differentiation in infinite dimensional spaces. In 1976, P. Malliavin[1] successfully extended the gradient, divergence and Ornstein-Uhlenbeck operators to infinite dimensional cases and created the stochastic calculus of variation (known as Malliavin calculus). Under his sense of differentiation, many important Wiener functionals become smooth (infinitely differentiable). Along this line S. Watanabe[1], I. Shigekawa[1], D. W. Stroock[1], P. A. Meyer[1] et al. established a Sobolev theory over infinite dimensional spaces. With fruitful applications to partial differential operators and heat kernels, to stochastic oscillatory integrals, to filtering and control of stochastic systems, etc., the Malliavin calculus has become one of the most significant successes in the field of stochastic analysis.

In 1975, T. Hida launched out the white noise analysis. Since Gaussian white noise is the derivative of Brownian motion in the distribution sense, its sample space lies in the space of Schwartz distributions. By regarding Wiener functionals as functionals of white noise, Hida established a Schwartz theory over infinite dimensional spaces. With profound background in physics and successful applications to Feynman integrals as well as quantum field theory, white noise analysis has attracted more and more attention from theoretical physicists.

These two frameworks of infinite dimensional analysis are essentially based on the quasi-invariance of Gaussian measures and could be unified into one setting of the so-called Gaussian probability spaces. Its origin goes back to the works on abstract integration on Hilbert spaces by I. E. Segal[1, 2] and on rigged Hilbert spaces by I. M. Gel'fand in the fifties (see Gel'fand & Vilenkin[1]), and the work on abstract Wiener spaces by L. Gross[1] in the sixties. The choice of frameworks depends naturally on the practical problems to be solved. In Malliavin calculus, for example, one requires the space of testing functionals to be rich enough so that many important functionals become smooth, while in Hida calculus, one hopes that the space of distributions would be sufficiently large to contain many singular objects in physics which are not rigorously defined so far. The relationship between these two kinds of calculus is quite similar to that of Sobolev and Schwartz theories in finite dimensions.

This book is intended to offer a quick introduction to the above mentioned rapidly developing research area —— infinite dimensional stochastic analysis. We

have attempted to make the book concise and self-contained so that readers, who have acquired the basic knowledge of stochastic analysis, can easily master the very core of the theory and methods and go quickly to the front of this research area.

The book is divided into five chapters. The first chapter contains the basic knowledge of infinite dimensional analysis including linear operators on Hilbert spaces, Fock spaces, countably normed spaces, nuclear spaces and their dual spaces, Borel measures on topological linear spaces which are the base of other chapters and also of independent interests. In order to make the book self-contained and for reader's convenience, we summarize some basic notions and general results of locally convex topological linear spaces in Appendix B. The second chapter is devoted to the fundamental theory of Malliavin calculus, including the chaos decomposition and differentiation for functionals on Gaussian probability spaces, the Ornstein-Uhlenbeck semi-group, Meyer's inequalities, Sobolev spaces, existence and smoothness of densities of non-degenerate functionals and so on. Chapter III treats some important applications of Malliavin calculus. The regularity of densities of solutions to Itô's stochastic differential equations, namely the fundamental solutions to the corresponding second order parabolic partial differential equations, is derived in detail and a probabilistic proof of the celebrated Hörmander theorem on hypoellipticity of differential operators is presented. The potential theory, quasi-sure analysis on Wiener spaces and the anticipating stochastic calculus are also briefly touched. The general theory of white noise analysis, which is the main goal of Chapter IV, consists of construction of general settings, characterization of functional spaces, the products and Wick products of functionals and the moment characterization of distributions. Applications to Feynman integrals, $P(\phi)_2$-quantum fields and local times of self-intersection for Brownian motion are also briefly discussed. The last chapter is a presentation of the theory of operators on spaces of distributions (including some analytical operations on distributions) and its applications in quantum physics. Applications to infinite dimensional harmonic analysis are not touched here for the sake of length, for which we refer the readers to N. Obata[2].

§1, §4 of Chapter I, Chapters IV and V were written by J. A. Yan; §2, §3 of Chapter I, Chapters II and III and the appendices were written by Z. Y. Huang. Both authors wish to thank Prof. J. G. Ren and Dr. S. L. Luo, who provided considerable assistance with the preparation of §2 of Chapter III and §4 of Chapter V respectively and made many useful comments. This work was financially supported by the National Natural Science Foundation of China (grant no. 19631030 and 79790130) and the Science Press Foundation of the Chinese Academy of Sciences to whom the authors would like to acknowledge.

Z. Y. Huang and J. A. Yan, 1999

Chapter I

Foundations of Infinite Dimensional Analysis

§1. Linear operators on Hilbert spaces

Let $\mathbb{K}$ denote the number field $\mathbb{R}$ or $\mathbb{C}$. The capital letters H, K and E will denote Hilbert spaces on $\mathbb{K}$. We use $(\cdot,\cdot)$ and $\|\cdot\|$ to denote the inner product and norm in any Hilbert space. By convention, (x,y) is linear in x and conjugate-linear in y. All results apply to both cases if the number field $\mathbb{R}$ or $\mathbb{C}$ is not specified.

1.1 Basic notions, notations and lemmas

Denote by $L(H,K)$ the space of linear operators from H to K and $\mathcal{L}(H,K)$ the subspace of bounded linear operators. $L(H,H)$ and $\mathcal{L}(H,H)$ are denoted by $L(H)$ and $\mathcal{L}(H)$ respectively. For $A \in L(H,K)$, denote by $\mathcal{D}(A)$ its domain. $\mathcal{D}(A)$ is a subspace of H. Henceforth, for $A \in \mathcal{L}(H,K)$, we assume that $\mathcal{D}(A)$ is dense in H, and consequently, we can assume further $\mathcal{D}(A) = H$. For an unbounded operator, its domain must be specified. For $A \in L(H,K)$, let

$$\mathcal{N}(A) = \{x \in \mathcal{D}(A):\ Ax = 0\}, \qquad \mathcal{R}(A) = \{Ax:\ x \in \mathcal{D}(A)\},$$

$\mathcal{N}(A)$ and $\mathcal{R}(A)$ are called the *kernel* (or *null space*) and *range* of A respectively. If $\mathcal{D}(A)$ is dense in H, then A is said to be *densely defined*. If $\mathcal{N}(A) = \{0\}$, then A is said to be *invertible*. For an invertible operator A, its *inverse* A^{-1} is defined as: $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$; $A^{-1}y = x$ if $Ax = y$.

The product space $H \times K$ is a Hilbert space endowed with the inner product:

$$(\{x,y\},\{z,w\}) = (x,z) + (y,w), \quad x,z \in H,\ y,w \in K.$$

This is the direct sum $H \oplus K$. For $A \in L(H,K)$, let

$$\mathcal{G}(A) = \{\{x, Ax\}\ :\ \ x \in \mathcal{D}(A)\}, \tag{1.1}$$

$$\mathcal{W}(A) = \{\{Ax, x\}\ :\ \ x \in \mathcal{D}(A)\}. \tag{1.2}$$

Then $\mathcal{G}(A)$ and $\mathcal{W}(A)$ are subspaces of $H \oplus K$ and $K \oplus H$, respectively, they are called the *graph* and *inverse graph* of A, respectively. If A is invertible, then $\mathcal{W}(A) = \mathcal{G}(A^{-1})$.

Let $A_1, A_2 \in L(H,K)$. If $\mathcal{G}(A_1) \subset \mathcal{G}(A_2)$, i.e., $\mathcal{D}(A_1) \subset \mathcal{D}(A_2)$ and the restriction of A_2 to $\mathcal{D}(A_1)$ coincides with A_1, then A_2 is called an *extension* of A_1 and A_1 is a *restriction* of A_2. It is denoted by $A_1 \subset A_2$ or $A_2 \supset A_1$.

Let $A \in L(H,K)$. If $\mathcal{G}(A)$ is a closed subspace of $H \oplus K$ (this is equivalent to the fact that $\mathcal{W}(A)$ is a closed subspace of $K \oplus H$), then A is called a *closed operator*. If the closure $\overline{\mathcal{G}(A)}$ of $\mathcal{G}(A)$ in $H \oplus K$ is the graph of some linear operator $\widetilde{A}$, then A is said to be *closable* and $\widetilde{A}$ is called the *closure* of A. Clearly, A is closable if and only if $\{0, y\} \in \overline{\mathcal{G}(A)}$ implies $y = 0$. If A is a closed operator and $\mathcal{D}(A) = H$, then A is a bounded operator by the closed graph theorem. The null space of any closed operator is closed.

Let $A \in L(H,K)$ be densely defined. Put

$$\mathcal{D}(A^*) = \{y \in K : \exists c_y > 0, \text{s.t.} \forall x \in \mathcal{D}(A), |(Ax, y)| \le c_y \|x\|\}.$$

Then by Riesz' representation theorem, $\forall\ y \in \mathcal{D}(A^*)$, there exists a unique element of H, denoted by A^*y, such that

$$(x, A^*y) = (Ax, y), \quad \forall x \in \mathcal{D}(A). \tag{1.3}$$

Clearly $A^* \in L(K,H)$. A^* is called the *adjoint* of A. If $A, B \in L(H,K)$ are densely defined and $A \subset B$, then $B^* \subset A^*$.

Let $A \in L(H)$. If A is densely defined and $A \subset A^*$, i.e.,

$$(Ax, y) = (x, Ay), \quad \forall x, y \in \mathcal{D}(A)\ ,$$

then A is said to be *symmetric*; if furthermore $A = A^*$, then A is said to be *self-adjoint*.

Lemma 1.1 *Let $A \in L(H)$ be densely defined and $(Ax, x) = 0$, $\forall x \in \mathcal{D}(A)$.*

(1) If H is a complex space, then A is the null operator (i.e., $Ax = 0, \forall x \in \mathcal{D}(A)$);

(2) If H is a real space and A is symmetric, then A is the null operator.

Proof. (1) Let $x, y \in \mathcal{D}(A)$. Then

$$(Ax, y) + (Ay, x) = (A(x+y), x+y) - (Ax, x) - (Ay, y) = 0. \tag{1.4}$$

Multiplying both sides by $i\ (= \sqrt{-1})$ and replacing y by iy, we have

$$(Ax, y) - (Ay, x) = 0. \tag{1.5}$$

By (1.4) and (1.5) we have $(Ax, y) = 0$, $\forall y \in \mathcal{D}(A)$. But $\mathcal{D}(A)$ is dense in H, hence $Ax = 0$.

(2) follows from (1.4) and the symmetry of A. ∎

For $A \in \mathcal{L}(H,K)$, denote by $\|A\|$ the *norm* of A, i.e.,

$$\|A\| = \sup\{\|Ax\|\ :\ \|x\| = 1\}.$$

The following lemma gives another expression for the norm of a bounded symmetric operator.

Lemma 1.2 *Let $A \in \mathcal{L}(H)$ be a symmetric operator. Then*

$$\|A\| = \sup_{\|x\|=1} |(Ax, x)|. \tag{1.6}$$

Proof. Since

$$\begin{aligned}(Ax, y) + (y, Ax) &= (Ax, y) + (Ay, x) \\ &= \frac{1}{2}[(A(x+y), x+y) - (A(x-y), x-y)],\end{aligned}$$

we have

$$\begin{aligned}|(Ax, y) + (y, Ax)| &\le \frac{1}{2}(\|x+y\|^2 + \|x-y\|^2) \sup_{\|z\|=1} |(Az, z)| \\ &= (\|x\|^2 + \|y\|^2) \sup_{\|z\|=1} |(Az, z)|.\end{aligned} \tag{1.7}$$

The last equality follows from the parallelogram law. Without lose of generality, we may assume that A is not a null operator. Put $a = \sup_{\|x\|=1} |(Ax, x)|$. Then by Lemma 1.1, $a > 0$. Put $y = a^{-1}Ax$ in (1.7). Then

$$2a^{-1}\|Ax\|^2 \le (\|x\|^2 + a^{-2}\|Ax\|^2)a,$$

that is, $\|Ax\|^2 \le a^2\|x\|^2$, hence $\|A\| \le a$. But the inverse inequality always holds, hence (1.6) follows. ∎

For $A \in L(H,K)$, $B \in L(K,E)$, the *product* of B and A is defined by

$$\mathcal{D}(BA) = \{x \in \mathcal{D}(A) :\ Ax \in \mathcal{D}(B)\}, \tag{1.8}$$

$$(BA)x = B(Ax), \quad x \in \mathcal{D}(BA). \tag{1.9}$$

Consequently, $BA \in L(H,E)$.

Lemma 1.3 *Let $A \in L(H,K)$, $B \in L(K,E)$. If A, B and BA are all densely defined, then*

$$A^*B^* \subset (BA)^*. \tag{1.10}$$

If furthermore B is bounded, then

$$A^*B^* = (BA)^*. \tag{1.11}$$

Proof. (1.10) can be easily verified from the definition of self-adjointness. For (1.11), it suffices to check $(BA)^* \subset A^*B^*$. Now let $B \in \mathcal{L}(K,E)$. Since $\mathcal{D}(A) = \mathcal{D}(BA)$, $\mathcal{D}(B^*) = E$, for any $y \in \mathcal{D}((BA)^*)$, we have

$$(Ax, B^*y) = ((BA)x, y) = (x, (BA)^*y), \quad \forall x \in \mathcal{D}(A).$$

This implies that $B^*y \in \mathcal{D}(A^*)$(thus $y \in \mathcal{D}(A^*B^*)$) and $A^*B^*y = (BA)^*y$, i.e., $(BA)^* \subset A^*B^*$. ∎

Let M be a closed subspace of H, $M^\perp$ the orthogonal complement of M in H. Then any $x \in H$ has the unique decomposition:

$$x = y + z, \qquad y \in M,\ z \in M^\perp.$$

Denote y by Px, Px is called the *orthogonal projection* of x onto M. Obviously, P is a symmetric bounded linear operator on H. Moreover, it is *idempotent*, i.e., $P^2 = P$. Such an operator is called a *projection*.

The projection operators are characterized as follows.

Lemma 1.4 *Let $P \in \mathcal{L}(H)$. Then P is a projection if and only if $\mathcal{R}(P) = \mathcal{N}(P)^\perp$ and $P^2 = P$.*

Proof. If $\mathcal{R}(P) = \mathcal{N}(P)^\perp$ and $P^2 = P$, then $\forall x, y \in H$, $x - Px \in \mathcal{N}(P)$, $y - Py \in \mathcal{N}(P)$. Thus

$$\begin{aligned}(Px, y) &= (Px, Py + (y - Py)) = (Px, Py)\\ &= (Px + (x - Px), Py) = (x, Py).\end{aligned}$$

This means that P is symmetric and consequently a projection. Conversely, if P is a projection, then

$$\begin{aligned}x \in \mathcal{N}(P) &\iff \forall y \in H,\ (x, Py) = (Px, y) = 0\\ &\iff x \perp \mathcal{R}(P),\end{aligned}$$

i.e., $\mathcal{N}(P) = \mathcal{R}(P)^\perp$. But $P^2 = P$ implies $\mathcal{R}(P) = \mathcal{N}(I - P)$, hence $\mathcal{R}(P)$ is a closed subspace of H. From this we conclude that $\mathcal{R}(P) = \mathcal{R}(P)^{\perp\perp} = \mathcal{N}(P)^\perp$. ∎

1.2 Closable, symmetric and self-adjoint operators

Theorem 1.5 Let $A \in L(H, K)$ be densely defined.

(1) A^* is closed, and $\mathcal{G}(A^*) = \mathcal{W}(-A)^\perp$.

(2) If A is closed, then A^* is densely defined and $A^{**} = A$.

(3) A is closable if and only if A^* is densely defined. In this case, A^{**} is the closure of A.

Proof. (1) Let $y \in K$, $z \in H$. Then

$$\begin{aligned}\{y, z\} \in \mathcal{G}(A^*) &\iff y \in \mathcal{D}(A^*), z = A^*y\\ &\iff (z, x) = (y, Ax),\ \forall x \in \mathcal{D}(A)\\ &\iff (\{y, z\}, \{-Ax, x\}) = 0,\ \forall x \in \mathcal{D}(A).\end{aligned}$$

Thus $\mathcal{G}(A^*) = \mathcal{W}(-A)^\perp$. In particular, $\mathcal{G}(A^*)$ is a closed subspace of $K \oplus H$, i.e., A^* is a closed operator.

(2) Since $-A$ is closed, $\mathcal{G}(-A)$ is a closed subspace of $H \oplus K$, and consequently, $\mathcal{W}(-A)$ is a closed subspace of $K \oplus H$. By (1) we know that $K \oplus H$ has the orthogonal decomposition:

$$K \oplus H = \mathcal{W}(-A) \oplus \mathcal{G}(A^*). \tag{1.12}$$

Now let $z \in K$ and $z \perp \mathcal{D}(A^*)$. Then $\{z, 0\} \perp \mathcal{G}(A^*)$. By (1.12), $\{z, 0\} \in \mathcal{W}(-A)$, thus $z = -A0 = 0$. This means that $\mathcal{D}(A^*)$ is dense in K. Applying (1.12) to A^* and $-A$, we obtain

$$H \oplus K = \mathcal{W}(-A^*) \oplus \mathcal{G}(A^{**}), \tag{1.13}$$

$$K \oplus H = \mathcal{W}(A) \oplus \mathcal{G}(-A^*). \tag{1.14}$$

But (1.14) is equivalent to the following orthogonal decomposition:

$$H \oplus K = \mathcal{G}(A) \oplus \mathcal{W}(-A^*). \tag{1.15}$$

Comparing (1.13) with (1.15), we get $\mathcal{G}(A) = \mathcal{G}(A^{**})$, i.e., $A = A^{**}$.

(3) Suppose that A is closable and $\tilde{A}$ is the closure of A. Then $\tilde{A} \supset A$. By the definition of self-adjointness, $A^* \supset \tilde{A}^*$. In particular, $\mathcal{D}(A^*) \supset \mathcal{D}(\tilde{A}^*)$. Thus by (2), A^* is densely defined. Applying (1) to $-A$ yields

$$K \oplus H = \overline{\mathcal{W}(A)} \oplus \mathcal{G}(-A^*), \tag{1.16}$$

which is equivalent to

$$H \oplus K = \overline{\mathcal{G}(A)} \oplus \mathcal{W}(-A^*). \tag{1.17}$$

Comparing (1.13) and (1.17), we get $\overline{\mathcal{G}(A)} = \mathcal{G}(A^{**})$, i.e., A^{**} is the closure of A.

Conversely, suppose A^* is densely defined. We are going to prove that A is closable. Since A^* is closed, (1.13) still holds. But (1.17) always holds, hence $\overline{\mathcal{G}(A)} = \mathcal{G}(A^{**})$, which means that A is closable. ∎

Theorem 1.6 *Let $A \in L(H)$ be symmetric.*

(1) A is closable, A^{**} is the closure of A, and A^{**} is symmetric.

(2) If $\mathcal{D}(A) = H$, then A is a bounded self-adjoint operator.

(3) If A is self-adjoint and invertible, then $\mathcal{R}(A)$ is dense in H and A^{-1} is self-adjoint.

(4) If $\mathcal{R}(A)$ is dense in H, then A is invertible.

(5) If $\mathcal{R}(A) = H$, then A is self-adjoint and A^{-1} is a bounded self-adjoint operator.

Proof. (1) Since $A^* \supset A$, A^* is densely defined. By Theorem 1.5(3) we conclude that A is closable and the closure of A is A^{**}. Moreover, since $A \subset A^*$, we have $A^{**} \subset A^{***}$, i.e., A^{**} is symmetric.

(2) Since $A \subset A^*$ and $\mathcal{D}(A) = H$, we have $A = A^*$, i.e., A is self-adjoint. In particular, A is closed. Now by the closed graph theorem, A is bounded.

(3) Let $y \in H$, $y \perp \mathcal{R}(A)$. Then $\forall x \in \mathcal{D}(A)$, $(Ax, y) = 0$. Thus $y \in \mathcal{D}(A^*) = \mathcal{D}(A)$. Consequently, $\forall x \in \mathcal{D}(A)$, $(x, Ay) = (Ax, y) = 0$. This implies that $Ay = 0$. But A is invertible, hence $y = 0$. Thus we have proved that $\mathcal{R}(A)$ is dense in H. Now we show that A^{-1} is self-adjoint. Since $A = A^*$, by (1.12),

$$\mathcal{G}((A^{-1})^*) = \mathcal{W}(-A^{-1})^{\perp} = \mathcal{G}(-A)^{\perp} = \mathcal{W}(A).$$

However, we have $\mathcal{W}(A) = \mathcal{G}(A^{-1})$, thus $\mathcal{G}((A^{-1})^*) = \mathcal{G}(A^{-1})$, i.e., $(A^{-1})^* = A^{-1}$.

(4) Let $y \in \mathcal{D}(A)$ and $Ay = 0$. Then $\forall x \in \mathcal{D}(A)$, $(Ax, y) = (x, Ay) = 0$, i.e., $y \perp \mathcal{R}(A)$. By assumption, $\mathcal{R}(A)$ is dense in H, hence $y = 0$ which means that $\mathcal{N}(A) = \{0\}$, i.e., A is invertible.

(5) By (4), A is invertible, and by the assumption, $\mathcal{D}(A^{-1}) = \mathcal{R}(A) = H$. Let $x, y \in H$. Then

$$(A^{-1}x, y) = (A^{-1}x, A(A^{-1}y)) = (x, A^{-1}y).$$

Thus A^{-1} is symmetric. Now the conclusion follows from (2) and (3). ∎

The following important theorem is due to von Neumann.

Theorem 1.7 *If $A \in L(H, K)$ is a densely defined closed operator, then A^*A is a self-adjoint operator on H, and $\mathcal{G}_0 \equiv \{\{y, Ay\} : y \in \mathcal{D}(A^*A)\}$ is dense in $\mathcal{G}(A)$. Moreover, AA^* is a self-adjoint operator on K.*

Proof. Let $x \in H$. By (1.12), there exist $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(A^*)$ such that

$$\{0, x\} = \{-Au, u\} + \{v, A^*v\}.$$

Hence $v = Au$, and

$$x = u + A^*v = (I + A^*A)u.$$

Put $S = I + A^*A$. Then S^{-1} is symmetric and $\|S^{-1}\| \le 1$. Thus S^{-1} is self-adjoint. Now by Theorem 1.6(3) we know that S is self-adjoint and consequently so is A^*A.

In order to prove that $\mathcal{G}_0$ is dense in $\mathcal{G}(A)$, it suffices to prove that: $\forall x \in \mathcal{D}(A)$, if $\{x, Ax\}$ is orthogonal to $\mathcal{G}_0$, then $x = 0$. This orthogonality implies that $(x, y) + (Ax, Ay) = (x, y + A^*Ay) = 0$, $\forall y \in \mathcal{D}(A^*A)$. But $\mathcal{R}(I + A^*A) = H$, thus $x = 0$.

Finally, by Theorem 1.5, A^* is a densely defined closed operator from K to H, and $A^{**} = A$. Applying the proved result to A^*, we know that AA^* is a self-adjoint operator on K. ∎

Let $A \in L(H)$ be symmetric. If the closure (i.e., A^{**}) of A is self-adjoint, then A is said to be *essentially self-adjoint*.

The following theorem gives an equivalent description of the essential self-adjointness.

Theorem 1.8 *Let $A \in L(H)$ be symmetric. Then A is essentially self-adjoint if and only if A^* is self-adjoint (or equivalently, A^* is symmetric).*

Proof. If A^* is self-adjoint, then $A^* = A^{**}$, hence $A^{***} = A^{**}$, and A^{**} is self-adjoint. On the other hand, if A^{**} is self-adjoint, then $A^{**} = A^{***}$. But A^* is densely defined and closed, by Theorem 1.5(2) we have $A^{***} = A^*$, thus $A^* = A^{**}$, i.e., A^* is self-adjoint. ∎

Let $A \in L(H)$ be symmetric. If there exists a real number c such that

$$(Ax, x) \ge c\|x\|^2, \quad \forall x \in \mathcal{D}(A),$$

then A is said to be *bounded below*, and we denote this by $A \ge c$. If c can be taken to be 0 (positive), then A is said to be *positive* (to *have positive lower bound*).

The following theorem gives a useful characterization for a symmetric, bounded below operator to be self-adjoint or essentially self-adjoint.

Theorem 1.9 *Let $A \in L(H)$ be a symmetric bounded below operator, $A \ge c, \epsilon > 0$. Put $B = (\epsilon - c)I + A$, $\mathcal{D}(B) = \mathcal{D}(A)$. Then*

(1) A is self-adjoint iff $\mathcal{R}(B) = H$;

(2) A is essentially self-adjoint iff $\mathcal{R}(B)$ is dense in H (or equivalently, $\mathcal{N}(B^*) = \{0\}$).

Proof. (1) If A is self-adjoint, then so is B. Thus by Theorem 1.6(3) we know that $\mathcal{R}(B)$ is dense in H. $\forall x \in H, \exists y_n \in \mathcal{D}(A)$ such that $\|x - By_n\| \to 0, n \to \infty$. Put $x_n = By_n$. Since $A - cI \ge 0$ and

$$x_n - x_m = \epsilon(y_n - y_m) + (A - c)(y_n - y_m),$$

we have

$$\epsilon\|y_n - y_m\|^2 \le (x_n - x_m, y_n - y_m) \le \|x_n - x_m\|\|y_n - y_m\|.$$

Thus there exists $y \in H$ such that $y_n \to y$. But $x_n = (\epsilon - c)y_n + Ay_n$, hence Ay_n converges in H. Since A is closed, there exists $y \in \mathcal{D}(A)$ such that $Ay_n \to Ay$. Consequently we have $x = (\epsilon - c)y + Ay \in \mathcal{R}(B)$. This means $\mathcal{R}(B) = H$.

Conversely, if $\mathcal{R}(B) = H$, then by Theorem 1.6(5), B is self-adjoint, thus so is A.

(2) Let A be essentially self-adjoint. Then the closure $\widetilde{A}$ of A is self-adjoint. Put $\widetilde{B} = (\epsilon - c)I + \widetilde{A}$, by (1), $\mathcal{R}(\widetilde{B}) = H$. Note $A - cI \ge 0$, proceed similarly as above, we can prove $\mathcal{R}(\widetilde{B}) = \overline{\mathcal{R}(B)}$. Hence $\mathcal{R}(B)$ is dense in H. On the other hand, if $\mathcal{R}(B)$ is dense in H and $\widetilde{A}$ denotes the closure of A, $\widetilde{B} = (\epsilon - c)I + \widetilde{A}$, then it can be proved that $\mathcal{R}(\widetilde{B}) = \overline{\mathcal{R}(B)}$. Consequently, by (1), $\widetilde{B}$ is self-adjoint, i.e., $\widetilde{A}$ is self-adjoint. By definition, A is essentially self-adjoint. ∎

Corollary 1.10 *Let A be a positive bounded below symmetric operator on H. Then A is essentially self-adjoint iff $\mathcal{R}(A)$ is dense in H (or equivalently, $\mathcal{N}(A^*) = \{0\}$).*

Proof. Let $A \ge \epsilon$, $\epsilon > 0$. Put $A_1 = A - \epsilon I$. Then $A_1 \ge 0$, $A = A_1 + \epsilon I$. The conclusion follows immediately from Theorem 1.9(2). ∎

Example. Let $H = L^2(\mathbb{R}^d)$, $A = -\Delta + I$, $\mathcal{D}(A) = C_0^\infty(\mathbb{R}^d)$ (here Δ is the Laplace operator, $C_0^\infty(\mathbb{R}^d)$ is the set of all C^∞-functions on $\mathbb{R}^d$ with compact supports). Then A is essentially self-adjoint. In fact, A is obviously a symmetric operator having positive lower bound. In order to prove that A is essentially self-adjoint, it suffices to prove $\mathcal{N}(A^*) = \{0\}$. Now let $g \in H$, $A^*g = 0$. Then $Ag = 0$ in the sense of Schwartz distribution, this is because $\forall f \in C_0^\infty(\mathbb{R}^d)$, $\langle Af, g\rangle = (Af, g) = (f, A^*g) = 0$. Here $\langle\cdot,\cdot\rangle$ denotes the canonical bilinear form on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}^*(\mathbb{R}^d)$. Denote by $\mathcal{F}f$ the Fourier transform of f. Then

$$\mathcal{F}(-\Delta + I)g(\xi) = (|\xi|^2 + 1)\mathcal{F}g(\xi).$$

Hence $\mathcal{F}g(\xi) = 0$, i.e., $g = 0$. This means $\mathcal{N}(A^*) = \{0\}$.

Let $\bar{A}$ be the closure of A. Then $\bar{A}$ is self-adjoint and

$$\mathcal{D}(\bar{A}) = \mathcal{H}^2(\mathbb{R}^d) \equiv \left\{ f \in L^2(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |\xi|^2 |\mathcal{F}f(\xi)|^2 d\xi < \infty \right\}.$$

1.3 Self-adjoint extension of a symmetric bounded below operator

A symmetric operator does not necessarily possess a self-adjoint extension. But the following Friedrichs theorem shows that any symmetric bounded below operator has a self-adjoint extension. In order to prove this theorem we need a representation theorem for closed, positive, symmetric sesquilinear forms, which is also due to Friedrichs[1].

Definition 1.11 Let H be a Hilbert space on the number field $\mathbb{K}$, V a dense subspace of H. A function $a(\cdot,\cdot) : V \times V \to \mathbb{K}$ is called a *sesquilinear form* (or *Hermitian form*) on H if

(1) $a(x, y)$ is linear in x, conjugate linear in y;

(2) $a(\cdot,\cdot)$ is symmetric, i.e., $a(x, y) = \overline{a(y, x)}$.

V is called the domain of a, and will be denoted by $\mathcal{D}(a)$. If $a(x, x) \geq 0, \forall x \in \mathcal{D}(a)$, then a is said to be *positive*. If furthermore,

$$x \neq 0 \Longrightarrow a(x, x) > 0, \tag{1.18}$$

then a is said to be *strictly positive*.

Let a be a positive sesquilinear form. Define an inner product on $\mathcal{D}(a)$ by

$$(x, y)_a \equiv a(x, y) + (x, y), \quad x, y \in \mathcal{D}(a). \tag{1.19}$$

Then $\mathcal{D}(a)$ is a space with inner product $(\cdot,\cdot)_a$. If $\mathcal{D}(a)$ is complete with respect to the norm $\|\cdot\|_a$, then a is said to be closed.

Theorem 1.12 *Let a be a positive sesquilinear form on H. Then there exists a unique positive self-adjoint operator A such that $\mathcal{D}(A) \subset \mathcal{D}(a)$, and*

$$(Ax, y) = a(x, y), \quad \forall x \in \mathcal{D}(A),\ y \in \mathcal{D}(a). \tag{1.20}$$

Proof. Put

$$\mathcal{D}(A) = \{x \in \mathcal{D}(a) : \exists c_x > 0, \text{s.t. } |a(x, y)| \leq c_x\|y\|, \ \forall y \in \mathcal{D}(a)\}. \tag{1.21}$$

By Riesz representation theorem, $\forall x \in \mathcal{D}(A)$, there exists a unique element of H, denoted by Ax, such that

$$a(x, y) = (Ax, y), \quad \forall y \in \mathcal{D}(a). \tag{1.22}$$

Obviously $A \in L(H)$ and A is positive.

Let $z \in H$. By (1.19)

$$|(z, y)| \leq \|z\|\|y\| \leq \|z\|\|y\|_a, \quad \forall y \in \mathcal{D}(a). \tag{1.23}$$

Once again by Riesz representation theorem, there exists a unique element in $\mathcal{D}(a)$, denoted by Bz, such that

$$(z, y) = (Bz, y)_a = a(Bz, y) + (Bz, y), \quad \forall y \in \mathcal{D}(a). \tag{1.24}$$

We shall prove that $\mathcal{D}(A) = \mathcal{R}(B)$ and $\mathcal{D}(A)$ is dense in H. By (1.24) and (1.21), $Bz \in \mathcal{D}(A)$, hence $\mathcal{R}(B) \subset \mathcal{D}(A)$. On the other hand, by (1.22), $\forall x \in \mathcal{D}(A)$,

$$(x + Ax, y) = a(x, y) + (x, y), \quad y \in \mathcal{D}(a).$$

Thus from (1.24) we see that $B(x + Ax) = x, \forall x \in \mathcal{D}(A)$. Consequently, we have $\mathcal{D}(A) \subset \mathcal{R}(B)$, and finally, $\mathcal{D}(A) = \mathcal{R}(B)$. But by (1.24), if $y \in \mathcal{D}(a)$ is orthogonal to $\mathcal{R}(B)$, then $y = 0$. Hence $\mathcal{R}(B)$ (i.e.,$\mathcal{D}(A)$) is dense in $\mathcal{D}(a)$ with respect to the norm $\|\cdot\|_a$. But the norm $\|\cdot\|$ of H is weaker than $\|\cdot\|_a$, thus $\mathcal{R}(B)$ is dense in H.

We now prove that A is a self-adjoint operator. By (1.24), $Bz = 0$ implies $(z, y) = 0, \forall y \in \mathcal{D}(a)$. Thus $z = 0$ and B is invertible. Since $\mathcal{D}(B^{-1}) = \mathcal{R}(B)$, B^{-1} is densely defined, and $\mathcal{D}(A) = \mathcal{D}(B^{-1})$. By (1.24) and (1.22),

$$(Ax, y) = a(x, y) = (B^{-1}x, y) - (x, y), \quad \forall x \in \mathcal{D}(A), y \in \mathcal{D}(a), \tag{1.25}$$

from which follows $A = B^{-1} - I$. Since $\mathcal{R}(B^{-1}) = H$ and B^{-1} is symmetric, by Theorem 1.6(5), B^{-1} is self-adjoint, thus A is also self-adjoint. The uniqueness of A verifying (1.20) follows from Lemma 1.1. ∎

With the above preparation, we can now prove

Theorem 1.13 (Friedrichs) *Let $A \in L(H), A \geq c$ be symmetric and bounded below. Then A has a self-adjoint extension $\widehat{A}$. Moreover, $\widehat{A}$ is bounded below and $\widehat{A} \geq c$.*

Proof. First assume $A \geq 1$. Put

$$a(x, y) = (Ax, y), \quad \forall x, y \in \mathcal{D}(A). \tag{1.26}$$

Then a is a strictly positive sesquilinear form on H. It determines an inner product $(\cdot,\cdot)^*$ on $\mathcal{D}(A)$. Denote by $\mathcal{D}(\overline{a})$ the completion of $\mathcal{D}(A)$ with respect to the norm $\|\cdot\|^*$ and denote by $\overline{a}(\cdot,\cdot)$ the continuous extension of $a(\cdot,\cdot)$ onto $\mathcal{D}(\overline{a})\times\mathcal{D}(\overline{a})$. Since the norm $\|\cdot\|^*$ is stronger than the norm $\|\cdot\|$ on H, $\mathcal{D}(\overline{a})$ can be taken to be a closed subspsce of H. Thus $\overline{a}$ is a closed strictly positive sesquilinear form on H. Now by Theorem 1.12, there exists a unique self-adjoint operator $\widetilde{A}$ such that

$$(\widetilde{A}x,y)=\overline{a}(x,y)\,,\quad x\in\mathcal{D}(\widetilde{A}),\ y\in\mathcal{D}(\overline{a}).$$

Let $x\in\mathcal{D}(A)$. Then by (1.26),

$$|\overline{a}(x,y)|=|(Ax,y)|\le\|Ax\|\|y\|\le\|Ax\|\|y\|_a,\quad\forall y\in\mathcal{D}(A).$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{D}(\overline{a})$ with respect to the norm $\|\cdot\|^*$, the above inequality holds true for any $y\in\mathcal{D}(\overline{a})$. Now by the definition of $\widetilde{A}$, $x\in\mathcal{D}(\widetilde{A})$ and $(\widetilde{A}x,y)=\overline{a}(x,y)$, $\forall y\in\mathcal{D}(\overline{a})$. In particular, by (1.26),

$$(\widetilde{A}x,y)=\overline{a}(x,y)=(Ax,y),\quad\forall y\in\mathcal{D}(A).$$

Consequently, $\widetilde{A}x=Ax,\forall x\in\mathcal{D}(A)$. This shows that $\widetilde{A}$ is an extension of A. Obviously $\widetilde{A}\ge 1$.

For general case where $A\ge c$, put $A_1=A+(1-c)I$. Then $A_1\ge 1$. By the above proof, there exists a self-adjoint extension $\widetilde{A}_1$ of A_1 such that $\widetilde{A}_1\ge 1$. Put $\widetilde{A}=\widetilde{A}_1-(1-c)I$. Then $\widetilde{A}\ge c$, and $\widetilde{A}$ is the self-adjoint extension of A. ∎

1.4 Spectral resolution of self-adjoint operators

Definition 1.14 Let H be a Hilbert space on the number field $\mathbb{K}$, A a closed operator on H. Put

$$\rho(A)=\{\lambda\in\mathbb{K}:\mathcal{N}(\lambda I-A)=\{0\},\overline{\mathcal{R}(\lambda I-A)}=H,(\lambda I-A)^{-1}\in\mathcal{L}(H)\}.$$

$\rho(A)$ is called the *resolvent set* of A. The complement of $\rho(A)$ in $\mathbb{K}$ is called the *spectral set* of A, and will be denoted by $\sigma(A)$. Put

$$\sigma_p(A)=\{\lambda\in\mathbb{K}\ :\ \mathcal{N}(\lambda I-A)\ne\{0\}\}.$$

$\sigma_p(A)$ is called the *eigenvalue set* of A. For $\lambda\in\sigma_p(A)$, $\mathcal{N}(\lambda I-A)$ is called the *eigenspace* of A corresponding to the eigenvalue λ. Any $x\in\mathcal{N}(\lambda I-A)$ is called an *eigenvector* of A corresponding to the eigenvalue λ.

The spectral resolution of a self-adjoint operators is usually established for complex Hilbert space case. In that case, by means of Cayley's transform the problem can be converted to that for bounded self-adjoint operators. For self-adjoint opertaors on a real Hilbert space, we may use the method of complexification to reduce the problem to the complex Hilbert space case. Thus there is

a unified statement about spectral resolutions of self-adjoint operators in both complex and real cases. In the following we shall present relevant results, but the proofs will be given only for compact self-adjoint opertaors and their inverses.

Denote by $\mathcal{P}(H)$ the set of all projection operators on H. For $P_1,P_2\in\mathcal{P}(H)$, if $P_1(H)\subset P_2(H)$, then we denote this by $P_1\le P_2$, in this case, $P_2-P_1\in\mathcal{P}(H)$.

Definition 1.15 $\{E_\lambda,\lambda\in\mathbb{R}\}\subset\mathcal{P}(H)$ is called a *resolution of the identity* on H, if it satisfies the following conditions:

(1) monotonicity: $\lambda_1\le\lambda_2\Longrightarrow E_{\lambda_1}\le E_{\lambda_2}$;

(2) right continuous: $E_{\lambda+}\equiv s\text{-}\lim_{s\downarrow\downarrow\lambda}E_s=E_\lambda$;

(3) $E_{-\infty}\equiv s\text{-}\lim_{\lambda\to-\infty}E_\lambda=0, E_\infty\equiv s\text{-}\lim_{\lambda\to+\infty}E_\lambda=I$.

Here s-lim denotes strong limit of operators.

The following result is the spectral resolution theorem for self-adjoint operators.

Theorem 1.16 (von Neumann) *Let A be a self-adjoint operator on H. Then there exists a unique resolution of the identity $\{E_\lambda,\lambda\in\mathbb{R}\}$ on H such that $\forall x,y\in\mathcal{D}(A)$,*

$$(Ax,y)=\int_{\mathbb{R}}\lambda d(E_\lambda x,y)\,. \tag{1.27}$$

Here the integral is of Lebesgue-Stieltjes type. $\{E_\lambda,\lambda\in\mathbb{R}\}$ is called the spectral family of A.

The operator A is usually represented by the spectral integral:

$$A=\int_{\mathbb{R}}\lambda\,dE_\lambda\,. \tag{1.28}$$

We call it the spectral resolution or spectral representation of A. We have

$$\mathcal{D}(A)=\left\{x\in H\ :\ \int_{\mathbb{R}}\lambda^2d(E_\lambda x,x)<\infty\right\}. \tag{1.29}$$

Remark. For any resolution of the identity $\{E_\lambda,\lambda\in\mathbb{R}\}$, we define $\mathcal{D}(A)$ by (1.29). Then $\mathcal{D}(A)$ is dense in H. For a given $x\in\mathcal{D}(A)$, by Riesz' representation theorem, there is a unique element of H, denoted by Ax, such that (1.27) holds for any $y\in\mathcal{D}(A)$. It can be easily verified that the operator A is a self-adjoint operator on H with spectral family $\{E_\lambda,\lambda\in\mathbb{R}\}$.

For a Borel function φ on $\mathbb{R}$, $\varphi(A)$ is defined by the following theorem.

Theorem 1.17 *Let A be a self-adjoint operator on H with spectral family $\{E_\lambda,\lambda\in\mathbb{R}\}$, φ a real Borel function on $\mathbb{R}$. Put*

$$\mathcal{D}(\varphi(A))\equiv\left\{x\in H\ :\ \int_{\mathbb{R}}\varphi(\lambda)^2d(E_\lambda x,x)<\infty\right\}. \tag{1.30}$$

Then $\mathcal{D}(\varphi(A))$ is dense in H, and $\forall x,y\in\mathcal{D}(\varphi(A))$,

$$\int_{\mathbb{R}}|\varphi(\lambda)||d(E_\lambda x,y)|\le\|y\|\left(\int_{\mathbb{R}}\varphi(\lambda)^2d(E_\lambda x,x)\right)^{1/2}.$$

For $x \in \mathcal{D}(\varphi(A))$, let $\varphi(A)x$ be the unique element of H such that

$$(\varphi(A)x, y) = \int_{\mathbb{R}} \varphi(\lambda) d(E_\lambda x, y) \ , \ \forall y \in \mathcal{D}(A) . \tag{1.31}$$

Then $\varphi(A)$ is a self-adjoint operator on H. We express $\varphi(A)$ as the following spectral integral:

$$\varphi(A) = \int_{\mathbb{R}} \varphi(\lambda) dE_\lambda \ . \tag{1.32}$$

The eigenvalue set of A is characterized by its spectral family as follows.

Theorem 1.18 *Let A be a self-adjoint operator on H with spectral family $\{E_\lambda, \lambda \in \mathbb{R}\}$. Then*

$$\sigma_p(A) = \{\lambda \in \mathbb{R} : \ E_\lambda \neq E_{\lambda-}\}. \tag{1.33}$$

Theorem 1.19 *Let A be a self-adjoint bounded below operator on H with spectral family $\{E_\lambda, \lambda \in \mathbb{R}\}$. Put*

$$c = \sup\{\lambda : \ E_\lambda = 0\}.$$

Then $c \in \mathbb{R}$. In this situation, the integration domain in (1.27) and (1.31) can be replaced by the interval $[c, \infty)$. In particular, if A is positive (i.e. $c \geq 0$), then for any $p \in \mathbb{R}$, we can define p-th power of A as:

$$A^p = \int_{[0,\infty)} \lambda^p dE_\lambda . \tag{1.34}$$

A^p is self-adjoint. We call $A^{1/2}$ the square root of A.

The following result is an important complement to Theorem 1.12.

Theorem 1.20 *Let a be a poistive, closed, sesquilinear form on H, and A the assocaited positive, self-adjoint operator determined by (1.20). Then $\mathcal{D}(A^{1/2}) = \mathcal{D}(a)$, and*

$$a(x, y) = (A^{1/2}x, A^{1/2}y), \qquad x, y \in \mathcal{D}(a).$$

Proof. Let $a'(x, y) = (A^{1/2}x, A^{1/2}y), x, y \in \mathcal{D}(A^{1/2})$. Then a' is a positive, closed, sesquilinear form on H. Since $\mathcal{D}(A)$ is obviously dense in $\mathcal{D}(A^{1/2})$, and a' coincides with a on $\mathcal{D}(A)$, we have $a = a'$. ∎

The following theorem gives the polar decomposition of a densely defined closed operator.

Theorem 1.21 *Let A be a densely defined closed operator from H to K. Put $T = (A^*A)^{1/2}$. Then T is a positive self-adjoint operator on H, and $\mathcal{D}(T) = \mathcal{D}(A)$. Moreover, there exists a unique linear isometry U from $\mathcal{R}(T)$ to K such that $A = UT$. We call this the polar decomposition of A, and T (denoted by $|A|$) the absolute value of A.*

Proof. By Theorem 1.7, A^*A is a positive self-adjoint operator on H. Put

$$a(x, y) \equiv (Ax, Ay), \quad x, y \in \mathcal{D}(A).$$

Then a is a positive closed Hermite form on H. By Theorem 1.12 and 1.20, we have $\mathcal{D}(T) = \mathcal{D}(a) = \mathcal{D}(A)$, and

$$\|Ax\|^2 = a(x, x) = \|Tx\|^2, \quad x \in \mathcal{D}(A) = \mathcal{D}(T).$$

Thus $Ax = 0 \Longleftrightarrow Tx = 0$. For $y = Tx \in \mathcal{R}(T)$, put $Uy = Ax$. Then U is unambiguously defined on $\mathcal{R}(T)$, $A = UT$, and $\|Uy\| = \|Ax\| = \|Tx\| = \|y\|$. ∎

Definition 1.22 Let $A \in \mathcal{L}(H, K)$, $\mathcal{D}(A) = H$. If A maps the unit ball (or any bounded subset) of H to a relatively compact subset of K, then A is called a *compact operator* (or *completely continuous operator*).

We shall denote by $\mathcal{K}(H, K)$ the set of all compact operators from H to K. Clearly, compact operators are bounded, and $\mathcal{K}(H, K)$ is a closed subspace of $\mathcal{L}(H, K)$.

The following is the spectral resolution of a compact operator. For reader's convenience we give its proof.

Theorem 1.23 *Let A be a non-null self-adjoint compact operator on H. Then there exists an orthonormal system $\{e_n\}$ of H and a sequence of non-zero numbers $\{\lambda_n\}$ such that $Ae_n = \lambda_n e_n$, and*

$$Ax = \sum_n \lambda_n (x, e_n) e_n \ , \ \forall x \in H \ . \tag{1.35}$$

If A is degenerate (i.e., $\mathcal{R}(A)$ is finite dimensional), the above series contains only finite terms; if A is non-degenerate, then $\lim_{n\to\infty} \lambda_n = 0$.

Proof. Since A is self-adjoint, (Ax, x) is a real number. Put

$$m = \inf_{\|x\|=1} (Ax, x) \ , \quad M = \sup_{\|x\|=1} (Ax, x) \ .$$

By Lemma 1.2, m and M cannot be both zero. Let λ_1 be the maximum of $|m|$ and $|M|$. Then $|\lambda_1| = \sup_{\|x\|=1} |(Ax, x)|$. Choose $x_n \in H$, $\|x_n\| = 1$, such that $\lambda_1 = \lim_n (Ax_n, x_n)$. Since A is compact, $\{Ax_n, n \geq 1\}$ is relatively compact in H. We may assume that $\{Ax_n\}$ converges to y in H (otherwise we can take a subsequence). By Lemma 1.2, $\|A\| = |\lambda_1|$, hence $\|y\| \leq \|A\| = |\lambda_1|$. On the other hand,

$$\begin{aligned}\lim_{n\to\infty} \|Ax_n - \lambda_1 x_n\|^2 &= \lim_{n\to\infty} (\|Ax_n\|^2 - 2\lambda_1 (Ax_n, x_n) + \lambda_1^2) \\ &= \|y\|^2 - \lambda_1^2.\end{aligned}$$

It follows that $\|y\| = \lambda_1$, and $\lim_{n\to\infty} \|Ax_n - \lambda_1 x_n\| = 0$. Put $e_1 = \lambda_1^{-1} y$. Then $\|e_1\| = 1$, $Ae_1 = \lambda_1 e_1$.

Now let $V(e_1)$ be the subspace spanned by e_1, i.e., $V(e_1) = \{\alpha e_1, \alpha \in \mathbb{R}\}$. Put $H_1 = V(e_1)^\perp$. Then

$$(Ay, e_1) = (y, Ae_1) = \lambda_1 (y, e_1).$$

Thus $y \in H_1 \iff Ay \in H_1$. Obviously, when restricted on H_1, A is a self-adjoint compact operator. If A is not a null operator on H_1, we may repeat the above procedure and obtain $e_2 \in H_1$ and non-zero number λ_2, such that $|\lambda_2| = \sup_{x \in H_1, \|x\|=1} |(Ax, x)|$, and $Ae_2 = \lambda_2 e_2$. We proceed continuously in such a manner. If A is degenerate and $\dim R(A) = N$, we can obtain an orthonormal system $\{e_1, e_2, \cdots, e_N\}$ of H and a sequence of real numbers $\{\lambda_1, \cdots, \lambda_N\}$, such atht $Ae_j = \lambda_j e_j, 1 \le j \le N$, $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_N|$, and for $1 \le j \le N$,

$$|\lambda_j| = \sup_{\|x\|=1, x \in H_{j-1}} |(Ax, x)|, \tag{1.36}$$

where $H_0 = H$, H_j is the orthogonal complement of the linear subspace $V(e_1, \cdots, e_j)$ spaned by $e_1, \cdots, e_j$. In this case, (1.35) obviously holds true. If A is non-degenerate, then we obtain an orthonormal system $\{e_n, n \ge 1\}$ and a sequence of real numbers $\{\lambda_n, n \ge 1\}$, such that $\{|\lambda_n|, n \ge 1\}$ is decreasing, $Ae_n = \lambda_n e_n$, and (1.36) holds true for any j. Put $x_n = \lambda_n^{-1} e_n$. Then $e_n = Ax_n$. Since $\{e_n, n \ge 1\}$ is not relatively compact in H, $\{x_n, n \ge 1\}$ cannot be bounded in H. Consequently $\lim_{n \to \infty} \lambda_n = 0$.

It remains to prove (1.35). Let $x \in H$, $y_m = x - \sum_{n=1}^{m} (x, e_n) e_n$. Then $y_m \in H_m$. By (1.36) and Lemma 1.2,

$$\|Ay_m\| \le |\lambda_{m+1}| \, \|y_m\| \le |\lambda_{m+1}| \, \|x\|.$$

Thus $\lim_{m \to \infty} \|Ay_m\| = 0$ and (1.35) follows. ∎

Remark. Let $\{e_n, n \ge 1\}$ be an orthonormal system of H, $\{\lambda_n, n \ge 1\}$ a sequence of non-zero numbers satisfying $\lim_{n \to \infty} \lambda_n = 0$. Define A by (1.35). Then A is self-adjoint and compact. Moreover, A is positive (i.e., $(Ax, x) \ge 0$, $\forall x \in H$) iff $\lambda_n > 0$, $\forall n \ge 1$.

1.5 Hilbert-Schmidt and trace class operators

In this section we shall only study bounded linear operators on a separable Hilbert space. We shall call a complete orthonormal system in H an *orthonormal base* (or simply, a *base*) of H.

Lemma 1.24 *Let $A \in \mathcal{L}(H, K)$, $\{e_n\}$ and $\{f_n\}$ be bases of H and K, respectively. Then*

$$\sum_n \|Ae_n\|^2 = \sum_n \|A^* f_n\|^2. \tag{1.37}$$

In particular, $\sum_n \|Ae_n\|^2$ does not depend on the choice of bases.

Proof. Since

$$Ae_n = \sum_m (Ae_n, f_m) f_m, \quad A^* f_m = \sum_n (A^* f_m, e_n) e_n,$$

it follows that

$$\sum_n \|Ae_n\|^2 = \sum_n \sum_m |(Ae_n, f_m)|^2$$
$$= \sum_n \sum_m |(e_n, A^* f_m)|^2 = \sum_m \|A^* f_m\|^2. \quad ∎$$

The above lemma leads to

Definition 1.25 *Let $A \in \mathcal{L}(H, K)$. If for some base $\{e_n\}$ of H,*

$$\sum_n \|Ae_n\|^2 < \infty,$$

then A is called a Hilbert-Schmidt operator (abbr. H-S operator). Put

$$\|A\|_2 \equiv \Big(\sum_n \|Ae_n\|^2\Big)^{1/2}. \tag{1.38}$$

$\|A\|_2$ is called the Hilbert-Schmidt norm of A. We also denote it by $\|A\|_{HS}$.

We shall denote by $\mathcal{L}_{(2)}(H, K)$ the set of all H-S operators from H to K. For $A, B \in \mathcal{L}_{(2)}(H, K)$, put

$$(A, B)_2 \equiv \sum_n (Ae_n, Be_n), \tag{1.39}$$

here $\{e_n\}$ is a base of H. Since $(A, B)_2 = \frac{1}{4}(\|A+B\|_2^2 - \|A-B\|_2^2)$, $(A, B)_2$ does not depend on the choice of $\{e_n\}$, and $(\cdot, \cdot)_2$ is an inner product on $\mathcal{L}_{(2)}(H, K)$.

Theorem 1.26 *$\mathcal{L}_{(2)}(H, K)$ is a separable Hilbert space under the inner product $(\cdot, \cdot)_2$.*

Proof. First we choose arbitrarily bases $\{e_n\}$ of H and $\{f_n\}$ of K. For $A \in \mathcal{L}_{(2)}(H, K)$, put $a_{n,k}(A) = (Ae_n, f_k)$. Then

$$\|A\|_2^2 = \sum_n \|Ae_n\|^2 = \sum_n \sum_k |a_{n,k}(A)|^2.$$

Put $M = \{(a_{n,k})_{n,k \ge 1} : a_{n,k} \in \mathbb{K}, \sum_{n,k=1}^{\infty} |a_{n,k}|^2 < \infty\}$. Then M is a separable Hilbert space under the following inner product:

$$((a_{n,k}), (b_{n,k})) = \sum_{n,k} a_{n,k} b_{n,k}.$$

But $A \mapsto (a_{n,k}(A))_{n,k \ge 1}$ is a linear isometry from $\mathcal{L}_{(2)}(H, K)$ onto M, thus $\mathcal{L}_{(2)}(H, K)$ is also a separable Hilbert space. ∎

Theorem 1.27 *H-S opeators are compact. Moreover, for any $A \in \mathcal{L}_{(2)}(H, K)$, we have*

$$\|A\| \le \|A\|_2. \tag{1.40}$$

Proof. We first prove (1.40). Let $\{f_n\}$ be a base of K. By (1.37) and (1.38),

$$\begin{aligned}\|Ax\|^2 &= \sum_n |(Ax, f_n)|^2 = \sum_n |(x, A^* f_n)|^2 \\ &\leq \|x\|^2 \sum_n \|A^* f_n\|^2 = \|x\|^2 \|A\|_2^2 .\end{aligned}$$

Hence (1.40) follows. Now let $\{e_n\}$ be a base of H. For any $k \geq 1$, put

$$A_k x = \sum_{n=1}^{k} (x, e_n) A e_n , \quad x \in H . \tag{1.41}$$

Then A_k is non-degenerate, and hence compact. But

$$\begin{aligned}\|A - A_k\|^2 &\leq \|A - A_k\|_2^2 = \sum_n \|(A - A_k)e_n\|^2 \\ &= \sum_{n \geq k+1} \|Ae_n\|^2 ,\end{aligned}$$

by the convergence of series $\sum_n \|Ae_n\|^2$, we know $\lim_{k\to\infty} \|A - A_k\| = 0$. Hence A is compact (since the space of compact operators is closed in $\mathcal{L}(H, K)$). ∎

Remark. From the above proof, we know that the set of degenerate operators is dense in $\mathcal{L}_{(2)}(H, K)$.

Definition 1.28 Let B be a positive self-adjoint operator on H. Put $\mathrm{Tr}B \equiv \|B^{1/2}\|_2^2$. $\mathrm{Tr}B$ is called the *trace of* B.

By Lemma 1.24, for any base $\{e_n\}$ of H, we have

$$\mathrm{Tr}\, B = \sum_n \|B^{1/2} e_n\|^2 = \sum_n (Be_n, e_n) . \tag{1.42}$$

The following theorem gives a characterization of H-S operators.

Theorem 1.29 *Let A be a compact operator from H to K. Then $A \in \mathcal{L}_{(2)}(H, K) \iff \mathrm{Tr}(A^*A) < \infty$. In this case, we have*

$$\|A\|_2^2 = \mathrm{Tr}(A^*A) . \tag{1.43}$$

Proof. Let $A = UT$ be the polar decomposition of A (Theorem 1.21), where $T = (A^*A)^{1/2}$. Let $\{e_n\}$ be a base of H. Then

$$\sum_n \|Ae_n\|^2 = \sum_n \|Te_n\|^2 = \sum_n (A^*Ae_n, e_n) , \tag{1.44}$$

and the conclusion follows. ∎

Remark. Let $A \in \mathcal{K}(H, K)$, $A = UT$ be the polar decomposition of A, and

$$Tx = \sum_n \lambda_n (x, e_n) e_n , \quad x \in H \tag{1.45}$$

be the spectral resolution of T (see Theorem 1.23). Then an equivalent formulation of Theorem 1.29 is

$$A \in \mathcal{L}_{(2)}(H, K) \iff \sum_n \lambda_n^2 < \infty ,$$

and

$$\|A\|_2^2 = \sum_n \lambda_n^2 . \tag{1.46}$$

Definition 1.30 Let $A \in \mathcal{K}(H, K)$, $A = UT$ be the polar decomposition of A, (1.45) be the spectral resolution of T. If $\sum_n \lambda_n < \infty$, then A is called a *trace class operator* (or *nuclear operator*). Put

$$\|A\|_1 = \sum_n \lambda_n . \tag{1.47}$$

$\|A\|_1$ is called the *trace norm* of A. We shall denote by $\mathcal{L}_{(1)}(H, K)$ the set of all trace class operators from H to K.

By (1.46), trace class operators are H-S operators, hence compact. Furthermore, let $A \in \mathcal{K}(H, K)$. Then

$$\begin{aligned}A \in \mathcal{L}_{(1)}(H, K) &\iff \mathrm{Tr}[(A^*A)^{1/2}] < \infty \\ &\iff (A^*A)^{1/4} \in \mathcal{L}_{(2)}(H, K),\end{aligned}$$

and

$$\|A\|_1 = \mathrm{Tr}[(A^*A)^{1/2}] = \|(A^*A)^{1/4}\|_2^2 . \tag{1.48}$$

The following theorem gives an expression of the trace norm.

Theorem 1.31 *Let $A \in \mathcal{L}_{(1)}(H, K)$. Then*

$$\|A\|_1 = \sup \sum_n |(Af_n, g_n)| , \tag{1.49}$$

where the supremum is taken over all bases $\{f_n\}$ of H and $\{g_n\}$ of K. Moreover, $\mathcal{L}_{(1)}(H, K)$ is a separable Banach space under the norm $\|\cdot\|_1$.

Proof. Let $A = UT$ be the polar decomposition of A and (1.45) a spectral resolution of T. Then $\forall x \in H$,

$$Ax = UTx = \sum_n \lambda_n (x, e_n) Ue_n . \tag{1.50}$$

Enlarge $\{e_n\}$ to a base $\{f'_n\}$ of H and $\{Ue_n\}$ to a base $\{g'_n\}$ of K, and keep the correspondence between e_n and Ue_n. Then by (1.50),

$$\sum_n |(Af'_n, g'_n)| = \sum_n \lambda_n = \|A\|_1 . \tag{1.51}$$

On the other hand, for any base $\{f_n\}$ of H and base $\{g_n\}$ of K, by (1.50) we have

$$\begin{aligned}\sum_n |(Af_n, g_n)| &= \sum_n \Big|\sum_m \lambda_m (f_n, e_m)(Ue_m, g_n)\Big| \\ &\le \sum_m \lambda_m \sum_n |(f_n,e_m)(Ue_m,g_n)| \\ &\le \frac{1}{2} \sum_m \lambda_m \sum_n (|(f_n,e_m)|^2 + |(Ue_m,g_n)|^2) \\ &= \sum_m \lambda_m = \|A\|_1 .\end{aligned} \tag{1.52}$$

Thus (1.49) follows from (1.51) and (1.52). ∎

Theorem 1.32 *Let $A \in \mathcal{L}_{(1)}(H)$. Then for any base $\{f_n\}$ of H,*

$$\sum_n |(Af_n, f_n)| \le \|A\|_1 \tag{1.53}$$

Furthermore, $\sum_n (Af_n,f_n)$ does not depend on the choice of $\{f_n\}$. Put

$$\mathrm{Tr} A = \sum_n (Af_n,f_n). \tag{1.54}$$

$\mathrm{Tr}A$ is called the trace of A (cf. Definition 1.28).

Proof. (1.53) follows immediately from (1.49). Let $A=UT$ be the polar decomposition of A and (1.45) the spectral resolution of T. Then $\forall x\in H$,

$$Ax = UTx = \sum_n \lambda_n (x,e_n)Ue_n = \sum_n (x,e_n)Ae_n . \tag{1.55}$$

By (1.52), the series $\sum_n\sum_m (f_n,e_m)(Ae_m,f_n)$ converges absolutely, and thus the summations can be interchanged. Hence by (1.55),

$$\begin{aligned}\sum_n (Af_n,f_n) &= \sum_n\sum_m (f_n,e_m)(Ae_m,f_n) \\ &= \sum_m\sum_n (f_n,e_m)(Ae_m,f_n) \\ &= \sum_m \Big(Ae_m, \sum_n (e_m,f_n)f_n\Big) = \sum_m (Ae_m,e_m) .\end{aligned}$$

This means that $\sum_n(Af_n,f_n)$ does not depends on the choice of base $\{f_n\}$. ∎

The proof of the following theorem is left to the reader as an exercise.

Theorem 1.33 *Let $B\in\mathcal{L}(H,K)$, $A\in\mathcal{L}(K,E)$. Then*

$$\begin{aligned}&\|AB\|_2 \le \|A\|\|B\|_2 , && \|AB\|_2\le\|A\|_2\|B\| , \\ &\|AB\|_1 \le \|A\|\|B\|_1 , && \|AB\|_1 \le \|A\|_1\|B\| , \\ &\|AB\|_1 \le \|A\|_2\|B\|_2 .\end{aligned}$$

To conclude this section, we mention the following important result (for a proof, see Meyer [3]).

Theorem 1.34 *$\mathcal{L}_{(1)}(H,K)$ is the topological dual of $\mathcal{K}(H,K)$, and $\mathcal{L}(H,K)$ is the topological dual of $\mathcal{L}_{(1)}(H,K)$. The canonical bilinear forms are*

$$(B,A) = \sum_n (Bf_n, \overline{Af_n}) , \quad A\in\mathcal{L}_{(1)}(H,K) , \ B\in\mathcal{K}(H,K) ;$$

$$(A,B) \equiv \sum_n (Af_n, \overline{Bf_n}) , \quad A\in\mathcal{L}_{(1)}(H,K) , \ B\in\mathcal{L}(H,K) .$$

respectively. Here $\{f_n\}$ is any base of H.

§2. Fock spaces and second quantization

In this paragraph we assume that all Hilbert spaces are *separable* Hilbert spaces on the field $\mathbb{K}$ (real field $\mathbb{R}$ or complex field $\mathbb{C}$); their norms are denoted by $\|\cdot\|$ (with or without subscripts); the orthonormal base is called base (or ONB) for short.

2.1 Tensor products of Hilbert spaces

Let H_1 and H_2 be Hilbert spaces with inner products $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ respectively. For $\varphi_1\in H_1$ and $\varphi_2\in H_2$, we define their tensor product as a conjugate bilinear form on $H_1\times H_2$:

$$\varphi_1\otimes\varphi_2(\xi_1,\xi_2) = (\varphi_1,\xi_1)_1(\varphi_2,\xi_2)_2 , \quad \xi_1\in H_1 , \ \xi_2\in H_2 . \tag{2.1}$$

Denote by $\mathcal{E}$ the linear span of $\{\varphi_1\otimes\varphi_2 : \varphi_1\in H_1, \varphi_2\in H_2\}$. For $\varphi_1\otimes\varphi_2, \psi_1\otimes\psi_2\in\mathcal{E}$, we define

$$b(\varphi_1\otimes\varphi_2,\psi_1\otimes\psi_2) \equiv (\varphi_1,\psi_1)_1(\varphi_2,\psi_2)_2 \tag{2.2}$$

and linearly extend it to $\mathcal{E}$.

Proposition 2.1 *Eq. (2.2) defines a strictly positive Hermitian form on $\mathcal{E}\times\mathcal{E}$, hence $(\mathcal{E}, b)$ is an inner product space.*

Proof. Firstly we should prove that b has a definite extension on $\mathcal{E}$. If $F\in\mathcal{E}$ has two different expressions:

$$F=\sum_{j=1}^n (\varphi_{1j}\otimes\varphi_{2j}) = \sum_{k=1}^m (\varphi'_{1k}\otimes\varphi'_{2k}),$$

then by eq. (2.1), $\forall\xi_1\in H_1, \xi_2\in H_2$, we have

$$\begin{aligned}F(\xi_1,\xi_2) &= \sum_{j=1}^n(\varphi_{1j},\xi_1)_1(\varphi_{2j},\xi_2)_2 \\ &= \sum_{k=1}^m (\varphi'_{1k},\xi_1)_1(\varphi'_{2k},\xi_2)_2 .\end{aligned}$$

Hence by eq. (2.2), $\forall \psi_1 \in H_1, \psi_2 \in H_2$, we have

$$\begin{aligned} b(\textstyle\sum_{j=1}^n(\varphi_{1j}\otimes\varphi_{2j}), \psi_1\otimes\psi_2) &= \sum_{j=1}^n(\varphi_{1j},\psi_1)_1(\varphi_{2j},\psi_2)_2 \\ &= \sum_{k=1}^m(\varphi'_{1k},\psi_1)_1(\varphi'_{2k},\psi_2)_2 \\ &= b(\textstyle\sum_{k=1}^m(\varphi'_{1k}\otimes\varphi'_{2k}),\psi_1\otimes\psi_2), \end{aligned}$$

which means that the extension of b is independent of expressions of elements in $\mathcal{E}$.

From eq. (2.2) we know that b is an Hermitian form. Now we prove its strict positivity.

Let $F = \sum_{j=1}^n(\varphi_{1j}\otimes\varphi_{2j}) \neq 0$, $(e_1,\cdots,e_n)$ be a base of subspace generated by $(\varphi_{11},\cdots,\varphi_{1n})$. Then, there exist $f_1,\cdots,f_n \in H_2$, not all zero, such that $F = \sum_{j=1}^n(e_j\otimes f_j)$. Therefore

$$\begin{aligned} b(F,F) &= \sum_{j,k=1}^n b(e_j\otimes f_j, e_k\otimes f_k) \\ &= \sum_{j,k=1}^n(e_j,e_k)_1(f_j,f_k)_2 \\ &= \sum_{j=1}^n\|f_j\|_2^2 > 0. \end{aligned}$$ ∎

Definition 2.2 The Hilbert space obtained by completion of $(\mathcal{E},b)$ is called the *Hilbertian tensor product (tensor product* for short) of H_1 and H_2 and denoted by $H_1\otimes H_2$.

Proposition 2.3 *If $\{e_j\}$ and $\{f_k\}$ are bases of Hilbert spaces H_1 and H_2 respectively, then $\{e_j\otimes f_k\}_{j,k\in\mathbb{N}}$ constitute a base of Hilbert space $H_1\otimes H_2$.*

Proof. Orthogonality follows from eq. (2.2). To prove the completeness, it suffices to prove that $\mathcal{E}$ is contained in the closed subspace S generated by $\{e_j\otimes f_k\}_{j,k\in\mathbb{N}}$.

For any $\varphi_1\otimes\varphi_2 \in \mathcal{E}$, let $\varphi_1 = \sum_j c_je_j$, $\varphi_2 = \sum_k d_kf_k$ whose coefficients satisfy that $\sum_j|c_j|^2<\infty$, $\sum_k|d_k|^2<\infty$. Then

$$\sum_{j,k}c_jd_k(e_j\otimes f_k)\in S.$$

By a direct computation we know that

$$\lim_{\substack{m\to\infty\\ n\to\infty}}\left\|\varphi_1\otimes\varphi_2-\sum_{j=1}^n\sum_{k=1}^m c_jd_k(e_j\otimes f_k)\right\| = 0.$$ ∎

The most common Hilbert spaces are those spaces of square integrable functions. Their tensor products have very intuitive and natural meaning. Let (X,μ) be a measure space, $L^2(X,\mu)$ be the Hilbert space of square integrable functions (equivalence classes) on X with respect to measure μ, equipped with inner product

$$(f,g) = \int_X f(x)\overline{g(x)}\mu(dx). \tag{2.3}$$

Let H be any separable Hilbert space. We denote by $L^2(X,\mu;H)$ the Hilbert space of H-valued square integrable functions (μ-equivalence classes) on X, equipped with inner product

$$(f,g) = \int_X (f(x),g(x))\mu(dx). \tag{2.4}$$

Theorem 2.4 *Let (X,μ) and (Y,ν) be measure spaces such that $L^2(X,\mu)$ and $L^2(Y,\nu)$ are separable. Then*

1° *there exists a unique isometric isomorphism*

$$L^2(X,\mu)\otimes L^2(Y,\nu)\cong L^2(X\times Y,\mu\times\nu),$$

which maps $f\otimes g$ to $f(x)g(y)$;

2° *for any separable Hilbert space H, there exists a unique isometric isomorphism*

$$L^2(X,\mu)\otimes H\cong L^2(X,\mu;H),$$

which maps $f\otimes h$ to $f(x)\cdot h$.

Proof. 1° Let $\{e_j(x)\}$ and $\{f_k(y)\}$ be bases of $L^2(X,\mu)$ and $L^2(Y,\nu)$ respectively. It is easy to see that family $\{e_j(x)f_k(y)\}_{j,k\in\mathbb{N}}$ constitutes a base of $L^2(X\times Y,\mu\times\nu)$, hence the map

$$U: f\otimes g\longrightarrow f(x)g(y)$$

extends uniquely to a unitary operator from $L^2(X,\mu)\otimes L^2(Y,\nu)$ onto $L^2(X\times Y,\mu\times\nu)$.

2° Let $\{e_j\}$ be a base of H. For every $g\in L^2(X,\mu;H)$, by a direct computation we obtain

$$\lim_{n\to\infty}\left\|g(x)-\sum_{j=1}^n(g(x),e_j)e_j\right\| = 0.$$

So the linear span of $\{g_j(x)\cdot e_j : g_j\in L^2(X,\mu), j\in\mathbb{N}\}$ is dense in $L^2(X,\mu;H)$. The map

$$U:\quad \sum_{j=1}^n(g_j\otimes e_j)\longmapsto\sum_{j=1}^n g_j(x)\cdot e_j$$

being defined on a dense subspace of $L^2(X,\mu)\otimes H$ and preserves the inner products, it extends uniquely to a unitary operator from $L^2(X,\mu)\otimes H$ onto $L^2(X,\mu;H)$. ∎

It follows from eq. (2.2) that

$$\|\varphi_1\otimes\varphi_2\| = \|\varphi_1\|_1\|\varphi_2\|_2, \tag{2.5}$$

therefore, the bilinear map $(\varphi_1,\varphi_2)\mapsto\varphi_1\otimes\varphi_2$ is continuous from $H_1\times H_2$ to $H_1\otimes H_2$. If $\{e_j\}$ and $\{f_k\}$ are bases of H_1 and H_2 respectively, letting $\xi_1=e_j,\xi_2=f_k$ in eq. (2.1), taking square of norm on each side and summing up for $j,k=1,2,\cdots$, we obtain

$$\begin{aligned}\sum_{j,k=1}^{\infty}|\varphi_1\otimes\varphi_2(e_j,f_k)|^2 &= \|\varphi_1\|_1^2\|\varphi_2\|_2^2\\ &= \|\varphi_1\otimes\varphi_2\|^2.\end{aligned} \tag{2.6}$$

Therefore, the left hand side of eq. (2.6) can be taken as definition of norm in $H_1\otimes H_2$ which is independent of choices of bases and is called *Hilbert-Schmidt norm*.

By induction we can define tensor products of any finite number of Hilbert spaces. However, in view of the above mentioned fact, we can give a direct definition.

Definition 2.5 Let $\{e_k^j\}_{k\in\mathbb{N}}$ be a base of Hilbert space H_j $(1\le j\le n)$. For any (conjugate) n-linear form F on the product space $\prod_{j=1}^n H_j$, its Hilbert-Schmidt norm is defined to be

$$\|F\|_{\mathrm{HS}}^2 \equiv \sum_{(k_1,\cdots,k_n)\in\mathbb{N}^n}|F(e_{k_1}^1,\cdots,e_{k_n}^n)|^2, \tag{2.7}$$

which is independent of choices of bases. The totality of (conjugate) n-linear forms with finite HS norms constitute the space of *Hilbertian tensor product* $H_1\otimes H_2\otimes\cdots\otimes H_n$, simply denoted by $\otimes_{j=1}^n H_j$.

Extending eq. (2.1) to tensor product of n elements, we know that $\{\otimes_{j=1}^n e_{k_j}^j : (k_1,\cdots,k_n)\in\mathbb{N}^n\}$ constitute a base of $\otimes_{j=1}^n H_j$. Especially, if $H_1=H_2=\cdots=H_n=H$, the n-fold tensor product is denoted by $H^{\otimes n}$. If $\{e_k\}_{k\in\mathbb{N}}$ is a base of H, then $\{\otimes_{j=1}^n e_{k_j} : (k_1,\cdots,k_n)\in\mathbb{N}^n\}$ is a base of $H^{\otimes n}$. However, in mathematical physics, one usually deals with its two subspaces: the subspaces of symmetric tensor products and antisymmetric tensor products. Here we restrict ourselves to symmetric ones.

Let $\mathfrak{S}$ be the permutation group on $\{1,2,\cdots,n\}$. For $\sigma\in\mathfrak{S}$, define

$$\pi_\sigma(\varphi_1\otimes\cdots\otimes\varphi_n) = \varphi_{\sigma(1)}\otimes\cdots\otimes\varphi_{\sigma(n)}.$$

Then π_σ extends to an automorphism of $H^{\otimes n}$ such that for $\sigma,\tau\in\mathfrak{S}$, $\pi_{\sigma\tau}=\pi_\sigma\pi_\tau$. Therefore, the endomorphism

$$\pi_n \equiv \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}}\pi_\sigma \tag{2.8}$$

is an orthogonal projection on $H^{\otimes n}$.

Definition 2.6 The range of the orthogonal projection π_n given by (2.8) (which is a closed subspace of $H^{\otimes n}$) is called the space of *n-fold symmetric tensor products* and is denoted by $H^{\widehat{\otimes} n}$. For $\varphi_1,\cdots,\varphi_n\in H$, the projection of $\otimes_{j=1}^n\varphi_j$:

$$\begin{aligned}\widehat{\otimes}_{j=1}^n\varphi_j &= \pi_n(\otimes_{j=1}^n\varphi_j)\\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}}\otimes_{j=1}^n\varphi_{\sigma(j)},\end{aligned} \tag{2.9}$$

is called the *symmetric tensor product* of $\varphi_1,\cdots,\varphi_n$. We make convention that for $F,G\in H^{\widehat{\otimes} n}$,

$$(F,G)_{H^{\widehat{\otimes} n}} \equiv n!(F,G)_{H^{\otimes n}}. \tag{2.10}$$

However, for the sake of convenience, we shall transfer that into inner products in $H^{\otimes n}$ and omit subscripts $H^{\widehat{\otimes} n}$ as well as $H^{\otimes n}$ of those inner products.

If $H=L^2(X,\mu)$, (X,μ) being a measure space, then by Theorem 2.4, $H^{\otimes n}\cong L^2(X^n,\mu^n)$, where (X^n,μ^n) stands for n-fold product measure space of (X,μ). Moreover,

$$H^{\widehat{\otimes} n}\cong \widehat{L^2}(X^n,n!\mu^n),$$

which means that $H^{\widehat{\otimes} n}$ is isometrically isomorphic to the Hilbert space consists of all symmetric functions (equivalence classes) with n variables in X which are square integrable with respect to measure $n!\mu^n$.

Remark. Let H be any linear space, F be a symmetric n-linear form on $\overbrace{H\times\cdots\times H}^{n}$. If we define

$$A(\varphi)\equiv F(\varphi,\cdots,\varphi), \qquad \varphi\in H,$$

then by a simple computation we obtain that

$$F(\varphi_1,\cdots,\varphi_n) = \frac{1}{2^n n!}\sum_{\substack{\epsilon_j^2=1\\ 1\le j\le n}}\epsilon_1\cdots\epsilon_n A\left(\textstyle\sum_{j=1}^n\epsilon_j\varphi_j\right), \tag{2.11}$$

where $\sum$ is taken over all 2^n possible cases for $\epsilon_j=\pm1$ $(j=1,\cdots,n)$. This is the so-called *polarization identity*. As a consequence, we have

$$\widehat{\otimes}_{j=1}^n\varphi_j = \frac{1}{2^n n!}\sum_{\substack{\epsilon_j^2=1\\ 1\le j\le n}}\epsilon_1\cdots\epsilon_n\left(\textstyle\sum_{j=1}^n\epsilon_j\varphi_j\right)^{\otimes n}, \qquad \varphi_1,\cdots,\varphi_n\in H. \tag{2.12}$$

It follows that $H^{\widehat{\otimes} n}$ is exactly the closed subspace generated by $\{h^{\otimes n} : h\in H\}$.

Let Λ be the totality of sequences $\alpha=\{\alpha_j\}_{j\in\mathbb{N}}$ of non-negative integers of which all but finite terms vanish. For $\alpha\in\Lambda$, denote $|\alpha|\equiv\sum_j\alpha_j$, $\alpha!=\prod_j(\alpha_j!)$

(note that they are finite sum and finite product). If for $n \in \mathbb{N}$, denote $\Lambda_n \equiv \{\alpha \in \Lambda : |\alpha| = n\}$, then $\Lambda = \sum_n \Lambda_n$.

Proposition 2.7 *Let $\{e_j\}_{j\in\mathbb{N}}$ be a base of Hilbert space H. For $\alpha \in \Lambda_n$, define*

$$\widehat{e}_\alpha = \pi_n\Big(\bigotimes_j e_j^{\otimes \alpha_j}\Big) \tag{2.13}$$

(where tensor product contains n factors in which e_j appears α_j times for $j = 1, 2, \cdots$). Then $\{(\alpha!)^{-1/2}\widehat{e}_\alpha : \alpha \in \Lambda_n\}$ constitute a base of $H^{\widehat{\otimes} n}$.

Proof. For $(k_1, \cdots, k_n) \in \mathbb{N}^n$, let β_j be the number of those nature numbers in $(k_1, \cdots, k_n)$ which are equal to j, namely

$$\beta_j = \#\{i : 1 \le i \le n, k_i = j\}, \quad j = 1, 2, \cdots.$$

Then we obtain a sequence $\beta = \{\beta_j\} \in \Lambda_n$. Denote this map by $\pi : \mathbb{N}^n \to \Lambda_n$. It is obvious that π is a surjection and for $(k_1, \cdots, k_n) \in \pi^{-1}(\alpha)$ we have

$$\pi_n(\otimes_{j=1}^n e_{k_j}) = \widehat{e}_\alpha.$$

If $\beta \in \Lambda_n, (k'_1, \cdots, k'_n) \in \pi^{-1}(\beta)$, then

$$\begin{aligned}(\widehat{e}_\alpha, \widehat{e}_\beta) &= \big(\pi_n(\otimes_{j=1}^n e_{k_j}), \pi_n(\otimes_{j=1}^n e_{k'_j})\big)\\ &= \big(\otimes_{j=1}^n e_{k_j}, \pi_n(\otimes_{j=1}^n e_{k'_j})\big)\\ &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}}\prod_{j=1}^n (e_{k_j}, e_{k'_{\sigma(j)}}).\end{aligned}$$

When $\alpha = \beta$, the above expression is equal to $\alpha!/(n!)$; when $\alpha \ne \beta$, it is equal to 0, hence $\{(\alpha!)^{-1/2}\widehat{e}_\alpha : \alpha \in \Lambda_n\}$ constitute a base of $H^{\widehat{\otimes} n}$. ∎

2.2 Fock spaces

In quantum physics, $H^{\otimes n}$ represents a system of n particles of same kind. Since particles of same kind are indistinguishable, they obey either Bose-Einstein or Fermi-Dirac statistics according to their spins, so we usually consider its n-fold symmetric or antisymmetric tensor product subspaces. However, particles in a system may be created or annihilated, hence their total number may change. The Fock spaces are suitable models for such systems. One of the most important facts in infinite dimensional stochastic analysis is that the space of square integrable functionals on infinite dimensional Gaussian probability space is isometrically isomorphic to the symmetric Fock space, that is, the Wiener-Itô-Segal chaos decomposition (cf. §1 of Chapter II and §1 of Chapter IV).

Firstly we introduce the notion of infinite direct sum of Hilbert spaces.

Let $(H_n, (\cdot,\cdot)_n)$, $n = 1, 2, \cdots$, be a sequence of Hilbert spaces, H be a linear subspace of product space $\prod_n H_n$ consists of sequences $\varphi = \{\varphi_n\}$ satisfying that

$\sum_n \|\varphi_n\|_n^2 < \infty$. For $\varphi = \{\varphi_n\}, \psi = \{\psi_n\} \in H$, define

$$b(\varphi, \psi) = \sum_{n=1}^\infty (\varphi_n, \psi_n)_n. \tag{2.14}$$

By Schwarz inequality

$$|(\varphi_n, \psi_n)_n| \le \|\varphi_n\|_n \|\psi_n\|_n \le \frac{1}{2}(\|\varphi_n\|_n^2 + \|\psi_n\|_n^2)$$

we know that (2.14) are well defined for all $\varphi, \psi \in H$.

Proposition 2.8 *The functional b defined by (2.14) is a strictly positive Hermitian form on $H \times H$ and (H, b) is a Hilbert space which is called the Hilbertian direct sum of $\{H_n\}_{n\in\mathbb{N}}$. Moreover, the space of algebraic direct sum is dense in H.*

Proof. Obviously b is an Hermitian form. If $\varphi \ne 0$, then

$$b(\varphi, \varphi) = \sum_{n=1}^\infty \|\varphi_n\|_n^2 > 0,$$

hence b is strictly positive. Since the space of algebraic direct sum consists of sequences $\varphi = \{\varphi_n\}$ which have only finite number of non-zero terms, it is obviously dense in H.

Let $\varphi^{(m)} = \{\varphi_n^{(m)}\}_{n\in\mathbb{N}}$, $m = 1, 2, \cdots$, be Cauchy sequences in H with respect to norm $\|\varphi\| = \sqrt{b(\varphi, \varphi)}$. Then $\forall n$, $\{\varphi_n^{(m)}\}_{m\in\mathbb{N}}$ is Cauchy sequence in H_n hence converges in H_n to some $\varphi_n^{(0)}$. It is easy to see that $\varphi^{(0)} \equiv \{\varphi_n^{(0)}\} \in H$ and that $\|\varphi^{(m)} - \varphi^{(0)}\| \longrightarrow 0$. therefore, H is complete. ∎

Hereafter we denote this infinite direct sum by

$$H = \bigoplus_{n=1}^\infty H_n. \tag{2.15}$$

The readers should not confuse it with algebraic direct sum of $\{H_n\}$.

Definition 2.9 Let H be a Hilbert space. Then

$$\mathcal{F}(H) \equiv \bigoplus_{n=0}^\infty H^{\otimes n} \tag{2.16}$$

(by convention $H^{\otimes 0} = \mathbb{K}$) is called the *Fock space* (or *complete Fock space*, *free Fock space*) over H, while

$$\Gamma(H) \equiv \bigoplus_{n=0}^\infty H^{\widehat{\otimes} n} \tag{2.17}$$

is called the *symmetric Fock space* or *Boson Fock space* over H.

If $\{e_k\}_{k\in\mathbb{N}}$ is a base of H, then sequences of the form

$$(0, 0, \cdots, \otimes_{j=1}^n e_{k_j}, 0, \cdots) \tag{2.18}$$

(where $(k_1,\cdots,k_n)\in \mathbb{N}^n$, $\otimes_{j=1}^n e_{k_j}$ is located at $(n+1)$-th term, $\forall n\in \mathbb{N}$) together with $(1,0,0,\cdots)$ constitute a base of $\mathcal{F}(H)$, while sequences of the form

$$(0,0,\cdots,(\alpha!)^{-1/2}\widetilde{e}_{\alpha^{(n)}},0,\cdots) \tag{2.19}$$

(where $\alpha^{(n)}\in\Lambda_n$, $\widetilde{e}_{\alpha^{(n)}}$ is located at $(n+1)$-th term, $\forall n\in\mathbb{N}$) together with $(1,0,0,\cdots)$ constitute a base of $\Gamma(H)$.

2.3 Second quantization of operators

The second quantization is meant a construction of operators on Fock spaces $\mathcal{F}(H)$ or $\Gamma(H)$ by virtue of operators on any Hilbert space H. The first step of this construction is to define tensor products of operators.

Let $H_i, K_i (i=1,2)$ be Hilbert spaces, A_i be a densely defined linear operator from H_i to K_i, $\mathcal{D}(A_i)\subset H_i$ be its domain. Denote by $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ the linear span of $\{\varphi_1\otimes\varphi_2 : \varphi_i\in\mathcal{D}(A_i), i=1,2\}$. Then $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ is dense in $H_1\otimes H_2$. Define

$$A_1\otimes A_2(\varphi_1\otimes\varphi_2)=A_1\varphi_1\otimes A_2\varphi_2,\quad \varphi_i\in\mathcal{D}(A_i),\ i=1,2, \tag{2.20}$$

and extend it to be a linear operator on $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$. It is easy to see that the extension is independent of expressions of elements in $\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$ (see the proof of Proposition 2.1). Define

$$A_1\dotplus A_2\equiv A_1\otimes I+I\otimes A_2. \tag{2.21}$$

Proposition 2.10 *The operators $A_1\otimes A_2$ and $A_1\dotplus A_2$ defined by (2.20) and (2.21) respectively are densely defined linear operators from $H_1\otimes H_2$ to $K_1\otimes K_2$. If A_1 and A_2 are closable, then so are the operators $A_1\otimes A_2$ and $A_1\dotplus A_2$.*

Proof. It suffices to prove the closability of $A_1\otimes A_2$. If A_i is closable, then, (by identifying K_i^* with K_i and H_i^* with H_i), its adjoint A_i^* is a densely defined linear operator from K_i to H_i (cf. Theorem 1.5), and $\forall F\in\mathcal{D}(A_1)\otimes\mathcal{D}(A_2)$, $G\in\mathcal{D}(A_1^*)\otimes\mathcal{D}(A_2^*)$ we have

$$((A_1\otimes A_2)F,\ G)=(F,\ (A_1^*\otimes A_2^*)G). \tag{2.22}$$

It follows that $\mathcal{D}(A_1^*)\otimes\mathcal{D}(A_2^*)\subset\mathcal{D}((A_1\otimes A_2)^*)$. Therefore, $(A_1\otimes A_2)^*$ is densely defined, or equivalently, $A_1\otimes A_2$ is closable. ∎

From Theorem 1.5 we know that $(A_1^*\otimes A_2^*)^*$ is the closure of $A_1\otimes A_2$ which is called the *tensor product* of A_1 and A_2 and denoted by the same notation $A_1\otimes A_2$. Similarly, the closure of $A_1\dotplus A_2$ is still denoted by $A_1\dotplus A_2$. Therefore,

$$(A_1\otimes A_2)^*=A_1^*\otimes A_2^*, \tag{2.23}$$

$$(A_1\dotplus A_2)^*=A_1^*\dotplus A_2^*. \tag{2.24}$$

Proposition 2.11 *If A_1 and A_2 are bounded operators on Hilbert spaces H_1 and H_2 respectively, then*

$$\|A_1\otimes A_2\|=\|A_1\|\|A_2\|. \tag{2.25}$$

Proof. Let $\{e_j\}$ and $\{f_k\}$ be bases of H_1 and H_2, respectively. For any finite sum $\sum_{j,k}c_{jk}(e_j\otimes f_k)$, we have

$$\begin{aligned}\|(A_1\otimes I)\textstyle\sum_{j,k}c_{jk}(e_j\otimes f_k)\|^2&=\sum_k\|\textstyle\sum_j c_{jk}A_1e_j\|^2\\&\le\sum_k\|A_1\|^2\sum_j|c_{jk}|^2\\&=\|A_1\|^2\|\textstyle\sum_{j,k}c_{jk}(e_j\otimes f_k)\|^2.\end{aligned}$$

The totality of such finite sums being dense in $H_1\otimes H_2$, we have that $\|A_1\otimes I\|\le\|A_1\|$. Hence by $A_1\otimes A_2=(A_1\otimes I)(I\otimes A_2)$ we have

$$\|A_1\otimes A_2\|\le\|A_1\otimes I\|\|I\otimes A_2\|\le\|A_1\|\|A_2\|.$$

Conversely, for any $\epsilon>0$, there exist unit vectors $\varphi\in H_1, \psi\in H_2$ such that $\|A_1\varphi\|\ge\|A_1\|-\epsilon$, $\|A_2\psi\|\ge\|A_2\|-\epsilon$. Hence

$$\begin{aligned}\|(A_1\otimes A_2)(\varphi\otimes\psi)\|&=\|A_1\varphi\|\|A_2\psi\|\\&\ge\|A_1\|\|A_2\|-\epsilon\|A_1\|-\epsilon\|A_2\|+\epsilon^2.\end{aligned}$$

Since ϵ is arbitrary, $\|A_1\otimes A_2\|\ge\|A_1\|\|A_2\|$. (2.25) is proved. ∎

The tensor product of any finite number of operators is defined by induction. In particular, if A is a closed linear operator on Hilbert space H, its n-fold tensor product $A^{\otimes n}$ is a closed linear operator on $H^{\otimes n}$; its restriction to $H^{\hat{\otimes} n}$ is also a closed operator. Similarly, we define

$$\begin{aligned}A^{(n)}\equiv&\overbrace{A\otimes I\otimes\cdots\otimes I}^{n}+\overbrace{I\otimes A\otimes\cdots\otimes I}^{n}+\cdots\\&+\overbrace{I\otimes I\otimes\cdots\otimes A}^{n}.\end{aligned} \tag{2.26}$$

It follows from eqs. (2.23) and (2.24) that, by identifying H with H^*, if A is selfadjoint, then so are the operators $A^{\otimes n}$ and $A^{(n)}$.

Now we proceed to construct operators on Fock spaces.

Definition 2.12 Let A be a closed linear operator densely defined on Hilbert space H. Define $\mathcal{D}\equiv\{\varphi\in\mathcal{F}(H):\varphi=\{\varphi_n\},\varphi_n\in\mathcal{D}(A)^{\otimes n},\forall n$, having only finite non-zero terms$\}$. For $\varphi=\{\varphi_n\}\in\mathcal{D}$, we define two operators on $\mathcal{D}$ by

$$\Gamma(A)\varphi=\{A^{\otimes n}\varphi_n\};\qquad d\Gamma(A)\varphi=\{A^{(n)}\varphi_n\}. \tag{2.27}$$

(by convention $A^{\otimes 0} = I, A^{[0]} = 0$). Then $\Gamma(A)$ and $d\Gamma(A)$ are densely defined closable linear operators on $\mathcal{F}(H)$ whose closures are still denoted by $\Gamma(A)$ and $d\Gamma(A)$ which are called the *second quantization* and *differential second quantization of A* respectively [1].

It is easy to see from the definition and eqs. (2.23) and (2.24) that

$$\Gamma(A^*) = \Gamma(A)^*, \qquad d\Gamma(A^*) = d\Gamma(A)^*. \tag{2.28}$$

Especially, if A is a selfadjoint operator on H, by identifying H with H^*, then $\Gamma(A)$ and $d\Gamma(A)$ are selfadjoint operators on $\mathcal{F}(H)$. Since they commute with the projection to the symmetric Fock space, their restrictions to $\Gamma(H)$ are also selfadjoint. In particular, $\Gamma(I) = I, d\Gamma(I) = N$ is the *number operator*, namely

$$d\Gamma(I)\Big|_{H^{\hat{\otimes} n}} = n\cdot, \tag{2.29}$$

where $n\cdot$ stands for the operator of multiplying by n.

It is easy to prove the following facts (for details see Cook[1], Reed-Simon[1] or Simon[1]).

Proposition 2.13 1° *If A is a contraction on H, then $\Gamma(A)$ is a contraction on $\Gamma(H)$;*

2° *If A generates a strongly continuous semigroup on H, then $d\Gamma(A)$ generates a contractive strongly continuous semigroup on $\Gamma(H)$ and we have*

$$\exp\{-td\Gamma(A)\} = \Gamma(\exp\{-tA\}), \quad t \geq 0; \tag{2.30}$$

3° *If A is a selfadjoint operator on H generating the group $\exp\{itA\}, t \in \mathbb{R}$, of unitary operators, then $d\Gamma(A)$ is a selfadjoint operator on $\Gamma(H)$ generating the following group of unitary operators:*

$$\exp\{itd\Gamma(A)\} = \Gamma(\exp\{itA\}), \quad t \in \mathbb{R}. \tag{2.31}$$

For analysis in Fock spaces, the following exponential vectors

$$\mathcal{E}(h) \equiv 1 \oplus h \oplus \frac{h^{\otimes 2}}{2!} \oplus \cdots \oplus \frac{h^{\otimes n}}{n!} \oplus \cdots, h \in H \tag{2.32}$$

play an important role. Obviously, for $h, g \in H$ we have

$$(\mathcal{E}(h), \mathcal{E}(g))_{\Gamma(H)} = \exp\{(h,g)_H\}. \tag{2.33}$$

If $h \in \mathcal{D}(A)$, then

$$\Gamma(A)\mathcal{E}(h) = \mathcal{E}(Ah). \tag{2.34}$$

[1] In physics literature, $d\Gamma(A)$ is usually called the second quantized operator of A, while $\Gamma(A)$ has no special name.

Moreover, we have

Proposition 2.14 *The exponential vectors $\{\mathcal{E}(h), h \in H\}$ are linearly independent and their linear span is dense in $\Gamma(H)$.*

Proof. Let $h_1, \cdots, h_n \in H$ be distinct. It is easy to see that $\exists g \in H$ so that $\{(h_j, g)_H, 1 \leq j \leq n\}$ are distinct. If $\exists a_1, \cdots, a_n \in \mathbb{K}$ such that $a_1\mathcal{E}(h_1) + \cdots + a_n\mathcal{E}(h_n) = 0$, then $\forall \lambda \in \mathbb{K}$,

$$0 = \left(\sum_{j=1}^{n} a_j \mathcal{E}(h_j), \mathcal{E}(\bar{\lambda} g)\right)_{\Gamma(H)} = \sum_{j=1}^{n} a_j e^{\lambda (h_j, g)_H}.$$

It follows that $a_1 = a_2 = \cdots = a_n = 0$, hence $\{\mathcal{E}(h_j), 1 \leq j \leq n\}$ are linearly independent.

Denote by S the closed subspace generated by $\{\mathcal{E}(h), h \in H\}$. Suppose that $\{h^{\otimes k}, k = 0, 1, \cdots, n-1, h \in H\} \subset S$, where $h^{\otimes k}$ stands for the vector $0 \oplus \cdots \oplus h^{\otimes k} \oplus 0 \oplus \cdots$, it follows from

$$h^{\otimes n} = n! \lim_{t \to 0} t^{-n} \left\{\mathcal{E}(th) - \bigoplus_{j=0}^{n-1} (j!)^{-1} t^j h^{\otimes j}\right\}$$

that $h^{\otimes n} \in S$. In view of polarization identity (2.12), any space $H^{\hat{\otimes} n}$ is contained in S, therefore, $S = \Gamma(H)$. ∎

§3. Countably normed spaces and nuclear spaces

One of the most important locally convex spaces in applications is the normed space. Many results in finite dimensional spaces can be directly extended to infinite dimensional normed spaces. However, the normed space seems restrictive in some applications. For example, in theory of distributions, we need to investigate topological linear spaces generated by a family of seminorms, of which the most important one is the so-called nuclear space.

The theory of nuclear spaces was established in 1955 by A. Grothendieck[1]. Its name came from L. Schwartz kernels theorem. Suppose that $H = L^2(\mathbb{R})$. Then $H \otimes H \cong L^2(\mathbb{R}^2)$. It is well known that any square integrable kernel $K \in L^2(\mathbb{R}^2)$ defines a bounded linear operator on $L^2(\mathbb{R})$:

$$\widetilde{K}f(x) = \int_{\mathbb{R}} K(x,y)f(y)dy, \tag{3.1}$$

that is, $\widetilde{K} \in \mathcal{L}(H)$. However, not every bounded linear operator has a square integrable kernel. For example, the identity operator I has no expression like (3.1), unless that $K(x,y)$ is $\delta(x-y)$, but the latter is a distribution never belonging to $L^2(\mathbb{R}^2)$.

Having introduced the distribution theory, Schwartz proved that $\forall \widetilde{K} \in \mathcal{L}(S(\mathbb{R}), S^*(\mathbb{R}))$, there exists a kernel $K \in S^*(\mathbb{R}^2)$ so that (3.1) holds, that is,

$$\mathcal{L}(S(\mathbb{R}), S^*(\mathbb{R})) \cong S^*(\mathbb{R}^2) \cong S^*(\mathbb{R}) \otimes S^*(\mathbb{R}),$$

where $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing C^∞ functions on $\mathbb{R}$ and $\mathcal{S}^*(\mathbb{R})$ is its dual space, the space of tempered distributions. Both of them are nuclear spaces. The topological tensor product of general locally convex spaces will be introduced in this paragraph.

The nuclear space has many nice properties. For example, in a complete nuclear space all bounded closed sets are compact (this is impossible in any infinite dimensional normed space) and the weak convergence is equivalent to strong convergence for sequences. In any case, from either theoretical or applied viewpoint, the importance of nuclear spaces is no less than that of normed spaces.

The notions and basic properties of locally convex spaces and their dual spaces can be found in Appendix B.

3.1 Countably normed spaces and their dual spaces

Suppose that X is a locally convex topological linear space, whose topology is generated by a family Γ of seminorms. For $p \in \Gamma$, $p^{-1}(0)$ is a linear subspace of X. If $Q_p : X \longrightarrow X/p^{-1}(0)$ is the quotient mapping, then $\tilde{p}(Q_p x) = p(x)$ defines a norm on the quotient space $X/p^{-1}(0)$. By completion with respect to this norm we obtain a Banach space X_p. If $p, q \in \Gamma$ such that $p \prec q$, then $q^{-1}(0) \subset p^{-1}(0)$, hence the map $I_{pq} \equiv Q_p Q_q^{-1}$ extends to a continuous linear operator from X_q to X_p. Obviously, the topology in X is the weakest locally convex topology under which $\{Q_p, p \in \Gamma\}$ are continuous. Thus it is the projective topology with respect to $\{X_p, Q_p; p \in \Gamma\}$ (cf. Appendix B).

Generally speaking, I_{pq} need not be an injection. Therefore, we cannot treat X_q as a subset of X_p. Gel'fand and Shilov[1] introduced an important class of locally convex spaces, whose topologies are generated by countable norms satisfying the so-called consistency condition.

Definition 3.1 Let p, q be two norms on a linear space X. For any Cauchy sequence with respect to both norms, if it converges to 0 in one norm whenever it converges to 0 in another norm, then p and q are called to be *consistent*.

Note that for norms p, q, $p^{-1}(0) = q^{-1}(0) = \{0\}$. If $p \prec q$, then all q-Cauchy sequences are p-Cauchy sequences and by consistency they converge to the same element in X with respect to both norms p and q. Hence $I_{pq} : X_q \longrightarrow X_p$ is an injection. We can treat X_q as a subspace of X_p and therefore I_{pq} continuously and densely imbeds X_q into X_p.

If the topology in X is generated by countable norms $\{p_n\}_{n \in \mathbb{N}}$, without loss of generality we assume that they are non-decreasing:

$$p_1 \prec p_2 \prec \cdots \prec p_n \prec \cdots ,$$

otherwise we may replace p_n by $p_n' \equiv \max\{p_1, \cdots, p_n\}$ to obtain an equivalent family of norms.

Definition 3.2 If X is a locally convex space whose topology is generated

by a sequence of consistent norms $\{\|\cdot\|_n, n \in \mathbb{N}\}$. Then X is called a *countably normed space*.

We may assume that they are non-decreasing:

$$\|\cdot\|_1 \le \|\cdot\|_2 \le \cdots \le \|\cdot\|_n \le \cdots . \tag{3.2}$$

Denote by X_n the Banach space obtained by completing X with respect to norm $\|\cdot\|_n$. Then

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots , \tag{3.3}$$

where X_{n+1} is continuously and densely imbedded into X_n. Obviously, a complete countably normed space is a Fréchet space. The space X, a projective limit of complete locally convex spaces, is complete. Moreover,

$$X = \bigcap_n X_n . \tag{3.4}$$

By introducing the metric

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}, \quad x, y \in X, \tag{3.5}$$

X becomes a complete metric space. The topology induced by this metric is equivalent to that in X. A sequence converges in X if and only if it converges in every X_n. But one should note that the bounded sets in X need not be those defined by this metric.

A set $B \subset X$ is bounded if and only if

$$\sup_{x \in B} \|x\|_n < \infty, \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

In other words, B is bounded in every Banach space X_n. However, unless that X is a normable space (i.e. its topology can be generated by a single norm), the unit balls in any Banach space X_n are not bounded sets in X (otherwise 0 has a bounded neighborhood hence X is normable). It follows that, in a non-normable countably normed space, all bounded sets are nowhere dense and the whole space cannot be a countable union of bounded sets (because any complete metric space is of second category). Therefore, the intuitive meanings of bounded sets are far from those in normed spaces!

Remark. As a Fréchet space, X is barreled, that is, the original topology coincides with strong topology. By Mackey's theorem, all topologies which are compatible with the duality (X, X^*) have same family of bounded sets, hence all weakly bounded sets are strongly bounded. We shall simply call them *bounded sets*.

Now consider the dual space X^* of countably normed space X, that is, the space of all continuous linear functionals on X, and investigate its topologies.

As we know, the dual space of Banach space X_n, denoted by X_n^*, is again a Banach space equipped with the norm

$$\|f\|_{-n} = \sup_{\|x\|_n \le 1} |\langle f, x\rangle|, \quad n \in \mathbb{N}. \tag{3.7}$$

If f is a continuous linear functional on X, then it must be bounded on some neighborhood $\{x : \|x\|_n < \epsilon\}$ and is therefore a continuous linear functional on X_n, so

$$X^* = \bigcup_n X_n^*. \tag{3.8}$$

It follows from (3.7) that

$$\|\cdot\|_{-1} \ge \|\cdot\|_{-2} \ge \cdots \ge \|\cdot\|_{-n} \cdots. \tag{3.9}$$

Therefore,

$$X_1^* \subset X_2^* \subset \cdots \subset X_n^* \subset \cdots, \tag{3.10}$$

and X^* is the inductive limit of $\{X_n^*\}$.

There are several different topologies in the space X^*, among them the most important ones are strong topology and weak* topology, namely the topologies of uniform convergence on bounded sets and pointwise convergence, which are generated by seminorms

$$\|f\|_B \equiv \sup_{x \in B} |\langle f, x\rangle| \tag{3.11}$$

and

$$\|f\|_x \equiv |\langle f, x\rangle|, \quad x \in X \tag{3.12}$$

respectively, where B runs over all bounded sets in X. Generally speaking, X^* is no longer a countably normed space, even has no countable neighborhood base. Therefore, the sequential convergence is not enough to describe the topology in X^*. However, the sequential convergence is still very important in the investigation of topologies in X^*.

Firstly we give another description of bounded sets in X.

Proposition 3.3 *A subset B of countably normed space X is bounded if and only if every continuous linear functional on X is bounded on B.*

Proof. The "only if" part is obvious. To prove the "if" part, firstly suppose that X is a normed space and consider the polar of B.

$$B^\circ \equiv \{f \in X^* : \sup_{x \in B} |\langle f, x\rangle| \le 1\},$$

which is a closed absolutely convex set in X^*. By assumption, every $f \in X^*$ is bounded on B, say, $\sup_{x \in B} |\langle f, x\rangle| \le c$. Then $c^{-1}f \in B^\circ$ which means that B° is an absorbing set hence a barrel. Since Banach space is barreled, B° must contain some neighborhood of 0 in X^*, for example, $\{f : \|f\| \le \epsilon\}$, which means that

$$\|f\| < \epsilon \Longrightarrow \sup_{x \in B} |\langle f, x\rangle| \le 1.$$

Therefore

$$\sup_{x \in B} \|x\| = \sup_{x \in B} \sup_{\|f\| \le \epsilon} |\langle f, x\rangle| / \epsilon \le 1/\epsilon,$$

that is, B is a bounded subset of X.

Now suppose that X is a countably normed space. Since subset B of X is a subset of every normed space X_n, by assumption every $f \in X_n^*$ is bounded on B. It follows that B is a bounded set in X_n for every n, hence a bounded set in X. ∎

In the dual space X^* of countably normed space X, weakly* bounded sets and strongly bounded sets are the same.

Proposition 3.4 *Let X be a countably normed space and X^* its dual. Then, any weakly* bounded set in X^* is strongly bounded.*

Proof. Let F be a weakly* bounded set in X^*. Then its polar

$$F^\circ \equiv \{x \in X : \sup_{f \in F} |\langle f, x\rangle| \le 1\}$$

is a barrel in X containing some neighborhood U of 0 in X, i.e.

$$\sup_{f \in F} \sup_{x \in U} |\langle f, x\rangle| \le 1.$$

Since for any bounded set B in X, there exists $\lambda > 0$ such that $B \subset \lambda U$, it follows that

$$\sup_{f \in F} \|f\|_B = \sup_{f \in F} \sup_{x \in B} |\langle f, x\rangle| < \infty,$$

which means that F is a strongly bounded set in X^*. ∎

Henceforth in X^* we shall not distinguish between weakly* and strongly bounded sets and simply call them *bounded sets*. Now we give another description of bounded sets in X^*.

Proposition 3.5 *In the dual space X^* of countably normed space X, a subset F is bounded if and only if $\exists n \in \mathbb{N}$ such that $F \subset X_n^*$ and F is bounded in X_n^*.*

Proof. Suppose that F is bounded in X^*. In view of proof of Proposition 3.4, there exists a neighborhood U of 0 in X, say, $\{x : \|x\|_n < \epsilon\}$, so that the seminorm $p_F(x) \equiv \sup_{f \in F} |\langle f, x\rangle|$ is bounded on U. In other words, $\exists n \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_n < \epsilon \Longrightarrow \sup_{f \in F} |\langle f, x\rangle| < C,$$

which means that F is bounded in X_n^*.

Conversely, if $\exists n \in \mathbb{N}$ and $C > 0$ such that $F \subset X_n^*$ and that

$$\sup_{f \in F} \|f\|_{-n} = \sup_{f \in F} \sup_{\|x\|_n \le 1} |\langle f, x\rangle| \le C,$$

then F is bounded on some neighborhood $U = \{x : \|x\|_n \leq 1\}$ of 0 in X. Therefore, F is a bounded set in X^*. ∎

Since any convergent sequence is bounded, by Proposition 3.5 we have

Theorem 3.6 *In the dual space X^* of countably normed space X, a sequence $\{f_n\}$ weakly* converges to f if and only if $\exists m \in \mathbb{N}$, such that $\{f_n\} \subset X_m^*$ and $\forall x \in X_m$,*

$$\lim_{n\to\infty} \langle f_n, x\rangle = \langle f, x\rangle . \tag{3.13}$$

Proof. The "if" part is obvious, it suffices to prove the "only if" part. If $\{f_n\}$ weakly* converges to f, then it is bounded, hence $\exists m \in \mathbb{N}$, such that $\{f_n\} \subset X_m^*$ and is bounded in X_m^*. By definition of weak* convergence, (3.13) holds for all $x \in X$. X being dense in X_m and X_m being a normed space, (3.13) extends to all $x \in X_m$. ∎

It is easy to see that *X^* is sequentially complete with respect to weak* topology*. In other words, if a sequence $\{f_n\} \subset X^*$ such that $\forall x \in X$, $\{\langle f_n, x\rangle : n \in \mathbb{N}\}$ is a numerical Cauchy sequence, then there exists $f \in X^*$ so that f_n weakly* converges to f. Moreover, every subspace X_n^* *is weakly* dense in X^**.

3.2 Nuclear spaces and their dual spaces

In this section firstly we give a general definition of nuclear spaces, then we pay more attention to countably Hilbertian nuclear spaces and their dual spaces which are very important in applications.

Definition 3.7 Suppose that the topology of a locally convex space X is generated by a family Γ of H seminorms. If $\forall p \in \Gamma$, $\exists q \in \Gamma$ with $p \prec q$ such that the map

$$I_{pq} : \quad X_q \longrightarrow X_p \tag{3.14}$$

is nuclear (i.e. $I_{pq} \in \mathcal{L}_{(1)}(X_q, X_p)$ is of trace class), then X is called a *nuclear space*.

Since the product of two Hilbert-Schmidt operators is an operator of trace class, the above definition may be restated as follows: "$\forall p \in \Gamma, \exists q \in \Gamma$ such that $p \prec_{\mathrm{HS}} q$", where $p \prec_{\mathrm{HS}} q$ means that p is HS bounded by q, namely $I_{pq} \in \mathcal{L}_{(2)}(X_q, X_p)$ (cf. Appendix B).

If Γ is countable, then X is a projective limit of a sequence of Hilbert spaces, hence metrizable. A complete nuclear space whose topology is generated by countable H seminorms is called a *Fréchet nuclear space*. In particular, we have

Definition 3.8 A complete countably H normed space is called *countably Hilbertian space* (we may assume that its H norms satisfy (3.2), hence $\{X_n\}$ in (3.3) are Hilbert spaces). If, moreover, $\forall n \in \mathbb{N}$, $\exists m > n$ so that the imbedding map $I_{nm} : X_m \to X_n$ is nuclear (or Hilbert-Schmidt operator), then X is called *countably Hilbertian nuclear space*.

Obviously, any countably Hilbertian nuclear space is a Fréchet nuclear space.

Proposition 3.9 *Countably Hilbertian spaces are reflexive spaces.*

Proof. A locally convex space is reflexive if and only if it is a barreled space of which all bounded sets are relatively weakly compact (cf. Appendix B). Countably Hilbertian spaces being Fréchet spaces, they are barreled. Let B be any bounded set in countably Hilbertian space X. Then it is also bounded in every Hilbert space X_n hence relatively $\sigma(X_n, X_n^*)$-compact. Since X can be looked as a subset of product space $\prod_n X_n$, the weak topology in X being product topology of those weak topologies in X_n, it follows from Tychonoff Theorem that B is relatively $\sigma(X, X^*)$-compact in X. ∎

Theorem 3.10 *In complete nuclear spaces, all bounded sets are relatively compact, hence all bounded closed sets are compact.*

Proof. Let B be a bounded set in a complete nuclear space X. Then for any continuous H seminorm p, there exists another continuous H seminorm q with $p \prec q$ such that $I_{pq} : X_q \longrightarrow X_p$ is nuclear (hence a compact operator). B being bounded in every X_q, I_{pq} maps B to a relatively compact set in X_p. Since p is arbitrary, by looking on B as a subset of product space $\prod_{p\in\Gamma} X_p$, it follows from Tychonoff Theorem that B is relatively compact in X. ∎

Corollary 3.11 *Nuclear Banach spaces are finite dimensional.*

Proof. Since every point in a Banach space has bounded neighborhoods, it follows from the nuclearity that every point has compact neighborhoods hence the space is locally compact. However, a Banach space is locally compact if and only if it is finite dimensional. ∎

Theorem 3.12 *In a Fréchet nuclear space X as well as in its dual space X^*, the strong convergence and weak convergence for sequences are equivalent.*

Proof. It suffices to prove that any weakly convergent sequence converges strongly. Suppose that $\{x_n\}$ converges weakly to 0 in X, $\{x_n\}$ being bounded and X being complete nuclear space, $\{x_n\}$ is relatively compact. Since its weak limit is 0, any strongly convergent subsequence must converge to 0. It follows that $\{x_n\}$ converges strongly to 0.

Suppose that $\{f_n\}$ weakly* converges to 0, that is, $\forall x \in X$, $\lim_{n\to\infty}\langle f_n, x\rangle = 0$, we claim that it converges uniformly to 0 on any bounded set in X. If this is not true, then there exist bounded set B, $\epsilon > 0$ and a sequence $\{x_n\}$ in B such that for all n, $|\langle f_n, x_n\rangle| > \epsilon$. Since B is relatively compact, we may assume that $x_n \to x_0 \in X$, hence the sequence $x_n' \equiv x_n - x_0 \to 0$. In view of above proof, it converges strongly to 0, that is, it converges uniformly to 0 on any bounded set in X^*. By choosing a special bounded set $\{f_n\}$, we have $\lim_{n\to\infty}\langle f_n, x_n'\rangle = 0$. Therefore,

$$\lim_{n\to\infty}\langle f_n, x_n\rangle = \lim_{n\to\infty}\langle f_n, x_n'\rangle + \lim_{n\to\infty}\langle f_n, x_0\rangle = 0,$$

which contradicts the assumption that $|\langle f_n, x_n\rangle| > \epsilon$, hence the theorem is proved. ∎

Corollary 3.13 *In the dual space of a countably Hilbertian nuclear space, any bounded set is relatively sequentially compact with respect to strong and weak**

topologies.

Proof. The weak* compactness is a consequence of reflexivity. Since for sequences the strong convergence and weak* convergence are equivalent, it follows that any bounded set is relatively sequentially compact for strong topology. ∎

It is proved that (for example, cf. Schaefer[1]), the strong dual space of a Fréchet nuclear space is a complete nuclear space; any closed subspace M of complete nuclear space X and the quotient space X/M are nuclear spaces; the direct product or projective limit of any family of nuclear spaces is a nuclear space; the direct sum or inductive limit of countable nuclear spaces is a nuclear space.

Let X be a countably Hilbertian nuclear space continuously and densely imbedding into a separable Hilbert space H, by identifying H with its dual space H^*. Then H is also continuously and densely imbedded into the dual space X^* of X. We refer such a triplet $X \hookrightarrow H \hookrightarrow X^*$ as a *Gel'fand triplet.*

Example. Denote by $|\cdot|_0$ the H norm of Hilbert space $L^2(\mathbb{R})$. Let A be the selfadjoint extension of operator $-\frac{d^2}{dt^2} + t^2 - 1$ which is a positive operator in $L^2(\mathbb{R})$ with eigen functions (Hermite functions)

$$e_j(t) \equiv \left((j-1)!\right)^{-1/2} \pi^{-1/4} e^{-t^2/2} H_{j-1}(\sqrt{2}t), \qquad j = 1, 2, \cdots.$$

Moreover,

$$Ae_j = 2je_j, \qquad j = 1, 2, \cdots, \tag{3.15}$$

and $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal base of $L^2(\mathbb{R})$.

Define a sequence of H norms as follows:

$$|\varphi|_k = |A^k\varphi|_0, \quad \varphi \in \mathcal{D}(A^k), k = 0, 1, 2, \cdots. \tag{3.16}$$

Obviously, they are consistent and non-decreasing. Let

$$S_k(\mathbb{R}) = \mathcal{D}(A^k), \qquad k = 0, 1, 2, \cdots. \tag{3.17}$$

Then $S_k(\mathbb{R})$ are Hilbert spaces with norms $|\cdot|_k$. Moreover,

$$L^2(\mathbb{R}) = S_0(\mathbb{R}) \supset S_1(\mathbb{R}) \supset S_2(\mathbb{R}) \supset \cdots,$$

and their projective limit

$$\mathcal{S}(\mathbb{R}) = \varprojlim_k S_k(\mathbb{R}) \tag{3.18}$$

is a countably Hilbertian space. The imbedding operator from $S_{k+1}(\mathbb{R})$ into $S_k(\mathbb{R})$ has HS norm as follows:

$$\|A^{-1}\|_{\mathrm{HS}}^2 = \sum_{j=1}^{\infty}(2j)^{-2} = \frac{\pi^2}{24} < \infty. \tag{3.19}$$

It follows that the imbedding is a Hilbert-Schmidt operator, hence $\mathcal{S}(\mathbb{R})$ is a countably Hilbertian nuclear space. It is (for example, cf. Simon[1]) the Schwartz space of rapidly decreasing C^∞ functions.

If we define on $L^2(\mathbb{R})$ a sequence of H norms:

$$|\varphi|_{-k} \equiv |A^{-k}\varphi|_0, \quad k = 0, 1, 2, \cdots, \tag{3.20}$$

and denote by $S_{-k}(\mathbb{R})$ the completion of $L^2(\mathbb{R})$ with respect to H norm $|\cdot|_{-k}$, by identifying $L^2(\mathbb{R})$ with its dual space we then have

$$S_{-k}(\mathbb{R}) = S_k(\mathbb{R})^*. \tag{3.21}$$

Moreover,

$$L^2(\mathbb{R}) = S_0(\mathbb{R}) \subset S_{-1}(\mathbb{R}) \subset S_{-2}(\mathbb{R}) \subset \cdots.$$

Their inductive limit

$$\mathcal{S}^*(\mathbb{R}) = \varinjlim_k S_{-k}(\mathbb{R}) \tag{3.22}$$

is the dual space of $\mathcal{S}(\mathbb{R})$, namely the space of Schwartz tempered distributions. Therefore, we have a Gel'fand triplet:

$$\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^*(\mathbb{R}). \tag{3.23}$$

Similarly, in $L^2(\mathbb{R}^d)$ we consider the selfadjoint extension A of operator $-\triangle + |x|^2 + 1$. Let

$$e_{k_1,\cdots,k_d} \equiv \prod_{j=1}^{d} e_{k_j}(x_j), \ (k_1, \cdots, k_d) \in \mathbb{N}^d.$$

Then

$$Ae_{k_1,\cdots,k_d} = \left(1 - d + 2\sum_{j=1}^{d} k_j\right) e_{k_1,\cdots,k_d}. \tag{3.24}$$

If $k > d/2$, then

$$\|A^{-k}\|_{\mathrm{HS}}^2 = \sum_{(k_1,\cdots,k_d)\in\mathbb{N}^d} \left(1 + d + 2\sum_{j=1}^{d}(k_j - 1)\right)^{-2k}. \tag{3.25}$$

Successively using inequality

$$\sum_{j=0}^{\infty}(a + j)^{-b} \le \text{const} \cdot a^{-(b-1)} \quad (a > 1, b > 1), \tag{3.26}$$

we know that A^{-k} is a Hilbert-Schmidt operator, hence $\mathcal{S}(\mathbb{R}^d)$ is a countably Hilbertian nuclear space, we obtain again a Gel'fand triplet:

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}^*(\mathbb{R}^d). \tag{3.27}$$

3.3 Topological tensor product, the Schwartz kernels theorem

We have defined in §2 the tensor product of Hilbert spaces. For general locally convex spaces we have the following definition:

Definition 3.14 Let X and Y be locally convex spaces, X_σ^* and Y_σ^* be their weak dual spaces respectively. For $x \in X$ and $y \in Y$, we define their tensor product $x \otimes y$ as the following continuous bilinear form on $X_\sigma^* \times Y_\sigma^*$:

$$x \otimes y(f, g) = \langle f, x \rangle \langle g, y \rangle, \quad f \in X^*, g \in Y^*, \tag{3.28}$$

and extend linearly to the linear subspace $\mathcal{E}$ generated by $\{x \otimes y : x \in X, y \in Y\}$. We equip $\mathcal{E}$ with the strongest locally convex topology for which each map

$$\chi : X \times Y \ni (x, y) \longmapsto x \otimes y \in \mathcal{E} \tag{3.29}$$

is continuous. Then the complete locally convex space obtained by completion of $\mathcal{E}$ is called the *projective tensor product space of X and Y* and denoted by $X \tilde{\otimes} Y$.

Remark. Let $\mathcal{U}$ and $\mathcal{V}$ be neighborhood bases of 0 in X and Y respectively. For $U \in \mathcal{U}$, $V \in \mathcal{V}$, let $U \otimes V = \chi(U \times V) = \{(x \otimes y) : x \in U, y \in V\}$, $\Gamma(U \otimes V)$ be its absolutely convex hull, that is, the smallest absolutely convex set which contains $U \otimes V$. Then $\{\Gamma(U \otimes V) : U \in \mathcal{U}, V \in \mathcal{V}\}$ constitute a neighborhood base of 0 in the projective tensor product space $X \tilde{\otimes} Y$. Or equivalently, if families of seminorms $\mathcal{P}$ and $\mathcal{Q}$ generate topologies of X and Y respectively, for $p \in \mathcal{P}, q \in \mathcal{Q}$, define on $\mathcal{E}$ the following seminorm:

$$p \otimes q(z) = \inf\{\textstyle\sum_j p(x_j) q(y_j) : z = \sum_j (x_j \otimes y_j)\}. \tag{3.30}$$

then seminorms $\{p \otimes q : p \in \mathcal{P}, q \in \mathcal{Q}\}$ generates the topology of $X \tilde{\otimes} Y$. In particular, $p \otimes q(x \otimes y) = p(x) q(y)$.

If X and Y are normed spaces, then $X \tilde{\otimes} Y$ is a normed space, and $\|x \otimes y\| = \|x\| \|y\|$. In particular, if X and Y are Hilbert spaces, one shows that the projective tensor product spare $X \tilde{\otimes} Y \cong \mathcal{L}_{(1)}(X^*, Y)$, while the Hilbertian tensor product space $X \otimes Y \cong \mathcal{L}_{(2)}(X^*, Y)$. Moreover, if X and Y are nuclear spaces, then $X \tilde{\otimes} Y$ is again nuclear (for example, cf. Treves[1]).

By introducing tensor products we may identify bilinear functionals on product spaces with linear functionals on tensor product spaces. For any locally convex spaces X and Y, we denote by $\mathcal{B}(X, Y)$ the linear space consists of all continuous bilinear functionals on $X \times Y$. Then we have

Proposition 3.15 *$\mathcal{B}(X, Y)$ is algebraically isomorphic to $(X \tilde{\otimes} Y)^*$.*

Proof. For $f \in (X \tilde{\otimes} Y)^*$, define $\varphi(x, y) \equiv f(x \otimes y)$. Then the map $f \longmapsto \varphi$ is a linear injection: $(X \tilde{\otimes} Y)^* \longrightarrow \mathcal{B}(X, Y)$ (in fact, if $f(x \otimes y) = 0, \forall x \in X, y \in Y$, then $f \equiv 0$). Conversely, for $\varphi \in \mathcal{B}(X, Y), z = \sum_j (x_j \otimes y_j) \in \mathcal{E}$, define $f(z) = \sum_j \varphi(x_j, y_j)$. It is easy to see that the definition of f is independent of expressions of z and f is a linear functional on $\mathcal{E}$. To prove its continuity, we note that for any $\epsilon > 0$, there exist neighborhoods U and V of 0 in X and Y respectively such that $U \times V \subset \{(x, y) : |\varphi(x, y)| < \epsilon\}$, that is, $U \otimes V \subset \{|f(x \otimes y)| < \epsilon\}$. The latter being absolutely convex, it follows that $\Gamma(U \otimes V) \subset \{|f(x \otimes y)| < \epsilon\}$, which means that f is continuous on $\mathcal{E}$ at 0 with respect to the topology of projective tensor product, hence extends to a continuous linear functional on $X \tilde{\otimes} Y$. Therefore, the map $\varphi \longmapsto f$ is just the inverse of above defined map. ∎

If X and Y are normed spaces, then the above isomorphism is also isometry (isomorphism for Banach spaces), where the norm in $\mathcal{B}(X, Y)$ is defined to be

$$\|\varphi\| = \sup_{\|x\| \le 1} \sup_{\|y\| \le 1} |\varphi(x, y)|. \tag{3.31}$$

Proposition 3.16 *If X and Y are normed spaces, then*

$$\mathcal{B}(X, Y) \cong (X \tilde{\otimes} Y)^*. \tag{3.32}$$

Proof. Consider the linear isomorphism: $\varphi = f \circ \chi$ in Proposition 3.15. Since

$$|\varphi(x, y)| = |f(x \otimes y)| \le \|f\| \|x \otimes y\| = \|f\| \|x\| \|y\|,$$

it follows that $\|\varphi\| \le \|f\|$. Conversely, suppose that $z \in \mathcal{E}$ with $\|z\| = 1$, by (3.30) we know that $\forall \epsilon > 0$, there exists an expression of z: $z = \sum_j (x_j \otimes y_j)$ such that $\sum_j \|x_j\| \|y_j\| < 1 + \epsilon$. Therefore

$$|f(z)| = \Big|\sum_j \varphi(x_j, y_j)\Big| \le \|\varphi\| \sum_j \|x_j\| \|y_j\| < \|\varphi\|(1 + \epsilon).$$

Letting $\epsilon \downarrow 0$, we obtain that $\|f\| \le \|\varphi\|$ and hence the map is an isometry. ∎

Remark. In the case that X is a Fréchet nuclear space and Y is a Fréchet space, eq.(3.32) still holds when $\mathcal{B}(X, Y)$ is equipped with the *topology of bi-bounded convergence*, that is, topology of uniform convergence on any product set of bounded sets in X and that in Y, where $(X \tilde{\otimes} Y)^*$ stands for strong dual space. For the proof we refer to Schaefer[1] or Treves[1], the key step of which is to construct, for any bounded set B in $X \tilde{\otimes} Y$, two bounded sets B_1 and B_2 in X and Y respectively such that B is contained in the closed absolutely convex hull $(\Gamma(B_1 \otimes B_2))^-$ of $B_1 \otimes B_2$.

If the space $\mathcal{L}(X, Y)$ of continuous linear maps from X to Y is equipped with the topology of uniform convergence on bounded stes, then we have the following abstract form of Schwartz kernels theorem.

Theorem 3.17 *Let X be a Fréchet nuclear space and Y a Fréchet space.*

Then

$$X \tilde{\otimes} Y \cong \mathcal{L}(X^*, Y), \tag{3.33}$$

$$X^* \tilde{\otimes} Y \cong \mathcal{L}(X, Y), \tag{3.34}$$

$$\begin{aligned} X^* \tilde{\otimes} Y^* &\cong \mathcal{L}(X, Y^*) \\ &\cong \mathcal{B}(X, Y) \\ &\cong (X \tilde{\otimes} Y)^*, \end{aligned} \tag{3.35}$$

where all dual spaces are strong ones, $\mathcal{B}(X,Y)$ being equipped with topology of bi-bounded convergence and $\mathcal{L}(X,Y)$ that of uniform convergence on bounded sets.

Proof. We only prove the last two isomorphisms in (3.35) under the assumption that X is countably Hilbertian nuclear space and Y is a countably Hilbertian space, for general case see Treves[1].

By the Remark above we know that $\mathcal{B}(X,Y) \cong (X \tilde{\otimes} Y)^*$. Hence for any $\varphi \in \mathcal{B}(X,Y)$, letting $\tilde{\varphi}$ be the linear map

$$\tilde{\varphi} : X \ni x \longmapsto \varphi(x, \cdot) \in Y^*, \tag{3.36}$$

we have $\tilde{\varphi} \in \mathcal{L}(X, Y^*)$. Moreover,

$$\varphi(x, y) = (\tilde{\varphi}x, y). \tag{3.37}$$

In fact, we have

$$\begin{aligned} \varphi \in \mathcal{B}(X,Y) &\Longleftrightarrow \exists n, m \in \mathbb{N}, C > 0, \quad \forall x \in X, y \in Y \\ &\quad\ |\varphi(x,y)| \le C\|x\|_n \|y\|_m \\ &\Longleftrightarrow \exists n, m \in \mathbb{N}, C > 0, \quad \forall x \in X \\ &\quad\ \|\tilde{\varphi}x\|_{-m} \le C\|x\|_n \\ &\Longleftrightarrow \tilde{\varphi} \in \mathcal{L}(X, Y^*), \end{aligned} \tag{3.38}$$

it follows that $\mathcal{B}(X,Y)$ is algebraically isomorphic to $\mathcal{L}(X, Y^*)$.

Let $B_1 \subset X, B_2 \subset Y$ be bounded sets. Then the polar of B_2:

$$B_2^\circ = \{g \in Y^* : \sup_{y \in B_2} |(g, y)| \le 1\}$$

is a neighborhood of 0 in Y^*. When B_2 runs over all bounded sets in Y, their polars constitute a neighborhood base of 0 in Y^*. However,

$$\sup_{x \in B_1} \sup_{y \in B_2} |\varphi(x, y)| \le 1 \Longleftrightarrow \tilde{\varphi}B_1 \subset B_2^\circ, \tag{3.39}$$

it follows that the map $\varphi \longmapsto \tilde{\varphi}$ is a homeomorphism, hence

$$\mathcal{B}(X,Y) \cong \mathcal{L}(X, Y^*).$$

∎

As a consequence, we have:

Theorem 3.18 (Gel'fand-Vilenkin[1]) *If X and Y are countably Hilbertian spaces and X is nuclear, then for any $\varphi \in \mathcal{B}(X,Y)$, there exist $m, n \in \mathbb{N}$ and $A \in \mathcal{L}_{(2)}(X_n, Y_m^*)$ such that*

$$\varphi(x, y) = (Ax, y), \quad \forall x \in X, y \in Y. \tag{3.40}$$

If $\{e_k\}$ and $\{f_k\}$ are bases of X_n and Y_m^* respectively, then

$$\varphi(x, y) = \sum_{k=1}^{\infty} \lambda_k (x, e_k)_n (f_k, y), \quad \forall x \in X, y \in Y, \tag{3.41}$$

where $\lambda_k \ge 0$ such that $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$.

Proof. In view of (3.38), $\tilde{\varphi} \in \mathcal{L}(X_n, Y_m^*)$. Since X is nuclear, there exists $n' > n$ so that $I_{nn'} : X_{n'} \longrightarrow X_n$ is a Hilbert-Schmidt operator, hence $A = \tilde{\varphi} \circ I_{nn'} \in \mathcal{L}_{(2)}(X_{n'}, Y_m^*)$. The rest of theorem is obvious. ∎

Applying to concrete spaces $X = \mathcal{S}(\mathbb{R}^m)$ and $Y = \mathcal{S}(\mathbb{R}^n)$, we have

Theorem 3.19 *It holds that*

$$\begin{aligned} \mathcal{S}^*(\mathbb{R}^m) \tilde{\otimes} \mathcal{S}^*(\mathbb{R}^n) &\cong \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}^*(\mathbb{R}^n)) \\ &\cong (\mathcal{S}(\mathbb{R}^m) \tilde{\otimes} \mathcal{S}(\mathbb{R}^n))^* \\ &\cong \mathcal{S}^*(\mathbb{R}^{m+n}). \end{aligned} \tag{3.42}$$

It follows that, $\forall \tilde{K} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}^*(\mathbb{R}^n))$, there exists a unique kernel $K \in \mathcal{S}^*(\mathbb{R}^{m+n})$ so that

$$(K, \varphi \otimes \psi) = (\tilde{K}\varphi, \psi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^m), \psi \in \mathcal{S}(\mathbb{R}^n), \tag{3.43}$$

or formally

$$\tilde{K}\varphi(x) = \int_{\mathbb{R}^m} K(x, y)\varphi(y)dy, \tag{3.44}$$

which is the original form of the Schwartz kernel theorem.

§4. Borel measures on topological linear spaces

4.1 Minlos-Sazanov theorem

Let H be a real separable Hilbert space, $\mathcal{B}(H)$ the Borel σ-algebra on H. Then $\mathcal{B}(H)$ is a separable σ-algebra (i.e. $\mathcal{B}(H)$ is countably generated). A measure on measurable space $(H, \mathcal{B}(H))$ is called a Borel measure on H. Below we only investigate finite Borel measures.

Definition 4.1 *Let μ be a finite Borel measure on H and*

$$\hat{\mu}(x) = \int_H e^{i(x,y)} \mu(dy), \quad x \in H. \tag{4.1}$$

$\widehat{\mu}$ is called the Fourier transform of μ.

Clearly, $\widehat{\mu}$ possesses the following properties:

(1) $\widehat{\mu}(0) = \mu(H)$;

(2) $\widehat{\mu}$ is continuous on H (even with respect to the weak topology of H);

(3) $\widehat{\mu}$ is positive definite, in the sense that for any $n \geq 2$, $x_1, \cdots, x_n \in H$ and complex numbers $\alpha_1, \cdots, \alpha_n$,

$$\sum_{l,k=1}^{n} \widehat{\mu}(x_l - x_k)\alpha_l\bar{\alpha}_k \geq 0. \tag{4.2}$$

In fact, (4.2) follows from

$$\sum_{l,k=1}^{n} \widehat{\mu}(x_l - x_k)\alpha_l\bar{\alpha}_k = \int_H \left|\sum_{k=1}^{n} \alpha_k e^{i(x_k, y)}\right|^2 \mu(dy).$$

A natural question arises: is any positive definite continuous functional on H the Fourier transform of some finite Borel measure? If H is a finite dimensional space, the answer is affirmative (classical Bochner theorem). But for infinite dimensional Hilbert spaces, the answer is negative. For example, let $\varphi(x) = \exp\{-\frac{1}{2}\|x\|^2\}$. Then φ is a positive definite functional on H, but φ is not the Fourier transform of any finite Borel measure on H. We shall give some characterizations for Fourier transforms of finite Borel measures. To this end we first prove some lemmas.

Lemma 4.2 *Let φ be a positive definite functional on H. Then*

(1) $|\varphi(x)| \leq \varphi(0)$, $\overline{\varphi(x)} = \varphi(-x)$, $\forall x \in H$;

(2) $|\varphi(x) - \varphi(y)| \leq 2\sqrt{\varphi(0)}\sqrt{|\varphi(0) - \varphi(x-y)|}$, $\forall x, y \in H$;

(3) $|\varphi(0) - \varphi(x)| \leq \sqrt{2\varphi(0)(\varphi(0) - \mathrm{Re}\,\varphi(x))}$, $\forall x \in H$.

Proof. For $x, y \in H$, put

$$A = \begin{pmatrix} \varphi(0) & \varphi(x) \\ \varphi(-x) & \varphi(0) \end{pmatrix}, \quad B = \begin{pmatrix} \varphi(0) & \varphi(x) & \varphi(y) \\ \varphi(-x) & \varphi(0) & \varphi(y-x) \\ \varphi(-y) & \varphi(x-y) & \varphi(0) \end{pmatrix}.$$

From the positive definiteness of φ we know that both A and B are positive definite matrices. In particular, $\bar{A}^\tau = A$, here A^τ denotes the transpose of A. Hence $\overline{\varphi(x)} = \varphi(-x)$. Moreover it follows from $\det A \geq 0$ that $|\varphi(x)| \leq \varphi(0)$. Thus (1) is proved. Now by (1), the elements $\varphi(-x)$, $\varphi(-y)$ and $\varphi(y-x)$ in the matrix B can be replaced by $\overline{\varphi(x)}$, $\overline{\varphi(y)}$ and $\overline{\varphi(x-y)}$. The determinant of B is

$$\begin{aligned}
\det B = {} & \varphi(0)^3 - \varphi(0)|\varphi(x-y)|^2 \\
& - \varphi(x)[\varphi(0)\overline{\varphi(x)} - \overline{\varphi(x-y)}\,\overline{\varphi(y)}] \\
& + \varphi(y)[\overline{\varphi(x)}\varphi(x-y) - \varphi(0)\overline{\varphi(y)}] \\
= {} & \varphi(0)^3 - \varphi(0)|\varphi(x-y)|^2 \\
& - \varphi(0)|\varphi(x) - \varphi(y)|^2 \\
& + 2\mathrm{Re}[\varphi(y)\overline{\varphi(x)}(\varphi(x-y) - \varphi(0))].
\end{aligned}$$

Since

$$\varphi(0)^3 - \varphi(0)|\varphi(x-y)|^2 \leq 2\varphi(0)^2|\varphi(0) - \varphi(x-y)|,$$

we have

$$0 \leq \det B \leq 4\varphi(0)^2|\varphi(0) - \varphi(x-y)| - \varphi(0)|\varphi(x) - \varphi(y)|^2,$$

and (2) follows. Finally, (3) follows from

$$\begin{aligned}
|\varphi(0) - \varphi(x)|^2 &= (\varphi(0) - \varphi(x))(\varphi(0) - \overline{\varphi(x)}) \\
&= \varphi(0)^2 - 2\varphi(0)\mathrm{Re}\,\varphi(x) + |\varphi(x)|^2 \\
&\leq 2\varphi(0)^2 - 2\varphi(0)\mathrm{Re}\,\varphi(x).
\end{aligned}$$

The lemma is proved. ∎

Lemma 4.3 *Let μ be a finite Borel measure on H. Then the following assertions are equivalent:*

(1) $\int_H \|x\|^2\mu(dx) < \infty$;

(2) *there exists a positive, symmetric, trace class operator S such that $\forall x, y \in H$,*

$$(Sx, y) = \int_H (x, z)(y, z)\mu(dz). \tag{4.3}$$

If (2) holds, then

$$\mathrm{Tr}\, S = \int_H \|x\|^2\mu(dx). \tag{4.4}$$

Proof. Assume that (2) holds. Let $\{e_n\}$ be an orthonormal base of H. Then

$$\begin{aligned}
\int_H \|x\|^2\mu(dx) &= \sum_{j=1}^{\infty}\int_H (x, e_j)^2\mu(dx) \\
&= \sum_{j=1}^{\infty}(Se_j, e_j) = \mathrm{Tr}\, S.
\end{aligned} \tag{4.5}$$

This means (1) holds, and we have (4.4). Conversely, assume that (1) holds. Then

$$\int_H |(x, z)(y, z)|\mu(dz) \leq \|x\|\,\|y\| \int_H \|z\|^2\mu(dz).$$

Thus there exists a bounded linear operator S such that (4.3) holds. S is obviously positive and symmetric. Furthermore, by (4.5),

$$\mathrm{Tr}\, S = \int_H \|x\|^2\mu(dx) < \infty.$$

Thus S is of trace class. ∎

The following result is the *Minlos-Sazanov theorem*.

Theorem 4.4 *Let φ be a positive definite functional on H. Then the following assertions are equivalent:*

(1) *φ is the Fourier transform of a finite Borel measure on H;*

(2) *$\forall \varepsilon > 0$, there exists a symmetric operator of trace class S_ε such that*

$$(S_\varepsilon x, x) < 1 \Longrightarrow \mathrm{Re}(\varphi(0) - \varphi(x)) < \varepsilon; \tag{4.6}$$

(3) *there exists a symmetric operator of trace class S on H such that φ is continuous (or, equivalently, continuous at $x = 0$) with respect to the following norm $\|\cdot\|_*$:*

$$\|x\|_* = (Sx, x)^{1/2} = \|S^{1/2}x\|. \tag{4.7}$$

Proof. (1) $\Longrightarrow$ (2). Let $\varphi = \hat{\mu}$. For any $\gamma > 0$,

$$\begin{aligned}\mathrm{Re}(\varphi(0) - \varphi(x)) &= \int_H (1 - \cos(x, z))\mu(dz) \\ &\le \frac{1}{2}\int_{\|z\| \le \gamma} (x, z)^2 \mu(dz) + 2\mu(\{z : \|z\| > \gamma\}).\end{aligned}$$

Put $\mu_1(A) = \mu(A \cap \{\|z\| \le \gamma\})$. Applying Lemma 4.3 to μ_1, we know that there exists a positive symmetric operator B_γ such that

$$(B_\gamma z_1, z_2) = \int_{\|z\| \le \gamma} (z, z_1)(z, z_2)\mu(dz).$$

For a given $\varepsilon > 0$, take $\gamma > 0$ such that $\mu(\{\|z\| > \gamma\}) < \varepsilon/4$, and put $S_\varepsilon = \varepsilon^{-1}B_\gamma$, then

$$\mathrm{Re}(\varphi(0) - \varphi(x)) < \frac{\varepsilon}{2}(S_\varepsilon x, x) + \frac{\varepsilon}{2}.$$

(2) $\Longrightarrow$ (1). Assume (2) holds. Then $\mathrm{Re}\,\varphi(x)$ is continuous at $x = 0$. By Lemma 4.2, φ is continuous on H. Now take any orthonormal base $\{e_n\}$ of H. For $n \ge 1$, put

$$f_{i_1, \cdots, i_n} = \varphi(\omega_1 e_{i_1} + \cdots + \omega_n e_{i_n}), \quad \omega_j \in \mathbb{R}, \ 1 \le j \le n. \tag{4.8}$$

Then $f_{i_1, \cdots, i_n}$ is a positive definite function on $\mathbb{R}^n$. By the classical Bochner theorem, $f_{i_1, \cdots, i_n}$ is the Fourier transform of a finite Borel measure $\mu_{i_1, \cdots, i_n}$ on $\mathbb{R}^n$. Clearly, the family $\{\mu_{i_1, \cdots, i_n}\}$ satisfies the consistency conditions of Kolmogorov's extension theorem for measures. Thus there exists a unique finite measure ν on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ such that

$$\mu_{i_1, \cdots, i_n} = \nu \circ (X_{i_1}, \cdots, X_{i_n})^{-1}, \tag{4.9}$$

where $X_j(\omega) = \omega_j$, $\omega = (\omega_1, \omega_2, \cdots) \in \mathbb{R}^\infty$.

We are going to prove that $\sum_{k=1}^\infty X_k^2 < \infty$, ν-a.e.. For this purpose, let P_n be the standard Gaussian measure on $\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} e^{i(a_1 y_1 + \cdots + a_n y_n)} P_n(dy) = \exp\Big\{-\frac{1}{2}\sum_{j=1}^n a_j^2\Big\}. \tag{4.10}$$

For any $\varepsilon > 0$, by assumption, there exists a positive symmetric operator of trace class S_ε satisfying (4.6). Thus

$$\varphi(0) - \mathrm{Re}\,\varphi(x) \le \varepsilon + 2\varphi(0)(S_\varepsilon x, x), \ \forall x \in H \tag{4.11}$$

By Fubini's theorem,

$$\begin{aligned}&\varphi(0) - \int_{\mathbb{R}^\infty} \exp\Big\{-\frac{1}{2}\sum_{j=1}^n X_{k+j}^2\Big\} d\nu \\ &= \varphi(0) - \int_{\mathbb{R}^\infty} d\nu \int_{\mathbb{R}^n} \exp\Big\{i\sum_{j=1}^n y_j X_{k+j}\Big\} P_n(dy) \\ &= \varphi(0) - \int_{\mathbb{R}^n} \varphi\Big(\sum_{j=1}^n y_j e_{k+j}\Big) P_n(dy) \\ &= \int_{\mathbb{R}^n} \Big[\varphi(0) - \mathrm{Re}\,\varphi\Big(\sum_{j=1}^n y_j e_{k+j}\Big)\Big] P_n(dy).\end{aligned}$$

By (4.11), the above is less than

$$\begin{aligned}&\varepsilon + 2\varphi(0)\int_{\mathbb{R}^n} \Big(S_\varepsilon \sum_{j=1}^n y_j e_{k+j}, \sum_{j=1}^n y_j e_{k+j}\Big) P_n(dy) \\ &= \varepsilon + 2\varphi(0)\sum_{j=1}^n (S_\varepsilon e_{k+j}, e_{k+j}).\end{aligned}$$

But since $n \ge 1$ is arbitrary, we have

$$\varphi(0) - \int_{\mathbb{R}^\infty} \exp\Big\{-\frac{1}{2}\sum_{j=k+1}^\infty X_j^2\Big\} d\nu \le \varepsilon + 2\varphi(0)\sum_{j=k+1}^\infty (S_\varepsilon e_j, e_j). \tag{4.12}$$

First let $k \to \infty$ and $\varepsilon \downarrow 0$ in (4.12). We get (noting that $\varphi(0) = \nu(\mathbb{R}^\infty)$)

$$\varphi(0) - \lim_{k \to \infty} \int_{\mathbb{R}^\infty} \exp\Big\{-\frac{1}{2}\sum_{j=k+1}^\infty X_j^2\Big\} d\nu = 0,$$

which implies $\sum_{j=1}^\infty X_j^2 < \infty$, ν-a.e..

Finally, let $X(\omega) = \sum_{j=1}^\infty X_j(\omega) e_j$. Then X is defined on $\mathbb{R}^\infty$, ν-a.e. and X is an H-valued measurable function. Put $\mu = \nu \circ X^{-1}$. Then μ is a finite Borel measure on H, and by (4.9), we have

$$\begin{aligned}\hat{\mu}\Big(\sum_{j=1}^n (x, e_j)e_j\Big) &= f_{1, \cdots, n}((x, e_1), \cdots, (x, e_n)) \\ &= \varphi\Big(\sum_{j=1}^n (x, e_j)e_j\Big).\end{aligned}$$

Let $n \to \infty$. We obtain $\hat{\mu} = \varphi$. (2) $\Longrightarrow$ (1) is proved.

(2) $\Longleftrightarrow$ (3). Assume (2) holds. Let $S_{1/k}$ be the positive symmetric operator corresponding to $\epsilon = 1/k$. Take $\lambda_k > 0$ such that $\sum_k \lambda_k \mathrm{Tr} S_{1/k} < \infty$, put $S = \sum_k \lambda_k S_{1/k}$. Then S is a positive symmetric operator of trace class and it holds that

$$\begin{aligned}(Sx, x) < \lambda_k &\Longrightarrow (S_{1/k}x, x) < 1\\ &\Longrightarrow \mathrm{Re}(\varphi(0) - \varphi(x)) < \frac{1}{k}.\end{aligned}$$

Thus $\mathrm{Re}\,\varphi(x)$ is continuous at $x = 0$ with respect to the norm $\|\cdot\|_s$. Consequently, by Lemma 4.2, φ is continuous on H with respect to the norm $\|\cdot\|_s$. This proves (2) $\Longrightarrow$ (3). Conversely, assume (3) holds. For any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x\|_s < \delta \Longrightarrow \mathrm{Re}(\varphi(0) - \varphi(x)) < \epsilon$. Put $S_\epsilon = \delta^{-1}S$. Then (4.6) follows. (3) $\Longrightarrow$ (2) is thus proved. ∎

In the sequel, we shall present a more useful form of the Minlos-Sazanov theorem —— the Minlos theorem. First let's introduce some notations and lemmas.

Let B be a positive, symmetric, invertible, trace class operator on H. Introduce a new inner product $(\cdot,\cdot)_-$ and norm $\|\cdot\|_-$ on H as follows:

$$(x,y)_- = (Bx, y), \qquad \|x\|_- = (Bx, x)^{1/2} = \|B^{1/2}x\| \tag{4.13}$$

Denote by H_- the completion of H with respect to the norm $\|\cdot\|_-$. Then the inner product $(\cdot,\cdot)_-$ can be continuously extended to H_-, and H_- is a separable Hilbert space with respect to $(\cdot,\cdot)_-$. On the other hand, denote by H_+ the domain of $B^{-1/2}$. Then it is easily seen that H_+ is the range of $B^{1/2}$ (i.e. $H_+ = B^{1/2}(H)$). Introduce an inner product $(\cdot,\cdot)_+$ and norm $\|\cdot\|_+$ on H_+ as follows:

$$(x,y)_+ = (B^{-1/2}x, B^{-1/2}y), \quad \|x\|_+ = \|B^{-1/2}x\|, \; x \in H_+. \tag{4.14}$$

Obviously,

$$\|Bx\|_+ = \|x\|_-, \quad x \in H, \tag{4.15}$$

$$\|B^{-1}x\|_- = \|x\|_+, \quad x \in B(H), \tag{4.16}$$

$$\|x\| \le \|B\|^{1/2}\|x\|_+, \quad x \in H_+. \tag{4.17}$$

Concerning the spaces H_- and H_+, we have

Lemma 4.5 *Under the above assumptions we have:*

(1) H_+ is a separable Hilbert space with respect to $(\cdot,\cdot)_+$;

(2) B can be extended to an isometry from H_- onto H_+, and B^{-1} can be extended to an isometry from H_+ onto H_-;

(3) as an operator on H_-, B is positive, symmetric and of trace class, and $\mathrm{Tr}_- B = \mathrm{Tr} B$. Here Tr_- denotes the trace of B computed on H_-.

(4) H_- and H_+ are mutually adjoint, and the canonical bilinear form $\langle\cdot,\cdot\rangle$ on $H_- \times H_+$ is

$$\langle x,y\rangle = (B^{-1}x, y)_-, \quad x \in H_-, \; y \in H_+. \tag{4.18}$$

Proof. Let $\{x_n\}$ be a Cauchy sequence in H_+ under the norm $\|\cdot\|_+$. By (4.17), $\{x_n\}$ is also a Cauchy sequence in H. Denote its limit by x. Put $y_n = B^{-1/2}x_n$. Then $\{y_n\}$ is a Cauchy sequence in H. Denote its limit by y. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} B^{1/2}y_n = B^{1/2}y,$$

which implies $x \in H_+$, and

$$\|x_n - x\|_+ = \|B^{-1/2}(x_n - x)\| = \|y_n - y\| \to 0.$$

Thus H_+ is complete with respect to $\|\cdot\|_+$, i.e., H_+ is a Hilbert space under $(\cdot,\cdot)_+$.

(2) follows immediately from (4.15) and (4.16).

(3) It is easily verified that B, as an operator on H_-, is positive and symmetric. We shall prove that B is of trace class on H_-. Suppose that the spectral resolution of B on H is

$$Bx = \sum_n \lambda_n (x, e_n) e_n, \quad x \in H.$$

Since B is invertible, $\{e_n\}$ constitutes an orthonormal base of H. Put $f_n = e_n/\sqrt{\lambda_n}$. Then $(Bf_n, f_m) = (\lambda_n\lambda_m)^{-1/2}(Be_n, e_m) = \delta_{nm}$. Thus $\{f_n\}$ is an orthonormal base of H_- and we have

$$\mathrm{Tr}_- B = \sum_{n=1}^{\infty}(Bf_n, f_n)_- = \sum_{n=1}^{\infty}\|Bf_n\|^2 = \sum_{n=1}^{\infty}\lambda_n = \mathrm{Tr}\, B.$$

(4) By (2), the bilinear form $\langle\cdot,\cdot\rangle$ in (4.18) is well defined. Moreover, we have

$$|\langle x,y\rangle| \le \|B^{-1}x\|_- \|y\|_- = \|x\|_+\|y\|_-.$$

This means that $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $H_+ \times H_-$ which makes them mutually adjoint. ∎

Now we are ready to prove the *Minlos theorem.*

Theorem 4.6 *Let φ be a continuous and positive functional on H, B a positive, symmetric, invertible, trace class operator on H, and H_- be defined as before. Then there exists a unique finite Borel measure μ on H_- such that*

$$\int_{H_-} e^{i\langle x, y\rangle}\mu(dx) = \varphi(x), \quad \forall x \in H_+. \tag{4.19}$$

Proof. For $x \in H_-$, put $\psi(x) = \varphi(Bx)$. Then ψ is obviously a positive definite functional on H_-. By Lemma 4.5, B is a positive, symmetric, invertible, trace class operator on H_-. Define a new norm $\|\cdot\|_*$ on H_- as follows:

$$\|x\|_* = \|B^{1/2}x\|_- = |Bx| .$$

The continuity of φ on H implies the continuity of ψ on H_- with respect to the norm $\|\cdot\|_*$. Thus by Theorem 4.4, ψ is the Fourier transform of some finite Borel measure μ on H_-, i.e.,

$$\int_{H_-} e^{i(y,z)_-}\mu(dz) = \psi(y) , \quad \forall y \in H_- . \tag{4.20}$$

Put $y = B^{-1}x$, $x \in H_+$. Then (4.19) follows from (4.18). ∎

Let X be a countably Hilbertian nuclear space and X^* its topological dual. By Definition 3.8, $\forall n \in \mathbb{N}, \exists m > n$, such that the imbedding $I_{nm} : X_m \hookrightarrow X_n$ is of trace class. Clearly, the adjoint $I^*_{nm} : X^*_n \hookrightarrow X^*_m$ is also of trace class. If φ is a positive definite continuous functional on X, then φ is continuous on certain Hilbert space X_n. Identify X^*_n with X_n (through Riesz' representation), and replace H and H_- by X_n and X^*_m in Theorem 4.6 respectively. Then there exists a unique finite Borel measure μ on X^*_m with Fourie transform φ. Noting that $X^* = \bigcup_m X^*_m$, we have

Theorem 4.7 *Any positive definite continuous functional on a countably Hilbertian nuclear space X is the Fourier transform of a finite Borel measure on the dual space X^*.*

In particular, if $X \hookrightarrow H \hookrightarrow X^*$ is a Gel'fand triplet (see §3), $(\cdot,\cdot)$ is the canonical bilinear form on $X \times X^*$, then for any positive definite continuous functional φ on X, there exists a unique finite Borel measure μ on X^* such that

$$\int_{X^*} e^{i(x,z)}\mu(dz) = \varphi(x) , \quad x \in X. \tag{4.21}$$

$\varphi(x)$ is called the characteristic functional of μ.

4.2 Gaussian measures on Hilbert spaces

We shall study a special class of Borel probability measures on H —— Gauss measures. Let's first introduce the notions of mean vector and convariance operator for general Borel probability measures on H.

Definition 4.8 Let μ be a Borel probability measure on H. If for any $x \in H$, the function $z \mapsto (x,z)$ is integrable with respect to μ, and there exists an element $m \in H$ such that

$$(m,x) = \int_H (x,z)\mu(dz) , \quad x \in H, \tag{4.22}$$

then m is called the *mean vector* of μ. If furthermore there exists a positive symmetric linear opertor B on H such that

$$(Bx,y) = \int_H (z - m,x)(z - m,y)\mu(dz) , \quad \forall x,y \in H, \tag{4.23}$$

then B is called the *covariance operator* of μ.

Mean vector and covariance operator do not necessarily exist in general. But if $\int_H \|z\|\mu(dz) < \infty$, then by Riesz' representation theorem, the mean vector m does exist, and $\|m\| \le \int_H \|x\|\mu(dx)$. If furthermore, $\int_H \|x\|^2\mu(dx) < \infty$, then by Lemma 4.3, there exists a positive, symmetric, trace class operator S such that

$$(Sx,y) = \int_H (x,z)(y,z)\mu(dz) , \quad \forall x,y \in H. \tag{4.24}$$

Put

$$Bx = Sx - (m,x)m . \tag{4.25}$$

It is easily verified that B satisfies (4.23), i.e., B is the covariance operator of μ. Note that B is also a positive, symmetric trace class operator.

Definition 4.8 *Let μ be a Borel probability measure on H. If for any $x \in H$, the random variable $(x,\cdot)$ has a Gaussian distribution, then μ is called a Gaussian measure.*

We shall characterize Gaussian measures by means of Fourier transform. We shall need

Lemma 4.10 *Let $\{\alpha_j\}$ be a sequence of real numbers satisfying $\sum_{j=1}^\infty \alpha_j^2 = \infty$. Then there exists a sequence of real numbers $\{\beta_j\}$ such that $\alpha_j\beta_j \ge 0$, $\forall j \ge 1$, $\sum_{j=1}^\infty \beta_j^2 < \infty$ and $\sum_{j=1}^\infty \alpha_j\beta_j = \infty$.*

Proof. Put $n_0 = 0$ and define n_k inductively as follows:

$$n_k \equiv \inf\{l : \textstyle\sum_{j=n_{k-1}+1}^{l} \alpha_j^2 \ge 1\} , \quad k \ge 1 .$$

Clearly, $n_k \uparrow \infty$. Put

$$\beta_j = \frac{\alpha_j}{k+1}\Big(\sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2\Big)^{-1/2} , \quad n_k + 1 \le j \le n_{k+1} , \quad k = 0,1,2,\cdots .$$

Then $\alpha_j\beta_j \ge 0$, $\forall j \ge 1$, and

$$\sum_{j=1}^\infty \beta_j^2 = \sum_{k=0}^\infty \sum_{j=n_k+1}^{n_{k+1}} \beta_j^2 = \sum_{k=0}^\infty \frac{1}{(k+1)^2} < \infty ,$$

$$\sum_{j=1}^\infty \alpha_j\beta_j = \sum_{k=0}^\infty \sum_{j=n_k+1}^{n_{k+1}} \alpha_j\beta_j = \sum_{k=0}^\infty \frac{1}{k+1}\Big(\sum_{j=n_k+1}^{n_{k+1}} \alpha_j^2\Big)^{1/2}$$

$$\ge \sum_{k=0}^\infty \frac{1}{k+1} = \infty .$$

The lemma is proved. ∎

The following theorem gives a characterization of Gaussian measures.

Theorem 4.11 *A Borel probability measure μ on H is a Gaussian measure if and only if its Fourier transform can be expressed as*

$$\widehat{\mu}(x) = \exp\{i(m,x) - \tfrac{1}{2}(Bx,x)\} , \tag{4.26}$$

where $m \in H$, B is a positive, symmetric, trace class operator on H. In this situation, m and B are the mean vector and covariance operator of μ respectively. Moreover,

$$\int_H \|x\|^2 \mu(dx) = \operatorname{Tr} B + \|m\|^2. \tag{4.27}$$

Proof. Necessity. Let μ be a Gaussian measure. We first prove $\int_H \|x\|^2\mu(dx) < \infty$. By assumption, for any x, $(x,\cdot)$ has a Gaussian distribution. Thus there exist a real number m_x and a positive number σ_x such that

$$\widehat{\mu}(x) = \int_H e^{i(x,z)}\mu(dz) = \exp\{im_x - \tfrac{1}{2}\sigma_x^2\} . \tag{4.28}$$

Let $\{e_j\}$ be an orthonormal base of H. Then

$$\int_H \|x\|^2\mu(dx) = \sum_{j=1}^{\infty}\int_H (e_j,x)^2\mu(dx) = \sum_{j=1}^{\infty}(\sigma_{e_j}^2 + m_{e_j}^2) . \tag{4.29}$$

Let $\{\beta_j\}$ be a sequence of real numbers such that $\beta_j m_{e_j} \geq 0$, $\sum\limits_{j=1}^{\infty}\beta_j^2 < \infty$. Put

$$\xi(x) = \sum_{j=1}^{\infty}\beta_j(e_j,x). \tag{4.30}$$

Then ξ is a Gaussian variable (since by Schwarz' inequality, the above series converges absolutely) with a finite mean, i.e., $\sum_{j=1}^{\infty}\beta_j m_{e_j} < \infty$. Now by Lemma 4.10, $\sum_{j=1}^{\infty} m_{e_j}^2 < \infty$. Thus in order to prove $\int_H \|x\|^2\mu(dx) < \infty$, it suffices to check $\sum_{j=1}^{\infty}\sigma_{e_j}^2 < \infty$. By Lemma 4.4, there exists a positive, symmetric, trace class operator S such that $(Sx,x) < 1 \Longrightarrow 1 - \operatorname{Re}\widehat{\mu}(x) < \frac{1}{3}$. Hence we have

$$1 - \exp\{-\frac{1}{2}\sigma_x^2\} \leq 1 - \operatorname{Re}\widehat{\mu}(x) \leq (Sx,x) + \frac{1}{3} , \ \forall x \in H. \tag{4.31}$$

Without loss of generality we may assume that the kernel of S is $\{0\}$. For $x \in H$, $x \neq 0$, put $y = [3(Sx,x)]^{-1/2}x$. Then $\sigma_y^2 = [3(Sx,x)]^{-1}\sigma_x^2$, $(Sy,y) = \frac{1}{3}$. Replacing x by y in (4.31), we obtain

$$1 - \exp\left\{-\frac{\sigma_x^2}{6(Sx,x)}\right\} \leq \frac{2}{3},$$

that is, $\sigma_x^2 \leq (6\log 3)(Sx,x)$, $\forall x \in H$. From this we have

$$\sum_{j=1}^{\infty}\sigma_{e_j}^2 \leq (6\log 3)\operatorname{Tr} S < \infty .$$

Hence $\int_H \|x\|^2\mu(dx) < \infty$ is proved. By the remark following Definition 4.8, the mean vector m and covariance operator B of μ exist. Use the above notations, we have

$$\begin{aligned} m_x &= \int_H (x,z)\mu(dz) = (m,x), \\ \sigma_x^2 &= \int_H (x,z)^2\mu(dz) - m_x^2 = \int_H [(x,z)^2 - (m,x)^2]\mu(dz) \\ &= \int_H (x,z-m)^2\mu(dz) = (Bx,x). \end{aligned}$$

Hence (4.26) follows from (4.28), and (4.27) follows from (4.29).

Sufficiency. Let $m \in H$, and B be a positive, symmetric, trace class operator,

$$\varphi(x) = \exp\{i(m,x) - \tfrac{1}{2}(Bx,x)\}.$$

Then it is easy to verify that φ is a positive definite functional on H. Put

$$Sx = Bx + (m,x)m.$$

Then S is a positive, symmetric, trace class operator on H. Define the norm $\|\cdot\|_*$ on H as follows:

$$\|x\|_* = \|S^{1/2}x\| = ((Bx,x) + (m,x)^2)^{1/2}.$$

Then $\varphi(x)$ is continuous at $x = 0$ with respect to the norm $\|\cdot\|_*$. By Theorem 4.4, φ is the Fourier transform of some Borel probability measure μ on H. Clearly for any $x \in H$, $(x,\cdot)$ is a Gaussian random variable with mean (m,x) and variance (Bx,x) under μ. Thus μ is a Gaussian measure. ∎

4.3 Gaussian measures on Banach spaces

We now study Gaussian measures on Banach spaces. First let us introduce the notions of cylinder sets and cylinder measures.

Let X be a separable Banach space with dual X^*. Denote by $\|\cdot\|$ and $\|\cdot\|_{X^*}$ the norms on X and on X^*, respectively, and by $\langle\cdot,\cdot\rangle$ the canonical bilinear form on $X \times X^*$. Denote by $\mathcal{F}(X^*)$ the set of all finite dimensional subspaces of X^*. For any $K \in \mathcal{F}(X^*)$, we call

$$C = \{x \in X : (\langle x,y_1\rangle, \cdots, \langle x,y_n\rangle) \in E\} \tag{4.32}$$

a *cylinder set based on* K. Here $n \geq 1$, E is a Borel subset of $\mathbb{R}^n$, $y_1, \cdots, y_n \in K$. Let $\mathcal{C}(K)$ be the σ-algebra generated by the cylinder sets based on K. Put

$$\mathcal{R}(X) = \bigcup_{K \in \mathcal{F}(X^*)} \mathcal{C}(K). \tag{4.33}$$

Then $\mathcal{R}(X)$ is an algebra.

Lemma 4.12 *Let X be a separable Banach space. Then $\sigma(\mathcal{R}(X)) = \mathcal{B}(X)$, here $\mathcal{B}(X)$ is the Borel σ-algebra on X.*

Proof. Clearly $\sigma(\mathcal{R}(X)) \subset \mathcal{B}(X)$. Since X is a separable metric space, every open set can be expressed as the union of countable closed sets. Thus in order to prove $\sigma(\mathcal{R}(X)) = \mathcal{B}(X)$, is suffices to prove that every closed ball belongs to $\sigma(\mathcal{R}(X))$. Let $S = \{x : \|x - x_0\| \leq r\}$, here $x_0 \in X$, $r > 0$. Let $\{a_n\}$ be a countable subset of X. By the Hahn-Banach theorem, for any $n \geq 1$, there exists $z_n \in X^*$ such that $\langle a_n, z_n \rangle = \|a_n\|$, $\|z_n\|_{X^*} = 1$. Put

$$T = \bigcap_{n=1}^{\infty} \{x \in X : |\langle x - x_0, z_n \rangle| \leq r\}.$$

Clearly $S \subset T$, $T \in \sigma(\mathcal{R}(X))$. We now prove $S = T$. If $x \notin S$, i.e., $\|x - x_0\| = r_1 > r$, then there exists n such that $\|x - x_0 - a_n\| < (r_1 - r)/2$. Thus $\|a_n\| > (r_1 + r)/2$, and

$$\begin{aligned} |\langle x - x_0, z_n \rangle| &\geq |\langle a_n, z_n \rangle| - |\langle x - x_0 - a_n, z_n \rangle| \\ &\geq \|a_n\| - \|x - x_0 - a_n\| > r. \end{aligned}$$

This implies that $x \notin T$. Consequently, $T \subset S$. Finally $S = T \in \sigma(\mathcal{R}(X))$. ∎

Definition 4.13 Let μ be a non-negative set function on $\mathcal{R}(X)$. If $\mu(X) = 1$, and for any $K \in \mathcal{F}(X^*)$, μ is a measure when restricted to the σ algebra $\mathcal{C}(K)$, then μ is called a *cylinder (probability) measure* on X. A complex function f on X is called a *cylinder function* if it is measurable with respect to $\mathcal{C}(K)$ for some $K \in \mathcal{F}(X^*)$.

The integration of a bounded cylinder function f with respect to the cylinder measure μ makes sense if the cylinder measure is viewed as a measure on a σ-algebra $\mathcal{C}(K)$ which makes f measurable. We use $\int_X f(x)\mu(dx)$ to denote this integral. In particular, for a cylinder measure μ we can define

$$\widehat{\mu}(z) = \int_X e^{i\langle x, z \rangle} \mu(dx), \quad z \in X^*. \tag{4.34}$$

$\widehat{\mu}$ is called the *characteristic functional* of μ.

Clearly, the characteristic funtional of any cylinder measure is a positive definite continuous functional on X^*. On the other hand, if φ is a positive definite continuous functional on X^* and $\varphi(0) = 1$, then there exists a unique cylinder measure μ such that its characteristic functional is φ.

A natural question arises: what kind of cylinder measures can be extended to Borel measures on X? We shall give an answer to this question in a particular case where the Banach space X is the completion of some Hilbert space H with respect to a weaker norm and the cylinder measure on X is "lifted" from that on H.

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. Let $\|\cdot\|$ be another norm on H satisfying $\|x\| \leq c|x|$ for some constant c. In this situation, we say that the norm $\|\cdot\|$ is weaker than $|\cdot|$. Let X be the completion of H with respect to $\|\cdot\|$. Then X is a separable Banach space, and H is a linear subspace of X. If we identify the dual H^* of H with H itself, then the dual X^* of X can be regarded as a linear subspace of H:

$$X^* = \left\{ y \in H : \sup_{x \in H, \|x\| = 1} |(x, y)| < \infty \right\}. \tag{4.35}$$

Denote by $\langle \cdot,\cdot \rangle$ the canonical bilinear form on $X \times X^*$. Then $\langle \cdot,\cdot \rangle$ coincides with the inner product $(\cdot,\cdot)$ when restricted on $H \times X^*$, i.e.,

$$\langle x, y \rangle = (x, y), \quad \forall x \in H, \ y \in X^*. \tag{4.36}$$

Denote by $\mathcal{F}(X^*)$ and $\mathcal{F}(H)$ the finite dimensional subspaces of X^* and H, respectively. Since $\mathcal{F}(X^*) \subset \mathcal{F}(H)$, and for any $K \in \mathcal{F}(X^*)$, if we denote by $\mathcal{C}_X(K)$ and $\mathcal{C}_H(K)$ the σ-algebras on X and on H generated by the cylinder sets based on K, respectively, then $\mathcal{C}_X(K) \cap H \subset \mathcal{C}_H(K)$. Consequently, we have $\mathcal{R}(X) \cap H \subset \mathcal{R}(H)$. Thus for any cylinder measure μ on H, we can define a cylinder measure μ^* on X as follows:

$$\mu^*(C) = \mu(C \cap H), \quad C \in \mathcal{R}(X). \tag{4.37}$$

We call μ^* the *lifting* of μ to X. Clearly, for $x \in X^*$, $\widehat{\mu^*}(x) = \widehat{\mu}(x)$. Thus the characteristic functional of μ^* is the restriction of μ to X^*. Henceforth we call (H, X, μ) a *fundamental triplet*. In order to answer the above question, we shall introduced the notion of measurabl norm, which is first introduced by Gross[1].

Denote by $\mathcal{P}$ the set of all finite dimensional orthogonal projections on H. For $P \in \mathcal{P}$, let $f(x) = \|Px\|$, $x \in H$. Then f is a cylinder function on H.

Definition 4.14 Let $(H, |\cdot|)$ be a Hilbert space, μ a cylinder measure on H, $\|\cdot\|$ another norm on H weaker than $|\cdot|$. If for any $\epsilon > 0$, there esists $P_\epsilon \in \mathcal{P}$ such that for any $P \in \mathcal{P}$ orthogonal to P_ϵ,

$$\mu\{x \in H : \|Px\| > \epsilon\} < \epsilon,$$

then $\|\cdot\|$ is said to be *measurable with respect to μ* (or, simply, *μ-measurable*).

Definition 4.15 Let μ be a cylinder measure on H. If $\widehat{\mu}(x) = \exp(-\frac{1}{2}|x|^2)$, then μ is called a (*standard*) *Gaussian cylinder measure* on H.

Obviously, μ is a Gaussian cylinder measure if and only if for any $P \in \mathcal{P}$, $\mu \circ P^{-1}$ is a Gaussian measure on $P(H)$.

Now we can state the famous *Gross theorem.*

Theorem 4.16 *Let (H, X, μ) be a fundamental triplet. If μ is a Gaussian cylinder measure, and the norm $\|\cdot\|$ is μ-measurable, then the lifting μ^* of μ to X can be extended to a Borel measure on X, called Gaussian measure on X.*

Proof. We follow Kallianpur[1]. Let $\{\xi_n\}$ be a sequence of independent standard Gaussian random variables on some probability space $(\Omega, \mathcal{F}, m)$. Since the norm $\|\cdot\|$ is μ-measurable, there exists a sequence of finite dimensional projections $\{P_n\}$ of H such that $P_n \uparrow I$ (I is the identity operator) and for any $P \in \mathcal{P}$ orthogonal to P_n,

$$\mu\{x \in H : \|Px\| > 2^{-n}\} < 2^{-n} .$$

We take an orthonormal base $\{e_n\}$ of H such that $\{e_1, \cdots, e_{n_k}\}$ is an orthonormal base of $P_k(H)$. Put

$$\eta_k(\omega) = \sum_{j=1}^{n_k} \xi_j(\omega) e_j$$

Then

$$\eta_{k+1} - \eta_k = \sum_{j=n_k+1}^{n_{k+1}} \xi_j(\omega) e_j .$$

Since $P_{k+1}x - P_k x = \sum_{j=n_k+1}^{n_{k+1}} (x, e_j) e_j$, and $\forall E \in \mathcal{B}(\mathbb{R}^{n_{k+1}-n_k})$,

$$\begin{aligned} &m\{\omega : (\xi_{n_k+1}(\omega), \cdots, \xi_{n_{k+1}}(\omega)) \in E\} \\ &= \mu\{x \in H : ((e_{n_k+1}, x), \cdots (e_{n_{k+1}}, x)) \in E\}, \end{aligned}$$

we have

$$m(\|\eta_{k+1} - \eta_k\| > 2^{-k}) = \mu\{x \in H : \|P_{k+1}x - P_k x\| > 2^{-k}\} < 2^{-k} .$$

Thus $\{\eta_k\}$ converges in probability to some X-valued random element η. Let ν be the distribution of η, i.e., $\nu = m \circ \eta^{-1}$. Then for any $z \in X^*$,

$$\begin{aligned} \hat{\nu}(z) &= \int_X e^{i(x,z)} \nu(dx) = \int_\Omega e^{i(\eta(\omega), z)} m(d\omega) \\ &= \lim_{k \to \infty} \int_\Omega \exp\Big\{ i\Big(\sum_{j=1}^{n_k} \xi_j(\omega) e_j, z\Big) \Big\} m(d\omega) \\ &= \lim_{k \to \infty} \prod_{j=1}^{n_k} e^{-(e_j, z)^2/2} = e^{-|z|^2/2} = \hat{\mu}^*(z) . \end{aligned}$$

Thus μ^* coincides with ν on $\mathcal{R}(X)$ and ν is an extension of μ^*. ∎

Definition 4.17 Let X be a separable Banach space, μ a Borel probability measure on X. If for any $z \in X^*$, $(\cdot, z)$ is a normal random variable on X with zero mean, then μ is called a *symmetric Gaussian measure* on X and $(X, \mathcal{B}(X), \mu)$ is called a *Gaussian measure space.*

Let $(X, \mathcal{B}(X), \mu)$ be a Gaussian measure space, H a Hilbert space densely imbedded in X. Assume that the norm $\|\cdot\|$ on X is weaker than the norm $|\cdot|$ on H when restricted to H. If we identify the dual of H with respect to itself, then the dual X^* of X can be regarded as a subspace of H. If furthermore the characteristic functional of μ is $\hat{\mu}(z) = \exp\{-\frac{1}{2}|z|^2\}$, $z \in X^*$, then (H, X, μ) is called an *abstract Wiener space.*

It can be proved that for any real separable Banach space X, there exists some dense linear subspace H of X, such that H is a Hilbert space with a norm stronger than that of X when restricted to H and that this norm on H is measurable with respect to the standard Gaussian cylinder measure on H. Thus by Theorem 4.16, there exists a unique Gaussian measure μ on X (which is the extension of the lifting to X of the cylinder measure on H) such that (H, X, μ) is an abstract Wiener space. One can further prove (see Kuo[1]):

Theorem 4.18 *If (H, X, μ) is an abstract Wiener space, then there exists another Banach space Y such that $Y \hookrightarrow X$ is compact and (H, Y, μ) is still an abstract Wiener space.*

The following result shows that the classical Wiener space together with its Cameron-Martin subspace constitute an abstract Wiener space.

Theorem 4.19 *Let $X = C_0([0,1]; \mathbb{R}^d)$ be the set of all $\mathbb{R}^d$-valued continuous functions on $[0,1]$, null at zero, $\|\cdot\|$ the supremum norm on X, i.e.,*

$$h = (h_1, \cdots, h_d) \in X, \quad \|h\| \equiv \sum_{i=1}^d \sup_{0 \le t \le 1} |h_i(t)| .$$

Put

$$H \equiv \Big\{ h \in X : h_i \text{ is absolutely continuous}, \sum_{i=1}^d \int_0^1 \dot{h}_i(s)^2 ds < \infty \Big\}$$

(H is called the Cameron-Martin space). Define an inner product on H as follows:

$$(h, g) = \sum_{i=1}^d \int_0^1 \dot{h}_i(s) \dot{g}_i(s) ds , \quad h, g \in H. \tag{4.38}$$

The norm on H is denoted by $|\cdot|$. Let μ be a standard Wiener measure on X (i.e., the distribution of the d-dimensional standard Brownian motion). Then (H, X, μ) is an abstract Wiener space.

Proof. First, the norm $\|\cdot\|$ of X is weaker than the norm $|\cdot|$ of H when restricted to H, and H is dense in X. In the sequel, we shall identify the dual X^*

with a subset of H via (4.35). By Riesz' representation theorem, X^* is the set of all signed-measures on $[0,1]$ with d components, the canonical bilinear form on $X \times X^*$ is

$$\langle x, \nu \rangle = \sum_{i=1}^{d} \int_0^1 x_i(s) d\nu_i(s) . \tag{4.39}$$

Put

$$g_i^\nu(t) = \nu_i((0,1])t - \int_0^t \nu_i((0,s])ds . \tag{4.40}$$

Then $g^\nu = (g_1^\nu, \cdots, g_d^\nu) \in H$, and

$$\dot{g}_i^\nu(t) = \nu_i((0,1]) - \nu_i((0,t]) , \; 0 \le t \le 1 . \tag{4.41}$$

In particular, $\dot{g}_i^\nu(1) = 0$. Hence by the integration by parts formula for functions of bounded variation,

$$\begin{aligned} \langle h, \nu \rangle &= -\sum_{i=1}^{d} \int_0^1 h_i(s) d\dot{g}_i^\nu(t) = \sum_{i=1}^{d} \int_0^1 \dot{g}_i^\nu(s)\dot{h}_i(s)ds \\ &= (h, g^\nu)_1 \qquad \forall h \in H . \end{aligned} \tag{4.42}$$

Thus we can identify ν in X^* with g^ν in H. On the other hand, put $B_t(x) = x(t)$, $x \in X$. Then under μ, $(B_t, 0 \le t \le 1)$ is a standard d-dimensional Brownian motion. Now by the integration by parts formula for stochastic integrals,

$$\sum_{j=1}^{d} \int_0^1 \dot{g}_j^\nu(s) dB_s^j(x) = -\sum_{j=1}^{d} \int_0^1 x_j(s) d\dot{g}_j^\nu(s) = \langle x, \nu \rangle , \; \mu\text{-a.e. } x .$$

Consequently, we have

$$\begin{aligned} \hat{\mu}(\nu) &= \int_X e^{i\langle x,\nu\rangle} \mu(dx) = \mathbf{E}_\mu\Big[\exp\Big\{i \int_0^1 \dot{g}^\nu(s) dB_s\Big\}\Big] \\ &= \exp\Big\{-\frac{1}{2}\sum_{j=1}^{d} \int_0^1 \dot{g}_j^\nu(s)^2 ds\Big\} = \exp\{-\tfrac{1}{2}|g^\nu|^2\} . \end{aligned}$$

By definition, (H, X, μ) is an abstract Wiener space. ∎

We conclude this section with an important result due to Fernique about the symmetric Gaussian measure on Banach space.

Theorem 4.20 *Let E be a separable Banach space, μ a symmetric Gaussian measure on $(E, \mathcal{B}(E))$. Then there exists $\lambda > 0$ such that*

$$\int_E e^{\lambda\|x\|^2} \mu(dx) < \infty . \tag{4.43}$$

Proof. Let X and Y be two independent E-valued stochastic elements on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with the same distribution μ. Put

$$\tilde{X} = \frac{1}{\sqrt{2}}(X + Y), \quad \tilde{Y} = \frac{1}{\sqrt{2}}(X - Y).$$

Clearly, $\tilde{X}$ and $\tilde{Y}$ are independent, and both have distribution μ. Let $t \ge s \ge 0$. Then

$$\begin{aligned} \mathbf{P}(\|X\| \le s)\mathbf{P}(\|X\| > t) &= \mathbf{P}(\|\tilde{Y}\| \le s)\mathbf{P}(\|\tilde{X}\| > t) \\ &= \mathbf{P}\Big(\frac{\|X - Y\|}{\sqrt{2}} \le s\Big)\mathbf{P}\Big(\frac{\|X + Y\|}{\sqrt{2}} > t\Big) \\ &= \mathbf{P}\Big(\frac{\|X - Y\|}{\sqrt{2}} \le s, \frac{\|X + Y\|}{\sqrt{2}} > t\Big) \\ &\le \mathbf{P}(|\,\|X\| - \|Y\|\,| \le \sqrt{2}s, \|X\| + \|Y\| > \sqrt{2}t) \\ &\le \mathbf{P}\Big(\|X\| > \frac{t-s}{\sqrt{2}}, \|Y\| > \frac{t-s}{\sqrt{2}}\Big) \\ &= \Big[\mathbf{P}\Big(\|X\| > \frac{t-s}{\sqrt{2}}\Big)\Big]^2 . \end{aligned} \tag{4.44}$$

Fix $r > 0$ and put $t_0 = r$, $t_{n+1} = r + \sqrt{2}t_n$, $n \ge 1$, and define

$$\alpha_n(r) = \frac{\mathbf{P}(\|X\| > t_n)}{\mathbf{P}(\|X\| \le r)} , \quad n = 0, 1, 2, \cdots .$$

Then by (4.44),

$$\begin{aligned} \alpha_{n+1}(r) &= \frac{\mathbf{P}(\|X\| > r + \sqrt{2}t_n)}{\mathbf{P}(\|X\| \le r)} \\ &\le \Big[\frac{\mathbf{P}(\|X\| > t_n)}{\mathbf{P}(\|X\| \le r)}\Big]^2 = \alpha_n(r)^2 , n = 0, 1, 2, \cdots . \end{aligned}$$

Consequently, $\alpha_n(r) \le \exp\{2n \log \alpha_0(r)\}$, $n = 0, 1, \cdots$. Moreover, since $(\sqrt{2})^{n+4}r > t_n$,

$$\begin{aligned} \mathbf{P}(\|X\| > (\sqrt{2})^{n+4}r) &< \mathbf{P}(\|X\| > t_n) = \alpha_n(r)\mathbf{P}(\|X\| \le r) \\ &\le \exp\{2n \log \alpha_0(r)\} , \quad n = 0, 1, 2, \cdots . \end{aligned}$$

Thus for $\lambda > 0$, putting

$$\Sigma_n = \{x \in E : (\sqrt{2})^{n+4}r < \|x\| \le (\sqrt{2})^{n+5}r\} ,$$

we have

$$\begin{aligned} \int_{\|x\| > 4r} e^{\lambda\|x\|^2} \mu(dx) &= \sum_{n=0}^{\infty} \int_{\Sigma_n} e^{\lambda\|x\|^2} \mu(dx) \\ &\le \sum_{n=0}^{\infty} \mathbf{P}(\|X\| > (\sqrt{2})^{n+4}r)\exp\{\lambda r^2 2^{n+5}\} \\ &\le \sum_{n=0}^{\infty} \exp\{2n(\log \alpha_0(r) + 32\lambda r^2)\} . \end{aligned}$$

First take r sufficiently large such that $\mathbb{P}(\|X\| > r) < e^{-1}\mathbb{P}(\|X\| \leq r)$, and then take λ sufficiently small such that

$$\log \frac{\mathbb{P}(\|X\| > r)}{\mathbb{P}(\|X\| \leq r)} + 32\lambda r^2 \leq -1.$$

Since $2n \leq 2^n$, we have

$$\int_E e^{\lambda\|x\|^2}\mu(dx) \leq e^{16\lambda r^2} + \frac{e^2}{e^2 - 1}.$$

∎

Chapter II

Malliavin Calculus

The stochastic calculus of variation initiated by P. Malliavin is a kind of infinite dimensional differential analysis on the Wiener space. Since N. Wiener constructed in 1923 a mathematical model for Brownian motion, namely the Wiener measure on the space of continuous functions, many attempts have been made to develop a theory of differential analysis for Wiener functionals. Unfortunately, they were not successful since most usual functionals such as Itô integrals and solutions of Itô stochastic differential equations may be not differentiable in the sense of Fréchet, even not continuous as functionals on the Wiener space. In 1976, by virtue of the quasi-invariance of Wiener measure, P. Malliavin introduced a kind of weak differential calculus for Wiener functionals such that the above mentioned important functionals became smooth under his sense of differentiation and thus opened up a new prospect. Using this kind of calculus, he investigated the smoothness of densities of Wiener functionals, invented a nice probabilistic proof to the celebrated Hörmander's theorem on hypoellipticity of differential operators and thus received widespread attention from mathematical society. This kind of differentiation was defined by "perturbation" of Brownian paths, hence obtained the name "*stochastic calculus of variation*" and now popularly known as "*Malliavin calculus*".

This kind of differential structure on Wiener space is completely determined by the Cameron-Martin subspace. K. Itô's recent work [3,4] showed that one could derive the calculus of variation only from a separable Hilbert space without any other additional structures. One of the advantages of this basic framework is that one can choose different models to fit different practical problems to be solved. Malliavin[5] called this framework as Gaussian probability space. We shall adopt this point of view in the present book and develope the basic theory of infinite dimensional stochastic analysis under this framework.

§1. Gaussian probability spaces and Wiener chaos decomposition

1.1 Functionals on Gaussian probability spaces

Let H be a real separable Hilbert space equipped with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. By Kolmogorov's theorem, there exist a probability space $(\Omega,\mathcal{F},\mu)$ and on which a family of Gaussian random variables $\mathcal{H}=\{W_h, h\in H\}$ such that

$$E[W_h]=0;\ E[W_hW_g]=(h,g)_H,\quad \forall h,g\in H. \tag{1.1}$$

It follows that the map $h\mapsto W_h$ is a linear isometry from H into $L^2(\Omega,\mathcal{F},\mu)$ so that H is isomorphic to the closed subspace $\mathcal{H}$ of $L^2(\Omega,\mathcal{F},\mu)$.

Definition 1.1 Let $(\Omega,\mathcal{F},\mu)$ be a complete probability space, H be a real separable Hilbert space and $\mathcal{H}=\{W_h, h\in H\}$ be a family of Gaussian random variables satisfying eq. (1.1). Then $(\Omega,\mathcal{F},\mu;H)$ is called a *Gaussian probability space*.

Here are some examples:

Example 1. (finite dimensional Gaussian space) Let $\mu=\gamma^n$ be standard Gaussian measure on $\mathbb{R}^n$:

$$\mu(dx)=(2\pi)^{-n/2}\exp\{-\tfrac{1}{2}|x|^2\}dx, \tag{1.2}$$

$\mathcal{F}$ be the completion of $\mathcal{B}(\mathbb{R}^n)$ with respect to μ. Then $(\mathbb{R}^n,\mathcal{F},\mu)$ is a complete probability space. Taking $H=\mathbb{R}^n, \forall h\in H$, define $W_h(x)\equiv h\cdot x=\sum_{k=1}^n h_kx_k$. Then $\mathcal{H}=\{W_h, h\in H\}$ is a family of Gaussian random variables satisfying eq. (1.1). Hence, $(\mathbb{R}^n,\mathcal{F},\mu;\mathbb{R}^n)$ is a Gaussian probability space.

Example 2. (classical Wiener space) Let $W=C^0[0,1]$ be the Banach space of all real valued continuous functions on $[0,1]$ such that $w(0)=0$, equipped with the norm

$$\|w\|_W\equiv\sup_{0\le t\le 1}|w(t)|, \tag{1.3}$$

μ be the Wiener measure and $\mathcal{F}$ the μ-completion of $\mathcal{B}(W)$. Taking $H=L^2[0,1]$, $\forall h\in H$, define

$$W_h(w)\equiv\int_0^1 h(t)dw(t) \tag{1.4}$$

to be the Wiener integral. Then $\mathcal{H}=\{W_h, h\in H\}$ satisfies eq. (1.1) and $(W,\mathcal{F},\mu;H)$ is a Gaussian probability space.

For any $h\in H$, denote $\tilde{h}(t)=\int_0^t h(s)ds\ (0\le t\le 1)$. Then $\tilde{h}\in W$, and

$$\begin{aligned}\|\tilde{h}\|_W&=\sup_{0\le t\le 1}\left|\int_0^t h(s)ds\right|\\ &\le\sup_{0\le t\le 1}\left(t\int_0^t|h(s)|^2ds\right)^{1/2}\\ &\le\left(\int_0^1|h(s)|^2ds\right)^{1/2}=\|h\|_H.\end{aligned}$$

Consequently, the map $J:h\mapsto\tilde{h}$ is a continuous linear injection from H into W such that $\tilde{H}\equiv J(H)$ is dense in W. $\tilde{H}$ is called the *Cameron-Martin subspace* of W.

Denote by W^* the dual space of W. By identifying H^* with H, we then have (see Theorem I.4.19)

$$W^*\hookrightarrow H^*\cong H\hookrightarrow W. \tag{1.5}$$

Example 3 (abstract Wiener space). Let X be a separable Banach space, H be a separable Hilbert space continuously and densely embeded into X. Denote the embeding map by $J:H\to X$. Then X^* is continuously and densely embeded into $H^*\cong H$ by its dual map J^*. Hence

$$X^*\hookrightarrow H^*\cong H\hookrightarrow X. \tag{1.6}$$

Let μ be a Gaussian measure on X satisfying that

$$\int_X\exp\{i\langle l,x\rangle\}\mu(dx)=\exp\{-\tfrac{1}{2}\|J^*l\|_H^2\},\quad\forall l\in X^*, \tag{1.7}$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $X^*\times X$. (H,X,μ) is called an *abstract Wiener space* (cf. Definition I.4.17). The classical Wiener space in Example 2 is a particular case of abstract Wiener space.

Let $\mathcal{F}$ be the μ-completion of $\mathcal{B}(X)$. $\forall l\in X^*$, define $W_l(x)\equiv\langle l,x\rangle$. It follows from eq. (1.7) that $\{W_l, l\in X^*\}$ is a family of Gaussian random variables on $(X,\mathcal{F},\mu)$ satisfying that

$$E[W_l]=0;\ E[W_lW_{l'}]=(J^*l,J^*l')_H,\quad\forall l,l'\in X^*.$$

Consequently the map $J^*l\mapsto W_l$ is a linear isometry from $J^*(X^*)$ into $L^2(X,\mathcal{F},\mu)$. Since $J^*(X^*)$ is dense in H, it can be extended to an isometry from H into $L^2(X,\mathcal{F},\mu)$ satisfying eq. (1.1). Therefore, $(X,\mathcal{F},\mu;H)$ is a Gaussian probability space.

Example 4 (white noise space). Let $H=L^2(\mathbb{R}),\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^*(\mathbb{R})$ be spaces of Schwartz rapidly decreasing C^∞ functions and tempered distributions respectively. We then have

$$\mathcal{S}(\mathbb{R})\hookrightarrow L^2(\mathbb{R})\hookrightarrow\mathcal{S}^*(\mathbb{R}).$$

$\mathcal{S}(\mathbb{R})$ being a Fréchet nuclear space (cf. example in Ch. I §3), by the Minlos theorem, there exists a unique Gaussian measure μ on $\mathcal{B}(\mathcal{S}^*(\mathbb{R}))$ such that $\forall\xi\in\mathcal{S}(\mathbb{R})$,

$$\int_{\mathcal{S}^*(\mathbb{R})}\exp\{i\langle\omega,\xi\rangle\}\mu(d\omega)=\exp\{-\tfrac{1}{2}\|\xi\|_H^2\}, \tag{1.8}$$

where $\langle\omega,\xi\rangle$ is the canonical bilinear form on $\mathcal{S}^*(\mathbb{R})\times\mathcal{S}(\mathbb{R})$ (cf. Theorem I.4.7). Denote by $\mathcal{F}$ the μ-completion of $\mathcal{B}(\mathcal{S}^*(\mathbb{R}))$. $\forall\xi\in\mathcal{S}(\mathbb{R})$, define $W_\xi(\omega)\equiv\langle\omega,\xi\rangle$. As in Example 3, the map $\xi\mapsto W_\xi$ can be extended to a linear isometry from

$L^2(\mathbb{R})$ into $L^2(\mathcal{S}^*(\mathbb{R}),\mathcal{F},\mu)$ so that $(\mathcal{S}^*(\mathbb{R}),\mathcal{F},\mu;L^2(\mathbb{R}))$ become a Gaussian probability space, namely *white noise space*. μ is referred to as *white noise measure*. We shall discuss white noise analysis on this space in Chapter IV.

Definition 1.2 Let $(\Omega,\mathcal{F},\mu;H)$ be a Gaussian probability space, $\mathcal{N}$ be the totality of all μ-null sets, $\mathcal{F}^0$ be the σ- algebra generated by $\mathcal{H}=\{W_h, h\in H\}$ If $\mathcal{F}=\sigma(\mathcal{F}^0\cup\mathcal{N})$. Then the space is called *irreducible*.

Consider functionals of the following forms:

$$F(\omega)=f(W_{h_1}(\omega),\cdots,W_{h_n}(\omega)),\quad n\in\mathbb{N},h_1,\cdots,h_n\in H. \tag{1.9}$$

If f is a polynomial of n variables, then F is called a *polynomial functional*. Denote by $\mathcal{P}$ the totality of all polynomial functionals, i.e. the algebra generated by $\mathcal{H}$; If f is a tempered C^∞ function, that is, an infinitely differentiable function which together with its derivatives are polynomially bounded, then F is called a *smooth functional*. The set of smooth functionals is denoted by $\mathcal{S}_M$.

Let E be a separable Hilbert space equipped with inner product $(\cdot,\cdot)_E$ and norm $|\cdot|_E$. For $p\in[1,\infty)$, $L^p(E)\equiv L^p(\Omega,\mathcal{F},\mu;E)$ stands for the Banach space of equivalence classes of E-valued measurable functionals with norm

$$\|F\|_p\equiv\Big(\int_\Omega|F(\omega)|_E^p\,\mu(d\omega)\Big)^{1/p}. \tag{1.10}$$

For $E=\mathbb{R}$ we simply denote $L^p(E)$ by L^p.

Consider E-valued functionals of the following forms:

$$F(\omega)=\sum_{k=1}^m F_k(\omega)e_k,\quad m\in\mathbb{N},e_1,\cdots,e_m\in E \tag{1.11}$$

If $F_1,\cdots,F_m\in\mathcal{P}$ (respectively, $\mathcal{S}_M$), then they are called *E-valued polynomials* (respectively, *E-valued smooth functionals*) whose totality is denoted by $\mathcal{P}(E)$ (respectively, $\mathcal{S}_M(E)$).

Remark. By Gram-Schmidt orthogonalization procedure, modifying forms of f if necessary, one can always suppose in eq. (1.9) that

$$(h_i,h_j)_H=\delta_{ij}\quad(i,j=1,\cdots,n),$$

hence the order of polynomial is determined by F and the joint distribution of $W_{h_1},\cdots,W_{h_n}$ is standard Gaussian on $\mathbb{R}^n$. Similarly, without loss of generality one may assume in eq. (1.11) that

$$(e_i,e_j)_E=\delta_{ij}\quad(i,j=1,\cdots,m).$$

Proposition 1.3 *If $(\Omega,\mathcal{F},\mu;H)$ is an irreducible Gaussian probability space and E is a separable Hilbert space, then*

$$\mathcal{P}(E)\subset\mathcal{S}_M(E)\subset L^p(E)\quad(1\le p<\infty),$$

and $\mathcal{P}(E)$ is dense in $L^p(E)$.

Proof. By constructing an auxiliary Gaussian probability space $(\Omega',\mathcal{F}',\mu';E)$ from E, one can treat the E-valued functional (1.11) as a real valued functional on the product space $(\Omega\times\Omega',\mathcal{F}\times\mathcal{F}',\mu\times\mu')$,

$$F'(\omega,\omega')=\sum_{k=1}^m F_k(\omega)W_{e_k}(\omega'), \tag{1.12}$$

hence it suffices to consider the case $E=\mathbb{R}$.

Choose an orthonormal base $\{h_j\}$ of H. Then $\{W_{h_j}\}$ is a sequence of independent standard Gaussian variables generating σ-algebra $\mathcal{F}$. For any $n\in\mathbb{N}, t_1,\cdots,t_n\in\mathbb{R}$ and $c>0$, we have

$$\mathbb{E}\Big[\exp\Big\{c\sum_{j=1}^n|t_jW_{h_j}|\Big\}\Big]<\infty. \tag{1.13}$$

It follows that $\mathcal{P}\subset\mathcal{S}_M\subset L^p$ $(1\le p<\infty)$.

Suppose that $\mathcal{P}$ is not dense in L^p. Then there exists $\xi\in L^q$ with $q^{-1}+p^{-1}=1$ such that

$$\mathbb{E}[\xi F]=0,\quad\forall F\in\mathcal{P}. \tag{1.14}$$

Since $q>1$, it follows from (1.13) that

$$\mathbb{E}\Big[|\xi|\exp\Big\{\sum_{j=1}^n|t_jW_{h_j}|\Big\}\Big]<\infty,$$

hence by (1.14) we have

$$\mathbb{E}\Big[\xi\exp\Big\{i\sum_{j=1}^n t_jW_{h_j}\Big\}\Big]=\sum_{m=0}^\infty\frac{i^m}{m!}\mathbb{E}\Big[\xi\Big(\sum_{j=1}^n t_jW_{h_j}\Big)^m\Big]=0. \tag{1.15}$$

Let $\xi_n\equiv\mathbb{E}[\xi|W_{h_1},\cdots,W_{h_n}]$. Then there exists a measurable function g on $\mathbb{R}^n$ such that $\xi_n=g(W_{h_1},\cdots,W_{h_n})$. By (1.15) we have

$$\mathbb{E}\Big[\xi_n\exp\Big\{i\sum_{j=1}^n t_jW_{h_j}\Big\}\Big]=\mathbb{E}\Big[\xi\exp\Big\{i\sum_{j=1}^n t_jW_{h_j}\Big\}\Big]=0,$$

namely

$$\int_{\mathbb{R}^n}g(x)\exp\Big\{i\sum_{j=1}^n t_jx_j\Big\}\gamma^n(dx)=0.$$

It follows from the uniqueness of Fourier transformation that $g=0$ a.e. $[\gamma^n]$, hence $\xi_n=0$ a.s.. However, according to the convergence theorem for martingales, $\xi_n\to\xi$ a.s., therefore $\xi=0$ a.s. which is a contradiction. Hence $\mathcal{P}$ is dense in L^p for all p. ∎

1.2 Numerical models

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$ be a one-dimensional Gaussian space, $\mathcal{P}$ be the totality of real polynomials. Then $\mathcal{P}$ is dense in $L^2(\mathbb{R}, \gamma)$. Define on $\mathcal{P}$ the following operators:

$$\partial \equiv \frac{d}{du}, \quad \partial^* \equiv -\frac{d}{du} + u\cdot. \tag{1.16}$$

Then

$$\partial^*\partial = -\frac{d^2}{du^2} + u\frac{d}{du}. \tag{1.17}$$

By the formula of integration by part, $\forall \varphi, \psi \in \mathcal{P}$, we have

$$(\partial\varphi, \psi)_{L^2(\mathbb{R},\gamma)} = (\varphi, \partial^*\psi)_{L^2(\mathbb{R},\gamma)}, \tag{1.18}$$

hence ∂, ∂^* and $\partial^*\partial$ can be extended to closed operators in $L^2(\mathbb{R}, \gamma)$ such that ∂ and ∂^* are mutually adjoint and $\partial^*\partial$ is selfadjoint (namely, number operator in one dimension). It is clear by eq. (A.6) of Appendix A that Hermite polynomials $H_n(u)$ are eigen functions of $\partial^*\partial$:

$$\partial^*\partial H_n = nH_n, \quad n \in \mathbb{N}_0. \tag{1.19}$$

By eqs. (A.5) and (A.7), we have the recursion formula:

$$H_{n+1} = \partial^* H_n, \quad n \in \mathbb{N}_0. \tag{1.20}$$

Therefore

$$H_n = (\partial^*)^n 1, \quad n \in \mathbb{N}_0, \tag{1.21}$$

and

$$\begin{aligned}(H_n, H_m)_{L^2(\mathbb{R},\gamma)} &= (H_n, (\partial^*)^m 1)_{L^2(\mathbb{R},\gamma)} \\ &= (\partial^m H_n, 1)_{L^2(\mathbb{R},\gamma)}.\end{aligned}$$

In case $m > n$, since H_n is a polynomial of order n, the last term vanishes; when $m = n$, by eq. (A.5) it becomes $n!$. Hence we come again to eq. (A.10) which means that $\{(n!)^{-1/2}H_n\}$ constitute an orthonormal base of $L^2(\mathbb{R}, \gamma)$.

Let $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$ be infinite product space of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma)$. For any sequence of nonnegative integers $\alpha = \{\alpha_j\}_{j\in\mathbb{N}}$, denote $|\alpha| = \sum_j \alpha_j$, $\alpha! = \prod_j(\alpha_j!)$ and by Λ the set of sequences α for which $|\alpha|$ are finite. For $\alpha = \{\alpha_j\} \in \Lambda$ and $x = \{x_j\} \in \mathbb{R}^\infty$ define

$$H_\alpha(x) \equiv \prod_j H_{\alpha_j}(x_j). \tag{1.22}$$

Now that $\{\alpha_j\}$ contains all but finite zero elements and $H_0(u) \equiv 1$, the product in (1.22) is in fact a finite one. Similarly to one-dimensional case, we have

Theorem 1.4 *For any $\alpha, \beta \in \Lambda$, it holds that*

$$\int_{\mathbb{R}^\infty} H_\alpha(x)H_\beta(x)\gamma^\infty(dx) = \alpha!\delta_{\alpha\beta}, \tag{1.23}$$

hence $\{(\alpha!)^{-1/2}H_\alpha : \alpha \in \Lambda\}$ constitute an orthonormal base of $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$. Moreover, the following product formula holds:

$$H_\alpha(x)H_\beta(x) = \sum_{\substack{\kappa\le\alpha\wedge\beta \\ \kappa\in\Lambda}} \kappa!\binom{\alpha}{\kappa}\binom{\beta}{\kappa}H_{\alpha+\beta-2\kappa}(x), \tag{1.24}$$

where $\binom{\alpha}{\kappa} \equiv \prod_j \binom{\alpha_j}{\kappa_j}$, $\alpha + \beta - 2\kappa \equiv \{\alpha_j + \beta_j - 2\kappa_j\}_{j\in\mathbb{N}}$, and $\kappa \le \alpha\wedge\beta$ stands for $\kappa_j \le \alpha_j \wedge \beta_j$, $\forall j \in \mathbb{N}$. Denote $i = \sqrt{-1}$, $(x \pm iy)^\alpha \equiv \prod_j (x_j \pm iy_j)^{\alpha_j}$. Then

$$H_\alpha(x) = \int_{\mathbb{R}^\infty} (x \pm iy)^\alpha \gamma^\infty(dy). \tag{1.25}$$

Proof. Due to the independence of components, eqs. (1.23) and (1.25) are consequences of eqs. (A.10) and (A.11) respectively. Eqs. (1.24) can be proved by a direct computation using eq. (A.8). ∎

For any $j \in \mathbb{N}$, denote by ∂_j and ∂_j^* differentiation with respect to j-th component and its adjoint operator respectively. Define

$$\mathcal{L} \equiv -\sum_{j=1}^\infty \partial_j^*\partial_j \tag{1.26}$$

to be infinite dimensional Ornstein-Uhlenbeck operator. For any $\alpha \in \Lambda$, denote $\partial_\alpha^* \equiv \prod_j (\partial_j^*)^{\alpha_j}$, similar to eq. (1.21), we have

$$H_\alpha = \partial_\alpha^* 1, \quad \alpha \in \Lambda, \tag{1.27}$$

and H_α are eigen functions for $\mathcal{L}$ whose characteristic equations are

$$\mathcal{L}H_\alpha = -|\alpha|H_\alpha, \quad \alpha \in \Lambda. \tag{1.28}$$

Let $\Lambda_n \equiv \{\alpha \in \Lambda : |\alpha| = n\}, \mathcal{H}_n$ be the closed subspace in $L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty)$ generated by $\{H_\alpha : \alpha \in \Lambda_n\}$. Then we have the following orthogonal decomposition:

$$L^2(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty) = \bigoplus_{n=0}^\infty \mathcal{H}_n. \tag{1.29}$$

Now let $(\Omega, \mathcal{F}, \mu; H)$ be any irreducible Gaussian probability space. By fixing a base $\{h_j\}_{j\in\mathbb{N}}$ in H, one sees that H is isometrically isomorphic to l^2, the space of square summable numerical sequences. For any $\omega \in \Omega$, let

$T\omega \equiv \{W_{h_j}(\omega)\}_{j\in\mathbb{N}}$. Then $T:\Omega\to\mathbb{R}^\infty$ is $\mathcal{F}/\mathcal{B}^\infty$ measurable and measure preserving: $\gamma^\infty = \mu\circ T^{-1}$. For $1\le p\le\infty, \varphi\in L^p(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty)$, let $T_*\varphi(\omega)\equiv\varphi(T\omega)$. Then $T_*:L^p(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty)\to L^p(\Omega,\mathcal{F},\mu)$ is an isomorphism. It is also an algebra homomorphism when restricted to corresponding spaces $\mathcal{P}$ of polynomial functionals. The Gaussian probability space $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$ is referred to as a *numerical model* of $(\Omega,\mathcal{F},\mu;H)$.

Obviously, numerical model depends on the choice of base in H. However, there are some properties of Gaussian probability spaces which are independent of choices of base. We call them the *intrinsic properties*. It is very convenient to investigate such properties by using a numerical model. For instance, the chaos decomposition is an intrinsic property.

Theorem 1.5 *Let $(\Omega,\mathcal{F},\mu;H)$ be an irreducible Gaussian probability space. For any base $\{h_j\}$ in H and $\alpha\in\Lambda$, define*

$$H_\alpha(\omega)\equiv\prod_j H_{\alpha_j}(W_{h_j}(\omega)). \tag{1.30}$$

Then $\{(\alpha!)^{-1/2}H_\alpha:\alpha\in\Lambda\}$ constitute a base of $L^2(\Omega,\mathcal{F},\mu)$. Let $\mathcal{H}_0\equiv\mathbb{R}$; for $n\ge1$, let $\mathcal{H}_n$ be the closed subspace generated by $\{H_\alpha:\alpha\in\Lambda_n\}$. Then

$$L^2(\Omega,\mathcal{F},\mu)=\bigoplus_{n=0}^\infty\mathcal{H}_n, \tag{1.31}$$

the decomposition is independent of choices of base in H. Moreover,

$$L^2(\Omega,\mathcal{F},\mu)\cong\Gamma(H), \tag{1.32}$$

namely, it is isomorphic to the symmetric Fock space over H.

Proof. Consider a numerical model $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$. Since $L^2(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty)$ is isomorphic to $L^2(\Omega,\mathcal{F},\mu)$, the existence of decomposition is evident by eq. (1.29). We have to prove that the decomposition is independent of choices of base in H.

Let $\{\widetilde{h}_j\}$ be another base of H. Define corresponding polynomial functionals $\{\widetilde{H}_\alpha\}$ by eq. (1.30) and obtain another decomposition

$$L^2(\Omega,\mathcal{F},\mu)=\bigoplus_{n=0}^\infty\widetilde{\mathcal{H}}_n.$$

If $\widetilde{h}_j=\sum_k a_{kj}h_k$ (convergence in H), then, by isomorphism, $W_{\widetilde{h}_j}=\sum_k a_{kj}W_{h_k}$ (convergence in L^2).

Since for centered Gaussian variables ξ_n and ξ we have

$$\mathbb{E}|\xi_n-\xi|^{2p}=(2p-1)!!(\mathbb{E}|\xi_n-\xi|^2)^p,\quad\forall p\in\mathbb{N},$$

L^2-convergence of $\{\xi_n\}$ implies their L^{2p}-convergence, and therefore L^2-convergence of their polynomials. Hence

$$\bigoplus_{k=0}^n\widetilde{\mathcal{H}}_k\subset\bigoplus_{k=0}^n\mathcal{H}_k,\quad\forall n\in\mathbb{N}.$$

By symmetry, the inverse inclusion also holds, therefore, $\widetilde{\mathcal{H}}_n=\mathcal{H}_n, \forall n\in\mathbb{N}$.

If we define

$$\widetilde{h}_\alpha=\widetilde{\bigotimes}_j h_j^{\otimes\alpha_j},\quad\alpha\in\Lambda, \tag{1.33}$$

then by Proposition I.2.7, $\{(\alpha!)^{-1/2}\widetilde{h}_\alpha:\alpha\in\Lambda_n\}$ constitute a base of $H^{\hat{\otimes}n}$, hence

$$\mathcal{H}_n\cong H^{\hat{\otimes}n},\quad n\in\mathbb{N}_0, \tag{1.34}$$

eq. (1.32) is proved. ∎

The decomposition is called *Wiener-Itô-Segal chaos decomposition* (especially, $\mathcal{H}_1$ is nothing but H), which means that the space of square integrable functionals on a Gaussian probability space is isomorphic to the symmetric Fock space over H. This is one of the most important facts in infinite dimensional stochastic analysis.

1.3 Multiple Wiener-Itô integral representation

Now we consider a special case, namely $H=L^2(T,\mathcal{B},\lambda)$, where $(T,\mathcal{B})$ is a measurable space and λ is a σ-finite non-atomic measure. It covers the case of classical Wiener spaces and white noise spaces mentioned before. We shall prove that, in this case, the chaos decomposition is represented as multiple Wiener-Itô integrals.

Firstly, we give a general definition of multiple Wiener-Itô integrals.

Let $\mathcal{B}_0\equiv\{A\in\mathcal{B}:\lambda(A)<\infty\}$ and define on $\mathcal{B}_0$ the following random set functions:

$$W(A)=W_{1_A},\quad A\in\mathcal{B}_0. \tag{1.35}$$

It follows from eq. (1.1) that $W(\cdot)$ as an $L^2(\Omega,\mathcal{F},\mu)$- valued random set function satisfies that, for all $A,B\in\mathcal{B}_0$,

1° $W(A)\sim N(0,\lambda(A))$;

2° $\mathbb{E}[W(A)W(B)]=\lambda(A\cap B)$.

Consequently, if $A_1,\cdots,A_n,\cdots$ are disjoint, then $W(A_1),\cdots,W(A_n),\cdots$ are mutually independent and if $\cup_nA_n\in\mathcal{B}_0$, then

$$W\Big(\bigcup_nA_n\Big)=\sum_nW(A_n)\quad(L^2-\text{convergence}).$$

Such a random set function is called *Gaussian orthogonal random measure* on $(T,\mathcal{B})$ with contructive measure λ. For $h\in H, W_h$ is just the stochastic integral of h with respect to this random measure:

$$W_h=\int_T h(t)W(dt). \tag{1.36}$$

Now we proceed to construct multiple stochastic integrals. Let $n \geq 1$, $f \in L^2(T^n, \mathcal{B}^n, \lambda^n)$ be of the following form:

$$f = \sum_{j_1,\cdots,j_n=1}^{N} a_{j_1\cdots j_n} 1_{A_{j_1}\times\cdots\times A_{j_n}}, \tag{1.37}$$

where $A_1, \cdots, A_N \in \mathcal{B}_0$ are disjoint, and if any two of indices $j_1, \cdots, j_n$ are equal, then $a_{j_1\cdots j_n} = 0$. We define its n-fold stochastic integral as:

$$I_n(f) \equiv \sum_{j_1,\cdots,j_n=1}^{N} a_{j_1\cdots j_n} W(A_{j_1})\cdots W(A_{j_n}). \tag{1.38}$$

It is easy to see that this definition is independent of representations of f and I_n is linear in f. Consider its symmetrization:

$$\tilde{f}(t_1, \cdots, t_n) \equiv \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}).$$

Since by definition, $I_n(f) = I_n(\tilde{f})$, without loss of generality we can assume that $f \in H^{\hat{\otimes} n}$, namely, f is symmetric with respect to $t_1, \cdots, t_n$:

$$a_{j_1\cdots j_n} = a_{\sigma(j_1)\cdots\sigma(j_n)}, \quad \forall\sigma \in \mathfrak{S}_n. \tag{1.39}$$

Proposition 1.6 *Under above stated conditions, $\forall n \in \mathbb{N}$, I_n uniquely extends to a linear isometry from $H^{\hat{\otimes} n}$ into $L^2(\Omega, \mathcal{F}, \mu)$ such that $\mathcal{R}(I_m) \perp \mathcal{R}(I_n)$ whenever $m \neq n$.*

$I_n(f)$ is referred to as n-fold Wiener-Itô stochastic integral of f with respect to random measure W and denoted by

$$I_n(f) = \int_{T^n} f(t_1, \cdots, t_n) W(dt_1)\cdots W(dt_n). \tag{1.40}$$

Proof. Let $f \in H^{\hat{\otimes} n}$, $g \in H^{\hat{\otimes} m}$, having forms (1.37) with same partition $\{A_1, \cdots, A_N\}$. Suppose that $n > m$. Since $W(\cdot)$ are independent centered random variables on disjoint sets, it follows that $\mathbb{E}[I_n(f)I_m(g)] = 0$.

Now suppose that $m = n$, g has form (1.37) except in which replacing $a_{j_1\cdots j_n}$ by $b_{j_1\cdots j_n}$. Since $\mathbb{E}[W(A_j)^2] = \lambda(A_j)$, $1 \leq j \leq n$, it follows that

$$\begin{aligned}\mathbb{E}[I_n(f)I_n(g)] &= (n!)^2 \sum_{j_1<\cdots<j_n} a_{j_1\cdots j_n} b_{j_1\cdots j_n} \lambda(A_{j_1})\cdots\lambda(A_{j_n}) \\ &= n!\,(f, g)_{H^{\otimes n}}.\end{aligned} \tag{1.41}$$

To prove that the set of functions of form (1.37) is dense in $H^{\hat{\otimes} n}$, it suffices to consider approximating indicator 1_A with $A = A_1 \times\cdots\times A_n$, $A_j \in \mathcal{B}_0$, $1 \leq j \leq n$. Since λ is non-atomic, $\forall\epsilon > 0$, every A_j $(1 \leq j \leq n)$ can be represented as union

of a finite number disjoint sets in $\mathcal{B}_0$ whose measures are less than ϵ. Denote these sets by $\{B_j\}_{1\leq j\leq M}$. Then

$$1_A = \sum_{j_1,\cdots,j_n=1}^{M} \epsilon_{j_1\cdots j_n} 1_{B_{j_1}\times\cdots\times B_{j_n}},$$

where $\epsilon_{j_1\cdots j_n} = 0$ or 1. Note that total measure of sets $B_{j_1} \times\cdots\times B_{j_n}$ which have repeated indices in $j_1, \cdots, j_n$ can be arbitrarily small along with ϵ, hence 1_A can be approximated in $H^{\otimes n}$ by functions of form (1.37). It follows from eq. (1.41) that I_n extends uniquely to a linear isometry from $H^{\hat{\otimes} n}$ into $L^2(\Omega, \mathcal{F}, \mu)$. ∎

For $f \in H^{\hat{\otimes} n}$, $g \in H^{\hat{\otimes} m}$, $1 \leq r \leq m \wedge n$, we define

$$\begin{aligned} &f \otimes_r g(t_1, \cdots, t_{m+n-2r}) \\ &\qquad \equiv \int_{T^r} f(t_1, \cdots, t_{n-r}, s) g(t_{n+1-r}, \cdots, t_{n+m-2r}, s) \lambda^r(ds). \end{aligned} \tag{1.42}$$

Then, $f \otimes_r g \in H^{\otimes(m+n-2r)}$, whose symmetrization is denoted by $f \hat{\otimes}_r g$. When $r = 0$, it reduces to $f \hat{\otimes} g$.

Proposition 1.7 *If $f \in H^{\hat{\otimes} n}$, $g \in H$, then*

$$I_n(f)I_1(g) = I_{n+1}(f \otimes g) + nI_{n-1}(f \otimes_1 g). \tag{1.43}$$

Proof. By linearity and continuity of I_n, it suffices to consider the case $f = \tilde{1}_{A_1\times\cdots\times A_n}$, where $A_1, \cdots, A_n \in \mathcal{B}_0$ are disjoint; $g = 1_A$, $A \in \mathcal{B}_0$, either disjoint from $A_1, \cdots, A_n$ or coincides with some A_j $(1 \leq j \leq n)$.

In the first case, $f \otimes_1 g = 0$, both sides of eq. (1.43) are equal to

$$W(A_1)\cdots W(A_n)W(A).$$

In the second case, without loss of generality we may assume that $A = A_1$, hence

$$f \otimes_1 g = \frac{1}{n}\lambda(A_1)\tilde{1}_{A_2\times\cdots\times A_n},$$

$$I_{n-1}(f \otimes_1 g) = \frac{1}{n}\lambda(A_1)W(A_2)\cdots W(A_n).$$

and there exists a sequence of partition for A_1, say $\{B_{m1}, B_{m2}, \cdots, B_{mk_m}\}_{m\in\mathbb{N}}$, such that

$$\lim_{m\to\infty} \sum_{j\neq k} 1_{B_{mj}\times B_{mk}} = 1_{A_1\times A_1}$$

holds in $L^2(\lambda^2)$. Therefore, it holds in $L^2(\Omega)$ that

$$\begin{aligned} &\lim_{m\to\infty} \sum_{j\neq k} W(B_{mj})W(B_{mk}) \\ &= \lim_{m\to\infty} \left\{ \Big(\sum_j W(B_{mj})\Big)^2 - \sum_j W(B_{mj})^2 \right\} \\ &= W(A_1)^2 - \lambda(A_1). \end{aligned}$$

Hence
$$I_{n+1}(f\otimes g)=(W(A_1)^2-\lambda(A_1))W(A_2)\cdots W(A_n),$$
which proves eq. (1.43). ∎

More generally, we have

Proposition 1.8 *If $f\in H^{\hat\otimes n}, g\in H^{\hat\otimes m}$, then*
$$I_n(f)I_m(g)=\sum_{r=0}^{n\wedge m} r!\binom{n}{r}\binom{m}{r}I_{n+m-2r}(f\hat\otimes_r g). \tag{1.44}$$

Proof. We prove it by induction on m and assume that $n\geq m$. For $m=1$, it reduces to eq. (1.43). Suppose that eq. (1.44) holds when m is replaced by $m-1$. Approximating by linear combinations, we may assume that $g=g_1\hat\otimes g_2$, where $g_1\in H^{\hat\otimes(m-1)}, g_2\in H$ such that $g_1\otimes_1 g_2=0$. By eq. (1.43) we have
$$I_m(g)=I_{m-1}(g_1)I_1(g_2).$$
Hence, by inductive hypothesis
$$\begin{aligned}I_n(f)I_m(g)&=\sum_{r=0}^{m-1} r!\binom{n}{r}\binom{m-1}{r}I_{n+m-1-2r}(f\otimes_r g_1)I_1(g_2)\\&=\sum_{r=0}^{m-1} r!\binom{n}{r}\binom{m-1}{r}[I_{n+m-2r}((f\hat\otimes_r g_1)\otimes g_2)\\&\qquad+(n+m-1-2r)I_{n+m-2r-2}((f\hat\otimes_r g_1)\otimes_1 g_2)]\\&=\sum_{r=0}^{m-1} r!\binom{n}{r}\binom{m-1}{r}I_{n+m-2r}((f\hat\otimes_r g_1)\otimes g_2)\\&\qquad+\sum_{r=1}^{m}(r-1)!\binom{n}{r-1}\binom{m-1}{r-1}(n+m+1-2r)\\&\qquad\times I_{n+m-2r}((f\hat\otimes_{r-1}g_1)\otimes_1 g_2).\end{aligned}$$
For $1\leq r\leq m$, we can verify that
$$m(f\hat\otimes_r g)=\frac{r(n+m-2r+1)}{n-r+1}((f\hat\otimes_{r-1}g_1)\otimes_1 g_2)+(m-r)(f\hat\otimes_r g_1)\hat\otimes g_2.$$
Substituting this into above expression, we obtain eq. (1.44). ∎

The following theorem shows the close relation between multiple stochastic integrals and Hermite polynomials.

Theorem 1.9 *If $h\in H=L^2(T)$ and $\|h\|=1$, then*
$$H_n(W_h)=I_n(h^{\otimes n}),\quad \forall n\in\mathbb{N}. \tag{1.45}$$
If $\{h_j\}_{j\in\mathbb{N}}$ is a base of H, then
$$H_\alpha=I_{|\alpha|}(\hat h_\alpha),\quad \forall\alpha\in\Lambda, \tag{1.46}$$

where H_α and $\hat h_\alpha$ are given by eqs. (1.30) and (1.33), respectively.

Proof. We prove eq. (1.45) by induction. For $n=1$ it reduces to eq. (1.36). Suppose that eq. (1.45) holds for $n\leq m$. By recursion formula (A.7) and eq. (1.43), we have
$$\begin{aligned}I_{m+1}(h^{\otimes(m+1)})&=I_m(h^{\otimes m})I_1(h)-mI_{m-1}(h^{\otimes(m-1)})\\&=H_m(W_h)W_h-mH_{m-1}(W_h)\\&=H_{m+1}(W_h),\end{aligned}$$
hence eq. (1.45) holds for $n=m+1$.

Since
$$H_\alpha=\prod_j H_{\alpha_j}(W_{h_j})=\prod_j I_{\alpha_j}(h_j^{\otimes\alpha_j}),$$
noting that for $j\neq k, r>0$, $h_j^{\otimes\alpha_j}\otimes_r h_k^{\otimes\alpha_k}=0$, by eq. (1.44) we know that
$$I_{\alpha_j}(h_j^{\otimes\alpha_j})I_{\alpha_k}(h_k^{\otimes\alpha_k})=I_{\alpha_j+\alpha_k}(h_j^{\otimes\alpha_j}\hat\otimes h_k^{\otimes\alpha_k}).$$
Therefore
$$H_\alpha=I_{|\alpha|}(\hat\otimes_j h_j^{\otimes\alpha_j})=I_{|\alpha|}(\hat h_\alpha),$$
and eq. (1.46) is proved. ∎

It is remarkable that, when $H=L^2(T,\mathcal{B},\lambda)$, the isomorphism (1.34) and (1.32) are represented as multiple stochastic integrals.

Theorem 1.10 *Let $(\Omega,\mathcal{F},\mu;H)$ be an irreducible Gaussian probability space, where $H=L^2(T,\mathcal{B},\lambda)$, λ being a σ-finite non-atomic measure on $(T,\mathcal{B})$. Then, any $F\in L^2(\Omega,\mathcal{F},\mu)$ has a unique orthogonal decomposition:*
$$F=\sum_{n=0}^{\infty}I_n(f_n), \tag{1.47}$$
where $f_n\in H^{\hat\otimes n}, n\geq 1, I_0(f_0)=\mathbb{E}[F]$, such that
$$\|F\|^2=(\mathbb{E}[F])^2+\sum_{n=1}^{\infty}n!\|f_n\|^2, \tag{1.48}$$
where $\|f_n\|$ stands for the norm in $L^2(T^n,\mathcal{B}^n,\lambda^n)$.

Proof. Straitforward consequence from Theorem 1.5, Proposition 1.6 and Theorem 1.9. ∎

Example. *Exponential functionals*
$$\mathcal{E}(h)=\exp\Big\{W_h-\tfrac{1}{2}\|h\|^2\Big\},\quad h\in H \tag{1.49}$$
have the following decomposition:
$$\mathcal{E}(h)=\sum_{n=0}^{\infty}\frac{1}{n!}I_n(h^{\otimes n}). \tag{1.50}$$

In fact, by eq. (1.45) we have

$$I_n(h^{\otimes n}) = \|h\|^n H_n(\|h\|^{-1} W_h).$$

In eq. (A.2) putting $t = \|h\|, u = \|h\|^{-1} W_h$, we obtain eq. (1.50). Note that in the isomorphism (1.32), exponential functionals correspond to exponential vectors (I.2.32). It follows from Proposition I.2.14 that exponential functionals $\{\mathcal{E}(h), h \in H\}$ are linearly independent and constitute a total set in $L^2(\Omega)$.

In the special case that $T = [0,1]$ and λ is the Lebesgue measure, we obtain a classical Wiener space. Here $W_t = W([0,t]), t \geq 0$, is just a Brownian motion and

$$I_n(f) = n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} \tag{1.51}$$

is just the n-fold iterated Itô integral.

§2. Differential calculus of functionals, gradient and divergence operators

2.1 Finite dimensional Gaussian probability spaces

In order to investigate differential calculus of functionals (random variables) on infinite dimensional Gaussian probability spaces, we first define the gradient, divergence and Ornstein-Uhlenbeck operators for polynomials or smooth functionals. By virtue of these operators, we then define a series of Sobolev norms. Finally, we extend their domains to corresponding Sobolev spaces. Two crucial facts in the procedure is the quasi-invariance of infinite-dimensional Gaussian measures and so-called hyper-contractivity of Ornstein-Uhlenbeck operators.

However, the analysis for polynomials or smooth functionals only involving a finite number of variables, it is essentially finite-dimensional analysis. For this reason we shall first discuss calculus on finite-dimensional Gaussian probability spaces.

As in Example 1 of previous paragraph, let $\mu = \gamma^n$ be standard Gaussian measure on $\mathbb{R}^n$, $(\mathbb{R}^n, \mathcal{F}, \mu; \mathbb{R}^n)$ be finite-dimensional Gaussian probability space, and E be any separable Hilbert space. Denote by $\mathcal{S}_M(E)$ the totality of E-valued smooth functionals. When $E = \mathbb{R}$ we simply write it as $\mathcal{S}_M$.

Let $\varphi \in \mathcal{S}_M$ be a smooth functional on $\mathbb{R}^n$. Its gradient $D\varphi \equiv \{\partial_j \varphi\}_{1 \leq j \leq n}$ is an $\mathbb{R}^n$-valued smooth functional, i.e. $D\varphi \in \mathcal{S}_M(\mathbb{R}^n)$. For $x, h \in \mathbb{R}^n$, the derivative of φ at x along direction h is defined to be

$$\begin{aligned} D_h \varphi(x) &= \lim_{\epsilon \to 0} \epsilon^{-1} [\varphi(x + \epsilon h) - \varphi(x)] \\ &= \sum_{j=1}^n h_j \partial_j \varphi(x) \\ &= (D\varphi(x), h). \end{aligned} \tag{2.1}$$

If $\psi \in \mathcal{S}_M(E)$ is an E-valued smooth functional on $\mathbb{R}^n$, then its *gradient* $D\psi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$ is determined by:

$$(D\psi(x), h \otimes e)_{\mathbb{R}^n \otimes E} = \frac{\partial}{\partial \epsilon} \Big[(\psi(x + \epsilon h), e)_E \Big]_{\epsilon=0}, \quad x, h \in \mathbb{R}^n, e \in E. \tag{2.2}$$

Especially, for $\varphi \in \mathcal{S}_M, D^2\varphi \equiv D(D\varphi) \in \mathcal{S}_M(\mathbb{R}^n \otimes \mathbb{R}^n), \cdots, D^m\varphi = D(D^{m-1}\varphi) \in \mathcal{S}_M((\mathbb{R}^n)^{\otimes m})$.

We know that if $\psi \in \mathcal{S}_M(\mathbb{R}^n)$, then $D\psi \in \mathcal{S}_M(\mathbb{R}^n \otimes \mathbb{R}^n)$ is its Jacobian matrix whose trace is denoted by

$$\operatorname{div} \psi(x) \equiv \operatorname{Tr}(D\psi(x)), \tag{2.3}$$

and is called divergence of ψ. Obviously, $\operatorname{div} \psi \in \mathcal{S}_M$. However, in Gaussian probability spaces it is more convenient to define the following operator:

$$\begin{aligned} \delta\psi(x) &= \sum_{j=1}^n (x_j \psi_j(x) - \partial_j \psi_j(x)) \\ &= (\psi(x), x) - \operatorname{Tr}(D\psi(x)). \end{aligned} \tag{2.4}$$

Using notations in (1.16), we write $\delta\psi = \sum_{j=1}^n \partial_j^* \psi_j$ where ∂_j^* is the adjoint operator of ∂_j in $L^2(\mathbb{R}, \gamma)$.

More generally, for $\psi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$, its *divergence* $\delta\psi \in \mathcal{S}_M(E)$ is defined by eq. (2.4). We have the following formula of integration by parts:

Proposition 2.1 If $\varphi \in \mathcal{S}_M(E), \psi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$, then $\delta\psi \in \mathcal{S}_M(E), D\varphi \in \mathcal{S}_M(\mathbb{R}^n \otimes E)$, and

$$\int_{\mathbb{R}^n} (D\varphi(x), \psi(x))_{\mathbb{R}^n \otimes E} \mu(dx) = \int_{\mathbb{R}^n} (\varphi(x), \delta\psi(x))_E \mu(dx). \tag{2.5}$$

Proof. It suffices to consider the case $E = \mathbb{R}$. On account of independence of all components, eq. (2.5) reduces to one-dimensional formula (1.18). ∎

For $\varphi \in \mathcal{S}_M(E)$, define

$$\begin{aligned} \mathcal{L}\varphi(x) &\equiv -\delta D\varphi(x) = -\sum_{j=1}^n \partial_j^* \partial_j \varphi(x) \\ &= \Delta\varphi(x) - \sum_{j=1}^n x_j \partial_j \varphi(x). \end{aligned} \tag{2.6}$$

$\mathcal{L}$ is called the *Ornstein-Uhlenbeck operator*. It follows from eq. (2.5) that D, δ and $\mathcal{L}$ are closable. We denote their closures again by D, δ and $\mathcal{L}$ respectively. Then δ and D are mutually adjoint and $\mathcal{L}$ is selfadjoint in $L^2(\mathbb{R}^n, \mu)$.

From theory of diffusion processes, $\mathcal{L}$-diffusion process is the unique strong solution of the following stochastic differential equation:

$$dX_t = -X_t\,dt + \sqrt{2}\,dW_t, \quad X_0 = x \in \mathbb{R}^n. \tag{2.7}$$

It has an explicit expression:

$$X_t = e^{-t}x + \sqrt{2}\int_0^t e^{-(t-s)}dW_s, \quad t \in \mathbb{R}_+. \tag{2.8}$$

Since random vector $(X_t - e^{-t}x)(1-e^{-2t})^{-1/2}$ is governed by n- dimensional standard Gaussian distribution, the transition semigroup generated by $\mathcal{L}$ has following form:

$$\begin{aligned}(T_t\varphi)(x) &= \mathbb{E}_x[\varphi(X_t)]\\ &= \int_{\mathbb{R}^n}\varphi(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(dy),\\ &\quad t \ge 0, x \in \mathbb{R}^n, \varphi \in L^2(\mu).\end{aligned} \tag{2.9}$$

Note that the right-hand side of eq. (2.9) still makes sense for $\varphi \in L^p(\mu), p \ge 1$ and defines a contraction semigroup in $L^p(\mu)$.

Proposition 2.2 *For any $p \ge 1$, operators $\{T_t, t \ge 0\}$ defined by the right-hand side of eq. (2.9) constitute a contraction semigroup in $L^p(\mathbb{R}^n, \mu)$.*

Proof. Consider the transformation from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, $Q_t : (x,y) \longmapsto e^{-t}x + \sqrt{1-e^{-2t}}y$. By rotation invariance of Gaussian measure we have $\mu = (\mu \times \mu)\circ Q_t^{-1}$. Therefore, $\forall p \ge 1$,

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} &|\varphi(e^{-t}x + \sqrt{1-e^{-2t}}y)|^p\mu(dx)\mu(dy)\\ &= \int_{\mathbb{R}^n}|\varphi(x)|^p\mu(dx).\end{aligned} \tag{2.10}$$

Hence, $\|T_t\varphi\|_p \le \|\varphi\|_p$. ∎

We shall prove a stronger contractivity for Ornstein-Uhlenbeck semigroup, namely the *hyper-contractivity*.

Theorem 2.3 (Nelson[1]) *Let $\{T_t, t \ge 0\}$ be the Ornstein-Uhlenbeck semigroup defined by eq. (2.9). For $p > 1, t > 0$, let $q(t) = e^{2t}(p-1)+1 > p$. Then for all $\varphi \in L^p(\mathbb{R}^n, \mu)$ it holds that*

$$\|T_t\varphi\|_{q(t)} \le \|\varphi\|_p. \tag{2.11}$$

Proof. Fixing any $t > 0$, let $q = q(t)$, q^* be the conjugate exponent of q (i.e. $q^{-1} + (q^*)^{-1} = 1$), and denote $a = e^{-t}$. To prove (2.11), it suffices to prove that $\forall \psi \in L^{q^*}(\mathbb{R}^n, \mu)$,

$$\left|\int_{\mathbb{R}^n}\psi T_t\varphi\,d\mu\right| \le \|\varphi\|_p\|\psi\|_{q^*}. \tag{2.12}$$

Since T_t preserves positivity and $|T_t\varphi| \le T_t(|\varphi|)$, we may assume that φ and ψ are nonnegative. Approximating by bounded functions if necessary, we may also assume that φ and ψ are bounded above and have positive lower bounds.

Let $\{B_t, 0 \le t \le 1\}$ and $\{\tilde{B}_t, 0 \le t \le 1\}$ be two mutually independent Brownian motions on some probability space $(\Omega, \mathcal{F}, P)$ generating filtration $\{\mathcal{F}_t, 0 \le t \le 1\}$. Take the following orthogonal transformation:

$$\begin{cases} B'_t = aB_t + \sqrt{1-a^2}\tilde{B}_t & (0 \le t \le 1),\\ \tilde{B}'_t = \sqrt{1-a^2}B_t - a\tilde{B}_t & (0 \le t \le 1).\end{cases}$$

Then B'_t and $\tilde{B}'_t$ are again two mutually independent Brownian motions with respect to $\mathcal{F}_t$. Noting that $\mathbb{E}[\varphi(B'_1)^p] = \|\varphi\|_p^p$, $\mathbb{E}[\psi(B_1)^{q^*}] = \|\psi\|_{q^*}^{q^*}$, by the theorem of stochastic integral representation, there exist n-dimensional progressive processes X_t and Y_t such that

$$\varphi(B'_1)^p = \|\varphi\|_p^p + \int_0^1 (X_s, dB'_s),$$

$$\psi(B_1)^{q^*} = \|\psi\|_{q^*}^{q^*} + \int_0^1 (Y_s, dB_s).$$

Applying Itō formula to the bounded positive martingales

$$M_t \equiv \|\varphi\|_p^p + \int_0^t (X_s, dB'_s), \quad 0 \le t \le 1$$

and

$$N_t \equiv \|\psi\|_{q^*}^{q^*} + \int_0^t (Y_s, dB_s), \quad 0 \le t \le 1$$

we have

$$\begin{aligned}\varphi(B'_1)\psi(B_1) &= M_1^{1/p}N_1^{1/q^*}\\ &= M_0^{1/p}N_0^{1/q^*} + \int_0^1 \frac{1}{p}M_s^{1/p-1}N_s^{1/q^*}dM_s\\ &\quad + \int_0^1 \frac{1}{q^*}M_s^{1/p}N_s^{1/q^*-1}dN_s\\ &\quad + \int_0^1 \frac{1}{2p}\Big(\frac{1}{p}-1\Big)M_s^{1/p-2}N_s^{1/q^*}|X_s|^2ds\\ &\quad + \int_0^1 \frac{a}{pq^*}M_s^{1/p-1}N_s^{1/q^*-1}(X_s, Y_s)ds\\ &\quad + \int_0^1 \frac{1}{2q^*}\Big(\frac{1}{q^*}-1\Big)M_s^{1/p}N_s^{1/q^*-2}|Y_s|^2ds,\end{aligned}$$

hence

$$\begin{aligned}\mathbb{E}[\varphi(B_1')\psi(B_1)] = {}& \|\varphi\|_p\|\psi\|_{q^*} \\ & - \frac{1}{2}\mathbb{E}\Big[\int_0^1 M_s^{1/p-2} N_s^{1/q^*-2}\Big\{\frac{1}{p}\Big(1-\frac{1}{p}\Big)N_s^2|X_s|^2 \\ & - \frac{2a}{pq^*}M_sN_s(X_s,Y_s) + \frac{1}{q^*}\Big(1-\frac{1}{q^*}\Big)M_s^2|Y_s|^2\Big\}ds\Big].\end{aligned}$$

Since $a^2 = (p-1)/(q-1) = (p-1)(q^*-1)$ and $(X_s,Y_s)^2 \le |X_s|^2|Y_s|^2$, the quadratic expression for M_s and N_s in the curly braces being nonnegative, it follows that

$$\mathbb{E}[\varphi(B_1')\psi(B_1)] \le \|\varphi\|_p\|\psi\|_{q^*}.$$

Noting that

$$\begin{aligned}\mathbb{E}[\varphi(B_1')\psi(B_1)] &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(ax+\sqrt{1-a^2}y)\psi(x)\mu(dx)\mu(dy) \\ &= \int_{\mathbb{R}^n} \psi(x)T_t\varphi(x)\mu(dx),\end{aligned}$$

we obtain (2.12) as desired. ∎

2.2 Gradient and divergence of smooth functionals

Now we extend the definitions of operators $D,\delta,\mathcal{L}$ to infinite dimensional Gaussian probability spaces. As we know, in finite dimensional spaces, the invariance under all translations of Lebesgue measure is crucial in the definition of differentiation. Since finite dimensional Gaussian measures are equivalent to Lebesgue measure, there will be no difficulties in defining derivatives in finite dimensional Gaussian probability spaces. However, on infinite dimensional spaces, no measure has property of invariance under all translations. We shall define differential calculus by virtue of quasi-invariance of Gaussian measure which means that sets of measure 0 will always be translated to sets of measure 0 under translations along a set of "directions" which constitute a dense subspace. This fact is a consequence of Cameron-Martin's theorem.

Let $(\Omega,\mathcal{F},\mu;H)$ be an infinite dimensional Gaussian probability space. We define

$$L^{\infty-} \equiv \bigcap_{1<p<\infty} L^p(\Omega,\mathcal{F},\mu), \tag{2.13}$$

$$L^{1+} \equiv \bigcup_{1<p<\infty} L^p(\Omega,\mathcal{F},\mu) \tag{2.14}$$

to be the projective and inductive limits respectively. Therefore, $L^{\infty-}$ is a countably normed space and L^{1+} is its topological dual space. It is easy from the Hölder inequality to verify that $L^{\infty-}$ is an algebra and the product operation is continuous.

In the abstract Wiener space (H,X,μ) in Example 3 of previous paragraph, H is a dense subspace of X, we can define translation operator along directions $h\in H$ for any functional f on X: $\tau_h f(x) \equiv f(x+h)$. While in general Gaussian probability spaces, by virtue of numerical model, we still have similar representation of additive group H.

Definition 2.4 Let $(\Omega,\mathcal{F},\mu;H)$ be an irreducible Gaussian probability space. Choosing a base of H, we obtain an isometry $J: H\to l^2$, a numerical model $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$ and an isomorphic mapping

$$T_*: L^{1+}(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty) \longrightarrow L^{1+}(\Omega,\mathcal{F},\mu).$$

For $h\in H$, define the following operator in $L^{1+}(\Omega,\mathcal{F},\mu)$:

$$\rho(h) \equiv T_* \circ \tau_{J(h)} \circ T_*^{-1}, \tag{2.15}$$

where τ is the translation operator acting on functionals on $\mathbb{R}^\infty$. It is easy to see that the restriction of $\rho(h)$ to $L^{\infty-}$ is an endomorphism and

$$\rho(h+g) = \rho(h)\rho(g), \quad h,g\in H. \tag{2.16}$$

Moreover, definition (2.15) is intrinsic, i.e. independent of choices of base in H. ρ is called the *canonical representation* of additive group H.

Theorem 2.5 (Cameron-Martin) *Let $(\Omega,\mathcal{F},\mu;H)$ be an irreducible Gaussian probability space, ρ the canonical representation of additive group H, $\mathcal{E}(h) = \exp\{W_h - \frac{1}{2}\|h\|^2\}$ $(h\in H)$ be exponential functionals. Then $\mathcal{E}(h)\in L^{\infty-}$, and*

$$\|\mathcal{E}(h)\|_p \le \exp\Big\{\frac{p-1}{2}\|h\|^2\Big\}, \quad 1<p<\infty. \tag{2.17}$$

For $f\in L^{1+}$ we have

$$\mathbb{E}[\rho(h)f] = \mathbb{E}[\mathcal{E}(h)f], \quad h\in H. \tag{2.18}$$

Moreover,

$$\lim_{t\to 0} t^{-1}(\mathcal{E}(th)-1) = W_h. \tag{2.19}$$

Proof. The canonical representation being intrinsic, without loss of generality we may take numerical model $(\mathbb{R}^\infty,\mathcal{B}^\infty,\gamma^\infty;l^2)$. For $h=\{h_j\}\in l^2, x=\{x_j\}\in\mathbb{R}^\infty$ and $n\in\mathbb{N}$, define

$$\mathcal{E}_n(h)(x) \equiv \exp\Big\{\sum_{j=1}^n h_jx_j - \frac{1}{2}\sum_{j=1}^n h_j^2\Big\}.$$

Denote by $\mathcal{F}_n$ the σ-algebra generated by the first n coordinates of x. Then $\{\mathcal{E}_n(h),\mathcal{F}_n, n\in\mathbb{N}\}$ is a martingale such that $\mathbb{E}[\mathcal{E}_n(h)] = 1$. If f only depends

on first m coordinates and $n \geq m$, then by transformation of variables in integrals over $\mathbb{R}^m$ we have

$$\mathbb{E}[\rho(h)f] = \mathbb{E}[\tau_h f] = \mathbb{E}[\mathcal{E}_n(h)f]. \tag{2.20}$$

Let $p \in (1,\infty)$. Since $\mathbb{E}[\mathcal{E}_n(ph)] = 1$, it follows that

$$\mathbb{E}\left[\exp\left\{p\sum_{j=1}^{n} h_j x_j\right\}\right] \leq \exp\left\{\frac{p^2}{2}\|h_i\|^2\right\},$$

therefore, $\{\mathcal{E}_n(h), \mathcal{F}_n; n \in \mathbb{N}\}$ is an L^p martingale converging to $\mathcal{E}(h)$ in L^p and satisfying that

$$\|\mathcal{E}(h)\|_p \leq \exp\left\{\frac{p-1}{2}\|h\|^2\right\}.$$

Upon taking limits in eq. (2.20), we obtain eq. (2.18). Eq. (2.19) is obvious. ∎

Eq. (2.18) means that, Gaussian measure μ has some kind of invariance under "translation" $\rho(h)$, i.e. $\rho(h)^*\mu$ is absolutely continuous with respect to μ, its Radon-Nikodym derivative is just $\mathcal{E}(h)$. It is this kind of invariance which enables us to define differential calculus on infinite dimensional Gaussian probability spaces.

Definition 2.6 Let $(\Omega, \mathcal{F}, \mu; H)$ be an irreducible Gaussian probability space, ρ the canonical representation of additive group H. For any smooth functional $F \in \mathcal{S}_M$, its *gradient* $DF \in \mathcal{S}_M(H)$ is determined by

$$(DF, h)_H = \lim_{\epsilon\to 0} \epsilon^{-1}[\rho(\epsilon h)F - F], \quad h \in H. \tag{2.21}$$

More generally, if E is a separable Hilbert space and $F \in \mathcal{S}_M(E)$, its *gradient* $DF \in \mathcal{S}_M(H \otimes E)$ is determined by

$$(DF, h\otimes e)_{H\otimes E} = \lim_{\epsilon\to 0} \epsilon^{-1}(\rho(\epsilon h)F - F, e)_E, \quad h \in H, e \in E. \tag{2.22}$$

Next proposition gives an explicit expression for gradient of any smooth functional thus shows that above definition is reasonable.

Proposition 2.7 *If $F \in \mathcal{S}_M$ has the form*

$$F = f(W_{h_1}, \cdots, W_{h_n}), \quad h_1, \cdots, h_n \in H,$$

then $\forall h \in H$,

$$(DF, h) = \sum_{j=1}^{n} \partial_j f(W_{h_1}, \cdots, W_{h_n})(h_j, h), \tag{2.23}$$

moreover,

$$\mathbb{E}[(DF, h)] = \mathbb{E}[FW_h]. \tag{2.24}$$

Proof. By orthogonalization if necessary, we may assume that $(h_i, h_j) = \delta_{ij}$ $(i,j = 1,\cdots,n)$ and extend them to a base $\{h_j\}_{j\in\mathbb{N}}$ of H. Hence, $h = \sum_j (h_j, h)h_j$. Taking numerical model, by definition we have

$$\begin{aligned}(DF, h) &= \frac{\partial}{\partial\epsilon}\Big[f(W_{h_1} + \epsilon(h_1, h), \cdots, W_{h_n} + \epsilon(h_n, h))\Big]_{\epsilon=0} \\ &= \sum_{j=1}^{n} \partial_j f(W_{h_1}, \cdots, W_{h_n})(h_j, h).\end{aligned}$$

Using formula of integration by parts, we obtain

$$\begin{aligned}\mathbb{E}[FW_h] &= \mathbb{E}\Big[F\sum_j (h_j, h)W_{h_j}\Big] \\ &= \sum_{j=1}^{n}(h_j, h)\int_{\mathbb{R}^n} x_j f(x_1, \cdots, x_n)\gamma^n(dx) \\ &= \sum_{j=1}^{n}(h_j, h)\int_{\mathbb{R}^n} \partial_j f(x_1, \cdots, x_n)\gamma^n(dx) \\ &= \mathbb{E}[(DF, h)].\end{aligned}$$

∎

Remark. The differential calculus for smooth functionals is essentially a finite dimensional calculus. It is easy to derive the following properties of gradient by using results in finite dimensional Gaussian probability spaces:

$$D(FG) = FDG + GDF, \quad G, F \in \mathcal{S}_M. \tag{2.25}$$

If $F = f(\varphi_1, \cdots, \varphi_n), \varphi_1, \cdots, \varphi_n \in \mathcal{S}_M, f$ is a smooth functional on $\mathbb{R}^n$, then

$$DF = \sum_{j=1}^{n} \partial_j f(\varphi_1, \cdots, \varphi_n)D\varphi_j. \tag{2.26}$$

If $F \in \mathcal{S}_M(E)$ has form (1.11), then

$$DF = \sum_{k=1}^{m} DF_k \otimes e_k. \tag{2.27}$$

Eq. (2.24) extends to the following form:

$$\mathbb{E}[(DF, h\otimes e)] = \mathbb{E}[(F, e)W_h], \quad h \in H, e \in E. \tag{2.28}$$

Moreover, if $G \in \mathcal{S}_M$, then $GF \in \mathcal{S}_M(E)$, and

$$(D(GF), h\otimes e) = G(DF, h\otimes e) + (DG, h)(F, e). \tag{2.29}$$

Hence

$$\mathbb{E}[G(DF, h\otimes e)] = \mathbb{E}[GW_h(F, e)] - \mathbb{E}[(DG, h)(F, e)]. \tag{2.30}$$

Proofs are left to the reader.

Definition (2.22) of E-valued functionals implies that of differentiation of higher order: if $F \in S_M(E)$, then $DF \in S_M(H \otimes E)$. Define $D^2F \equiv D(DF) \in S_M(H \otimes H \otimes E)$, $\cdots$, etc.. Then for any $k \geq 1$, we can define $D^kF = D(D^{k-1}F) \in S_M(H^{\otimes k} \otimes E)$.

Now consider adjoint operator of D. Since the gradient DF is an H-valued functional, H may be considered as tangent space. Hence H-valued functionals are vector fields and the adjoint operator δ is the divergence of vector fields.

Definition 2.8 For any smooth vector field $V \in S_M(H)$, its *divergence* $\delta V \in S_M$ is determined by:

$$\mathbb{E}[G\delta V] = \mathbb{E}[(DG, V)_H], \quad \forall G \in S_M. \tag{2.31}$$

More generally, if E is a separable Hilbert space, then for $V \in S_M(H \otimes E)$, its *divergence* $\delta V \in S_M(E)$ is determined by

$$\mathbb{E}[(G, \delta V)_E] = \mathbb{E}[(DG, V)_{H\otimes E}], \quad \forall G \in S_M(E). \tag{2.32}$$

The following proposition gives an explicit expression for divergence of smooth vector fields.

Proposition 2.9 *If $V \in S_M(H)$ has form*

$$V = \sum_{k=1}^m F_k h_k, \quad F_k \in S_M, h_k \in H, k = 1, \cdots, m,$$

then

$$\delta V = \sum_{k=1}^m (F_k W_{h_k} - (DF_k, h_k)). \tag{2.33}$$

If $F \in S_M, V \in S_M(H)$, then $FV \in S_M(H)$, and

$$\delta(FV) = F\delta V - (DF, V). \tag{2.34}$$

Proof. In view of eq. (2.30), $\forall G \in S_M$ and $k = 1, \cdots, m$, we have

$$\mathbb{E}[G(DF_k, h_k)] = \mathbb{E}[GF_k W_{h_k}] - \mathbb{E}[F_k(DG, h_k)].$$

Summing up over k, it is readily seen that the δV given by (2.33) satisfies eq. (2.31). Replacing F_k by FF_k in (2.33) and using formula (2.25), we obtain eq. (2.34). ∎

Remark. If $V \equiv h \in H$ is a constant vector field, then from (2.33) we have $\delta h = W_h$. Hence eq. (2.24) is a special case of eq. (2.31). In view of eq. (2.34), $\forall F \in S_M$ we have

$$\delta(Fh) = FW_h - (DF, h). \tag{2.35}$$

If we denote $D_hF \equiv (DF, h), \delta_h F \equiv \delta(Fh)$. Then it becomes

$$\delta_h + D_h = W_h\cdot, \tag{2.36}$$

where $W_h\cdot$ stands for the multiplication operator. Letting $V = Fh$ in eq. (2.31), we obtain

$$\mathbb{E}[G\delta_h F] = \mathbb{E}[FD_hG], \quad F, G \in S_M. \tag{2.37}$$

Since S_M is dense in L^2, it follows from Theorem I.1.5 that D_h and δ_h are closable, their closures are mutually adjoint.

By the same reason, for any separable Hilbert space E, it follows from eq (2.32) that D as densely defined operator from $L^2(E)$ to $L^2(H \otimes E)$ and δ as that from $L^2(H \otimes E)$ to $L^2(E)$ are all closable, their closures, denoted again by D and δ, are mutually adjoint. We shall specify their domains.

2.3 Sobolev spaces of functionals

For notational simplicity, if it causes no confusion, we shall denote norm in $L^p(\Omega; E)$ by $\|\cdot\|_p$ no matter what the Hilbert space E would be.

Definition 2.10 For $k \in \mathbb{N}, 1 \leq p < \infty, F \in S_M(E)$, define

$$\|F\|_{k,p} \equiv \left(\|F\|_p^p + \sum_{j=1}^k \|D^jF\|_p^p\right)^{1/p}. \tag{2.38}$$

Denote by $\mathbb{D}_k^p(E)$ the Banach space which is completion of $S_M(E)$ with respect to norm $\|\cdot\|_{k,p}$ and simply by $\mathbb{D}_k^p$ when $E = \mathbb{R}$.

Especially, when $p = 2$, $\mathbb{D}_k^2$ is a Hilbert space with inner product

$$(F, G)_{k,2} = (F, G)_2 + \sum_{j=1}^k (D^jF, D^jG)_2. \tag{2.39}$$

It is evident that, as an operator from L^2 into $L^2(H)$, the gradient has domain $\mathbb{D}_1^2$.

Proposition 2.11 *The norms $\{\|\cdot\|_{k,p}; k \in \mathbb{N}, 1 \leq p < \infty\}$ have the following properties:*

1° *monotonicity: if $1 \leq p \leq q < \infty, k \leq l$, then $\|\cdot\|_{k,p} \leq \|\cdot\|_{l,q}$;*

2° *consistence: for any $p, q \in [1, \infty), k, l \in \mathbb{N}$, if $\{F_n\} \subset S_M$ is Cauchy sequence with respect to both norms $\|\cdot\|_{k,p}$ and $\|\cdot\|_{l,q}$ satisfying that $\lim_{n\to\infty} \|F_n\|_{k,p} = 0$, then $\lim_{n\to\infty} \|F_n\|_{l,q} = 0$.*

Proof. The monotonicity is obvious, we prove the consistence. In the given case, for $j \in \mathbb{N}, j \leq l$, $\{D^jF_n\}$ is a Cauchy sequence in $L^q(H^{\otimes j})$. By the completeness of $L^q(H^{\otimes j})$, there exists some limit G_j. We shall prove that $G_j = 0$ by induction. It is obvious that $G_0 = 0$. Suppose that $G_1 = \cdots = G_{j-1} = 0$.

Then for all bounded smooth functional F and any $h_1, \cdots, h_j \in H$, by eq. (2.30) we have

$$\begin{aligned}
&\mathbb{E}[F(G_j, \otimes_{i=1}^{j} h_i)] \\
&= \lim_{n \to \infty} \mathbb{E}[F(D^j F_n, \otimes_{i=1}^{j} h_i)] \\
&= \lim_{n\to\infty} \{\mathbb{E}[F W_{h_j}(D^{j-1}F_n, \otimes_{i=1}^{j-1} h_i)] \\
&\qquad - \mathbb{E}[(DF, h_j)(D^{j-1}F_n, \otimes_{i=1}^{j-1} h_i)]\} \\
&= \mathbb{E}[F W_{h_j}(G_{j-1}, \otimes_{i=1}^{j-1} h_i)] \\
&\qquad - \mathbb{E}[(DF, h_j)(G_{j-1}, \otimes_{i=1}^{j-1} h_i)] \\
&= 0,
\end{aligned}$$

hence $G_j = 0$. ∎

The family $\{\mathbb{D}_k^p(E)\}$ of norms being consistent, we can define their projective limits.

Definition 2.12 For $k \in \mathbb{N}$, define

$$\mathbb{D}_k^{\infty}(E) \equiv \bigcap_{1<p<\infty} \mathbb{D}_k^p(E), \tag{2.40}$$

$$\mathbb{D}^{\infty}(E) \equiv \bigcap_{k\in\mathbb{N}} \bigcap_{1<p<\infty} \mathbb{D}_k^p(E) \tag{2.41}$$

to be topological projective limits and denote simply by $\mathbb{D}_k^{\infty}$ and $\mathbb{D}^{\infty}$, respectively when $E = \mathbb{R}$. These are all countably normed spaces.

When $k = 0$ we make conventions that $\mathbb{D}_0^p(E) \equiv L^p(E)$, and $\mathbb{D}_0^{\infty}(E) \equiv L^{\infty-}(E)$.

It follows from definition (2.38) and density of $S_M(E)$ in all $L^p(E)$ that, $\forall k \in \mathbb{N}$ and $p \geq 1$, the gradient D extends to a continuous linear operator from $\mathbb{D}_k^p(E)$ to $\mathbb{D}_{k-1}^p(H \otimes E)$ hence from $\mathbb{D}_k^{\infty}(E)$ to $\mathbb{D}_{k-1}^{\infty}(H \otimes E)$ as well as from $\mathbb{D}^{\infty}(E)$ to $\mathbb{D}^{\infty}(H \otimes E)$. Especially, formulas (2.24)–(2.26) as well as (2.28)–(2.30) can be extended to the cases of $\mathbb{D}_1^{\infty}$ and $\mathbb{D}_1^{\infty}(E)$. Hence $\mathbb{D}_1^{\infty}$ is an algebra on account of eq. (2.25).

Now suppose that $H = L^2(T, \mathcal{B}, \lambda)$ where λ is a σ-finite non-atomic measure on $(T, \mathcal{B})$. In this case, by identifying $L^2(\Omega; H)$ with $L^2(T \times \Omega)$, any H-valued functional is equal to a stochastic process parametered by T. Especially, if $F \in \mathbb{D}_1^2$, then $DF \in L^2(H)$ is equal to a stochastic process $\{D_t F; t \in T\}$. Therefore

$$(DF, h) = \int_T (D_t F) h(t) \lambda(dt). \tag{2.42}$$

If $F \in S_M$ has form $F = f(W_{h_1}, \cdots, W_{h_n}), h_1, \cdots, h_n \in H$, then

$$D_t F = \sum_{j=1}^{n} \partial_j f(W_{h_1}, \cdots, W_{h_n}) h_j(t). \tag{2.43}$$

More generally, $D^k F$ is equal to a k-parameter process:

$$D^k_{t_1 \cdots t_k} F = D_{t_1} \cdots D_{t_k} F, \quad (t_1, \cdots, t_k) \in T^k.$$

Naturally, it is defined for a.e. $(t_1, \cdots, t_k, \omega)$ $[\lambda^k \times \mu]$.

We have seen that $F \in L^2$ has a decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in H^{\hat{\otimes} n}. \tag{2.44}$$

The next proposition tells us what conditions imposed by $\{f_n\}$ imply existence of derivatives of F and how to compute them.

Proposition 2.13 *Let $(\Omega, \mathcal{F}, \mu; H)$ be an irreducible Gaussian probability space, $H = L^2(T, \mathcal{B}, \lambda)$. If $F \in L^2(\Omega)$ has decomposition (2.44), then $F \in \mathbb{D}_1^2$ if and only if*

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2 < \infty. \tag{2.45}$$

In that case,

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)) \tag{2.46}$$

converges in $L^2(\Omega)$ for a.e. t, where $f_n(\cdot, t)$ stands for the symmetric function of $n-1$ variables by fixing t. Moreover,

$$\begin{aligned}
\mathbb{E}[\|DF\|^2] &= \mathbb{E}\left[\int_T (D_t F)^2 \lambda(dt)\right] \\
&= \sum_{n=1}^{\infty} n n! \|f_n\|^2.
\end{aligned} \tag{2.47}$$

Proof. Firstly we suppose that $F = I_n(h^{\otimes n}) = H_n(W_h)$ where $h \in H$ such that $\|h\| = 1$. By eq. (2.43) we have

$$\begin{aligned}
D_t F &= H_n'(W_h) h(t) = n H_{n-1}(W_h) h(t) \\
&= n I_{n-1}(h^{\otimes(n-1)}) h(t), \qquad \text{a.e. } t,
\end{aligned}$$

which means that eq. (2.46) holds for $F = I_n(f_n), f_n(t_1, \cdots, t_n) = h(t_1) \cdots h(t_n)$ and

$$\begin{aligned}
\mathbb{E}\left[\int_T (D_t F)^2 \lambda(dt)\right] &= n^2 (n-1)! \|h\|^{2n} \\
&= n n! \|f_n\|^2.
\end{aligned}$$

Since by polarization identity (I.2.12), the linear subspace generated by $\{h^{\otimes n}, h \in H\}$ is dense in $H^{\hat{\otimes} n}$, it follows that proposition holds for any multiple Wiener integral $F = I_n(f_n), f_n \in H^{\hat{\otimes} n}$.

Now suppose that F has decomposition (2.44). For $m \in \mathbb{N}$, let $F^{(m)} = \sum_{n=0}^{m} I_n(f_n)$. Then $F^{(m)} \in \mathbb{D}_1^2$ and

$$\mathbb{E}[\|DF^{(m)}\|^2] = \sum_{n=1}^{m} nn!\|f_n\|^2.$$

If condition (2.45) is satisfied, then $\{F^{(m)}\}_{m\in\mathbb{N}}$ converges in $\mathbb{D}_1^2$ to F, hence $F \in \mathbb{D}_1^2$. Conversely, if $F \in \mathbb{D}_1^2, G = I_n(h^{\otimes n}), h \in H$, taking a base $\{e_j\}$ of H, by formula (2.30) we then have

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{E}[G(DF^{(m)}, e_j)] &= \lim_{m\to\infty} (\mathbb{E}[GW_{e_j}F^{(m)}] - \mathbb{E}[F^{(m)}(DG, e_j)]) \\
&= \mathbb{E}[GW_{e_j}F] - \mathbb{E}[F(DG, e_j)] \\
&= \mathbb{E}[G(DF, e_j)], \qquad j = 1, 2, \cdots.
\end{aligned}$$

If $m > n$, then

$$\mathbb{E}[G(DF^{(m)}, e_j)] = \mathbb{E}\Big[(n+1)GI_n\Big(\int_T f_{n+1}(\cdot, t)e_j(t)\lambda(dt)\Big)\Big],$$

which means that projection of (DF, e_j) in $\mathcal{H}_n$ is nothing but $(n+1)I_n(\int_T f_{n+1}(\cdot, t)e_j(t)\lambda(dt))$, hence

$$\begin{aligned}
\sum_{n=1}^{\infty} nn!\|f_n\|^2 &= \sum_{j=1}^{\infty} \mathbb{E}[(DF, e_j)^2] \\
&= \mathbb{E}[\|DF\|^2] < \infty.
\end{aligned}$$

∎

Now consider the divergence of vector fields. Let $V \in L^2(\Omega; H)$, which means the process $V = \{V_t, t \in T\}$ satisfying that

$$\int_T \mathbb{E}[V_t^2]\lambda(dt) < \infty.$$

The domain of D being $\mathbb{D}_1^2$, by definition (2.31) the domain of its adjoint operator δ is

$$\begin{aligned}
\mathcal{D}(\delta) = \Big\{ V \in L^2(\Omega; H) :\ &\exists c \ge 0 \text{ s.t. } \forall G \in \mathbb{D}_1^2 \\
&\Big|\mathbb{E}\Big[\int_T (D_tG)V_t\lambda(dt)\Big]\Big| \le c\|G\|_2 \text{ holds}\Big\}.
\end{aligned} \tag{2.48}$$

If $V \in L^2(\Omega; H)$, then for λ-a.e. $t, V_t \in L^2$ has following decomposition:

$$V_t = \sum_{n=0}^{\infty} I_n(f_{n+1}(\cdot, t)), \quad \text{a.e. } t, \tag{2.49}$$

where $f_{n+1}(\cdot, t) \in H^{\widehat{\otimes} n}$ is symmetric for first n variables and depends on t, which always has a version such that f_{n+1} is a measurable function of $n+1$ variables, hence its symmetrization

$$\begin{aligned}
\tilde{f}_{n+1}(t_1, \cdots, t_n, t) = \frac{1}{n+1}\Big[& f_{n+1}(t_1, \cdots, t_n, t) \\
&+ \sum_{j=1}^{n} f_{n+1}(t_1, \cdots, t_{j-1}, t, t_{j+1}, \cdots, t_n, t_j)\Big]
\end{aligned} \tag{2.50}$$

belongs to $H^{\widehat{\otimes}(n+1)}$ and it holds that

$$\mathbb{E}\Big[\int_T V_t^2\lambda(dt)\Big] = \sum_{n=1}^{\infty} n!\|\tilde{f}_{n+1}\|^2 < \infty. \tag{2.51}$$

Proposition 2.14 *Let $(\Omega, \mathcal{F}, \mu; H)$ be an irreducible Gaussian probability space, $H = L^2(T, \mathcal{B}, \lambda)$. If $V \in L^2(\Omega; H)$ has decomposition (2.49), then $V \in \mathcal{D}(\delta)$ if and only if*

$$\sum_{n=1}^{\infty} n!\|\tilde{f}_n\|^2 < \infty. \tag{2.52}$$

In that case, we have

$$\delta V = \sum_{n=1}^{\infty} I_n(\tilde{f}_n), \tag{2.53}$$

and

$$\mathbb{E}[(\delta V)^2] = \sum_{n=1}^{\infty} n!\|\tilde{f}_n\|^2. \tag{2.54}$$

Proof. Denote the sum of first $m+1$ terms in (2.49) by $V_t^{(m)}$ and sum of first m terms in (2.53) by $F^{(m)}$. Then the sum of first m terms in (2.52) is $\mathbb{E}[(F^{(m)})^2]$. It is easy to see that condition (2.52) is necessary and sufficient for L^2 convergence of series in (2.53).

Let $G \in \mathbb{D}_1^2, G = \sum_n I_n(g_n), g_n \in H^{\widehat{\otimes} n}, n \in \mathbb{N}$. By eq. (2.46) we have

$$\begin{aligned}
\Big|\mathbb{E}\Big[\int_T (D_tG)V_t^{(m)}\lambda(dt)\Big]\Big| &= \Big|\sum_{n=1}^{m} n!(\tilde{f}_n, g_n)\Big| \\
&\le \|F^{(m)}\|_2\|G\|_2.
\end{aligned}$$

It follows from (2.48) that the condition (2.52) is necessary and sufficient for $V \in \mathcal{D}(\delta)$, and in that case δV is given by (2.53). ∎

Remark. If $V \equiv h \in H$ is a constant vector field, then

$$\delta h = W_h = \int_T h(t)W(dt).$$

More generally, we may denote δV by $\int_T V_t W(dt)$. As we shall see in Chapter III, it coincide with the *Skorohod integral* which is an extension of Itô's integral.

§3. Meyer's inequalities and some consequences

3.1 Ornstein-Uhlenbeck semigroup

Let $(\Omega, \mathcal{F}, \mu; H)$ be an irreducible Gaussian probability space, E be a separable Hilbert space. The space of E-valued square integrable functionals, $L^2(\Omega, \mathcal{F}, \mu; E) \simeq L^2(\Omega, \mathcal{F}, \mu) \otimes E$, still has chaos decomposition (1.31). For simplicity of statement, in this paragraph we only give proofs in the case $E = \mathbf{R}$, but all results can be extended to general E-valued functionals. Denote by J_n the orthogonal projection onto subspace $\mathcal{H}_n$. As in finite dimensional case, we shall construct the Ornstein-Uhlenbeck semigroup on infinite dimensional spaces.

Definition 3.1 For $F \in \mathcal{S}_M$, define

$$\mathcal{L}F \equiv -\delta DF. \tag{3.1}$$

From the adjointness of δ and D we know that $\mathcal{L}$ is essential selfadjoint whose closure is a selfadjoint operator in L^2 and is called *Ornstein-Uhlenbeck operator* (*OU operator* for short).

Proposition 3.2 *If $F \in \mathcal{S}_M$ has form*

$$F = f(W_{h_1}, \cdots, W_{h_n}), \quad h_1, \cdots, h_n \in H,$$

then

$$\begin{aligned}\mathcal{L}F = &\sum_{j,k=1}^{n} \partial_k \partial_j f(W_{h_1}, \cdots, W_{h_n})(h_k, h_j) \\ &- \sum_{j=1}^{n} \partial_j f(W_{h_1}, \cdots, W_{h_n}) W_{h_j}.\end{aligned} \tag{3.2}$$

More generally we have

$$\mathcal{L} = \sum_{n=0}^{\infty} nJ_n, \tag{3.3}$$

$$\mathcal{D}(\mathcal{L}) = \{F \in L^2 : \textstyle\sum_n n^2 \|J_n F\|^2 < \infty\}. \tag{3.4}$$

Proof. By eq. (2.23) we have

$$DF = \sum_{j=1}^{n} \partial_j f(W_{h_1}, \cdots, W_{h_n}) h_j.$$

Applying eq. (2.33) we obtain eq. (3.2). Consider the numerical model and the Hermite polynomials (1.22) on $\mathbf{R}^\infty$. By eq. (1.28),

$$\mathcal{L}H_\alpha = -|\alpha| H_\alpha, \quad \alpha \in \Lambda. \tag{3.5}$$

Since $\mathcal{H}_n$ is generated by $\{H_\alpha : \alpha \in \Lambda, |\alpha| = n\}$ and chaos decomposition is an intrinsic property of Gaussian probability space, it follows that $\mathcal{L}$ has form (3.3) and its domain in L^2 is given by (3.4). ∎

Remark. $N \equiv -\mathcal{L} = \sum_n nJ_n$ is called *number operator* which is a positive selfadjoint operator in L^2.

Now consider the semigroup generated by $\mathcal{L}$.

Definition 3.3 The contraction semigroup on L^2

$$T_t = \sum_{n=0}^{\infty} e^{-nt} J_n, \quad t \geq 0 \tag{3.6}$$

is called *Ornstein-Uhlenbeck semigroup* (*OU semigroup for short*).

The semigroup $\{T_t, t \geq 0\}$ has following properties:

1° positivity preserving: $F \geq 0 \Longrightarrow T_t(F) \geq 0$, (cf. eq. (3.9));

2° symmetry: $\mathbf{E}[GT_tF] = \mathbf{E}[FT_tG]$.

It is easy to see that $\mathcal{L}$ is the generator of $\{T_t, t \geq 0\}$. In fact, if $F \in \mathcal{D}(\mathcal{L})$, then

$$\mathbf{E}[|t^{-1}(T_tF - F) - \mathcal{L}F|^2] = \sum_{n=0}^{\infty} (t^{-1}(e^{-nt} - 1) + n)^2 \mathbf{E}[|J_nF|^2].$$

Since $|t^{-1}(e^{-nt} - 1)| \leq n$ and $t^{-1}(e^{-nt} - 1) + n \longrightarrow 0$ when $t \downarrow 0$, the above expectation goes to 0. Conversely, if $t^{-1}(T_tF - F)$ converges in L^2 to some functional G, then $\forall n$,

$$J_nG = \lim_{t \downarrow 0} t^{-1}(T_tJ_nF - J_nF) = -nJ_nF,$$

therefore, $F \in \mathcal{D}(\mathcal{L})$ and $\mathcal{L}F = G$.

Using numerical model and finite dimensional approximation, we can prove contractivity of OU semigroup in all spaces $L^p (1 \leq p < \infty)$. Taking numerical model $(\mathbf{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$, for $h = \{h_j\} \in l^2, x = \{x_j\} \in \mathbf{R}^\infty$ and $n \in \mathbf{N}$, define

$$\begin{aligned}\mathcal{E}_n(h)(x) &\equiv \exp\Big\{\sum_{j=1}^{n} h_j x_j - \frac{1}{2}\sum_{j=1}^{n} h_j^2\Big\} \\ &= \prod_{j=1}^{n} \sum_{k=0}^{\infty} \frac{h_j^k}{k!} H_k(x_j) \\ &= \sum_{\alpha \in \mathbf{N}_0^n} \prod_{j=1}^{n} \frac{h_j^{\alpha_j}}{\alpha_j!} H_{\alpha_j}(x_j).\end{aligned}$$

We have proved in Theorem 2.5 that, when $n \to \infty$, $\mathcal{E}_n(h)$ converges in L^p to exponential functional $\mathcal{E}(h)$, hence

$$\mathcal{E}(h)(x) = \sum_{\alpha \in \Lambda} \frac{h^\alpha}{\alpha!} H_\alpha(x), \tag{3.7}$$

where $h^\alpha \equiv \prod_j h_j^{\alpha_j}, \alpha = \{\alpha_j\} \in \Lambda$. Therefore

$$J_n \mathcal{E}(h) = \sum_{\alpha \in \Lambda_n} \frac{h^\alpha}{\alpha!} H_\alpha, \quad n \in \mathbb{N}_0, \tag{3.8}$$

where $\Lambda_n = \{\alpha \in \Lambda : |\alpha| = n\}$.

Proposition 3.4 *In any numerical model $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$, for $t \geq 0, F \in L^2$ it holds that*

$$(T_t F)(x) = \int_{\mathbb{R}^\infty} F(e^{-t}x + \sqrt{1-e^{-2t}}y)\gamma^\infty(dy). \tag{3.9}$$

Consequently, for all $p \geq 1, T_t$ is a contraction in L^p.

Proof. Since the linear span of exponential functionals is dense in L^2, it suffices to verify eq. (3.9) for $F = \mathcal{E}(h), h \in l^2$. Firstly let $F = \mathcal{E}_n(h)$. By a straightforward calculation, the right-hand side of eq. (3.9) is equal to $\exp\{\sum_{j=1}^n x_j e^{-t}h_j - \frac{1}{2}\sum_{j=1}^n (e^{-t}h_j)^2\}$. Letting $n \to \infty$, in view of eq. (3.8), we obtain

$$\begin{aligned} \mathcal{E}(e^{-t}h)(x) &= \sum_{n=0}^\infty e^{-nt} J_n \mathcal{E}(h)(x) \\ &= T_t \mathcal{E}(h)(x), \end{aligned} \tag{3.10}$$

hence eq. (3.9) holds. As in the finite dimensional case (cf. Proposition 2.2) we can prove the contractivity of T_t in all spaces L^p. ∎

The definition of OU semigroup being intrinsic, we may define it via eq. (3.9) as well. It even makes sense for $F \in L^1$. Taking this definition, by finite dimensional approximation we obtain the hyper-contractivity of OU semigroup (cf. Theorem 2.3).

Theorem 3.5 Let $\{T_t, t \geq 0\}$ be the OU semigroup. For $p > 1, t > 0$, let $q(t) = e^{2t}(p-1) + 1 > p$. Then, for all $F \in L^p$ we have

$$\|T_t F\|_{q(t)} \leq \|F\|_p. \tag{3.11}$$

We have two useful consequences of the hyper-contractivity:

Corollary 3.6 *For all $p \in (1, \infty)$, every subspace $\mathcal{H}_n$ in the Wiener chaos decomposition is closed in L^p and in which all L^p norms are equivalent.*

Proof. For any given $q > p > 1$, choose $t > 0$ such that $q = e^{2t}(p-1) + 1$. Hence for any given n and $F \in \mathcal{H}_n$ we have

$$e^{-nt}\|F\|_q = \|T_t F\|_q \leq \|F\|_p,$$

which means that, in $\mathcal{H}_n$, L^p norm is equivalent to L^q norm. ∎

Corollary 3.7 *For all $p \in (1, \infty)$ and $n \in \mathbb{N}_0$, J_n is a bounded operator in L^p.*

Proof. For $p = 2$ the assertion is obvious. If $p > 2$, taking $t > 0$ such that $p = e^{2t} + 1$, then for $F \in L^p$ we have

$$\begin{aligned} \|J_n F\|_p &= e^{nt}\|T_t J_n F\|_p \leq e^{nt}\|J_n F\|_2 \\ &\leq e^{nt}\|F\|_2 \leq e^{nt}\|F\|_p. \end{aligned}$$

If $p < 2$ and q is its conjugate exponent, then $q > 2$. Take $t > 0$ such that $q = e^{2t} + 1$. Then for any $F, G \in L^2$ we have

$$\begin{aligned} |\mathbb{E}[GJ_n F]| &= |\mathbb{E}[FJ_n G]| \\ &\leq \|F\|_p \|J_n G\|_q \\ &\leq \|F\|_p e^{nt}\|G\|_q, \end{aligned}$$

which means that

$$\|J_n F\|_p \leq e^{nt}\|F\|_p,$$

hence J_n extends continuously to L^p. ∎

3.2 L^p-multiplier theorem

Generally speaking, given a sequence $\rho = \{\rho_n\}_{n \in \mathbb{N}_0}$ of real numbers, we can define a linear operator on $\mathcal{P}$:

$$T_\rho \equiv \sum_{n=0}^\infty \rho_n J_n. \tag{3.12}$$

For example, if $\rho_n = e^{-nt}, -n$, then T_ρ extends to an OU semigroup, an OU operator, respectively. The question is: for what kind of sequences ρ, T_ρ can be extended to bounded linear operators in L^p? The following *L^p-multiplier theorem* gives an answer:

Theorem 3.8 (Meyer) *Let $\rho = \{\rho_n\}$ be a sequence of real numbers, $T_\rho \equiv \sum_n \rho_n J_n$. If $\exists n_0 \in \mathbb{N}$ and $\beta > 0$ such that*

$$\rho_n = \sum_{k=0}^\infty a_k (n^{-\beta})^k, \quad n \geq n_0, \tag{3.13}$$

where $\{a_k\}$ is a sequence of real numbers satisfying that

$$\sum_{k=0}^\infty |a_k|(n_0^{-\beta})^k < \infty. \tag{3.14}$$

Then $\forall p \in (1, \infty), T_\rho$ extends uniquely to a bounded linear operator in L^p.

Remark. This condition means that there exist a function φ which is analytic in some neighbourhood of 0 and some $\beta > 0$ such that

$$\rho_n = \varphi(n^{-\beta}). \tag{3.15}$$

We adopt the simplified proof given by Shigekawa[4]. Firstly we prepare a lemma:

Lemma 3.9 *Let I be the identity operator, $n \in \mathbb{N}$. Denote*

$$I_n \equiv I - J_0 - \cdots - J_{n-1} = \sum_{k=n}^{\infty} J_k,$$

$$R_n \equiv \int_0^{\infty} T_t I_n dt = \sum_{k=n}^{\infty} \frac{1}{k} J_k.$$

Then, $\forall p \in (1, \infty), n \in \mathbb{N}, \exists c = c(p,n) > 0$, such that $\forall F \in L^p, k \in \mathbb{N}$,

$$\|T_t I_n F\|_p \leq c e^{-nt} \|F\|_p \quad t \geq 0; \tag{3.16}$$

$$\|R_n^k F\|_p \leq c n^{-k} \|F\|_p. \tag{3.17}$$

Proof. For $p = 2$ (3.16) is obvious. If $p > 2$, taking $t_0 > 0$ such that $p = e^{2t_0} + 1$, then $\forall t \geq 0$ by hyper-contractivity of OU semigroup we have

$$\begin{aligned}
\|T_{t+t_0} I_n F\|_p &= \|T_{t_0} T_t I_n F\|_p \leq \|T_t I_n F\|_2 \\
&= \Big(\sum_{k=n}^{\infty} \|e^{-kt} J_k F\|_2^2\Big)^{1/2} \\
&\leq e^{-nt} \Big(\sum_{k=n}^{\infty} \|J_k F\|_2^2\Big)^{1/2} \\
&\leq e^{-nt} \|F\|_2 \leq e^{-nt} \|F\|_p,
\end{aligned}$$

which means that (3.16) holds for $t \geq t_0$. For $t < t_0$, by contractivity we have

$$\begin{aligned}
\|T_t I_n F\|_p &\leq \|I_n F\|_p \leq \|I_n\| \|F\|_p \\
&\leq e^{nt_0} \|I_n\| e^{-nt} \|F\|_p,
\end{aligned}$$

hence (3.16) still holds.

If $p < 2$, by considering its conjugate exponent, as in the proof of Corollary 3.7 we establish (3.16).

From (3.16) we have $\|R_n F\|_p \leq c n^{-1} \|F\|_p$,

$$\begin{aligned}
\|R_n^2 F\|_p &= \Big\| \int_0^{\infty} \int_0^{\infty} T_t I_n T_s I_n F dt ds \Big\|_p \\
&= \Big\| \int_0^{\infty} \int_0^{\infty} T_{t+s} I_n F dt ds \Big\|_p \\
&\leq \int_0^{\infty} \int_0^{\infty} \|T_{t+s} I_n F\|_p dt ds \\
&\leq \|F\|_p \int_0^{\infty} \int_0^{\infty} c e^{-n(t+s)} dt ds \\
&= c n^{-2} \|F\|_p,
\end{aligned}$$

and so on, thus establish (3.17). ∎

Proof. of Theorem 3.8. It suffices to consider the case $\beta \leq 1$. Firstly suppose that $\beta = 1$. Denote $T_\rho = T_1 + T_2$, where

$$T_1 = \sum_{n=0}^{n_0 - 1} \rho_n J_n, \quad T_2 = \sum_{n=n_0}^{\infty} \rho_n J_n.$$

J_n being bounded operators in L^p, T_1 is obviously bounded. Let $n \geq n_0$, $F_n \in \mathcal{H}_n$. Since $R_{n_0} F_n = n^{-1} F_n$, we have

$$\sum_{k=0}^{\infty} a_k R_{n_0}^k F_n = \sum_{k=0}^{\infty} a_k n^{-k} F_n = \rho_n F_n,$$

hence

$$T_2 = \sum_{k=0}^{\infty} a_k R_{n_0}^k.$$

In view of (3.17) and condition (3.14), this series converges in norm of operators in L^p, hence T_2 is a bounded operator in L^p.

For the case $\beta < 1$, let $\lambda_t^\beta(ds)$ be a probability measure on $\mathcal{B}(\mathbb{R}_+)$ satisfying that

$$\int_0^{\infty} e^{-us} \lambda_t^\beta(ds) = \exp\{-u^\beta t\}, \quad u \geq 0.$$

Namely, λ_t^β is a single side stable distribution of order β. Define

$$T_t^\beta \equiv \int_0^{\infty} T_s \lambda_t^\beta(ds), \quad t \geq 0. \tag{3.18}$$

It can be proved from the relation $\lambda_t^\beta * \lambda_s^\beta = \lambda_{t+s}^\beta$ that $\{T_t^\beta, t \geq 0\}$ is a strongly continuous contraction semigroup in L^p, and for $F_n \in \mathcal{H}_n$ we have

$$\begin{aligned}
T_t^\beta F_n &= \int_0^{\infty} T_s F_n \lambda_t^\beta(ds) \\
&= \int_0^{\infty} e^{-ns} F_n \lambda_t^\beta(ds) \\
&= \exp\{-n^\beta t\} F_n.
\end{aligned}$$

Replacing T_t by T_t^β, similarly we obtain the desired conclusion. ∎

As a consequence of L^p-multiplier theorem, we have

Proposition 3.10 *For $s \in \mathbb{R}$, define*

$$(I - \mathcal{L})^{s/2} \equiv \sum_n (1 + n)^{s/2} J_n. \tag{3.19}$$

Then in the case $s \leq 0$, $(I - \mathcal{L})^{s/2}$ extends to a bounded linear operator in L^p $(1 < p < \infty)$.

Proof. Since $\varphi(x) \equiv (x/(1+x))^{-s/2}$ is analytic in the neighbourhood of $x=0$, taking $\rho_n = (1+n)^{s/2} = \varphi(n^{-1})$, the proposition follows from Theorem 3.8. ∎

3.3 Meyer's inequalities

In this section we will prove the important Meyer's inequalities (cf. Meyer[1]) which constitute the cornerstone of theory of infinite dimensional Sobolev spaces. Here we adopt the much simplified proof due to Pisier[1]. We shall use an important property of so called *Hilbert transformation*

$$Tf(x) = \int_{\mathbb{R}} \frac{f(x+t) - f(x-t)}{t} dt. \tag{3.20}$$

Namely, the transformation T is a bounded linear operator in all $L^p(\mathbb{R})$ $(1 < p < \infty)$ (for its proof we refer to Stein[1], Dunford - Schwartz[1] or Bass[1]).

Firstly we prove two simple lemmas.

Lemma 3.11 *It holds on $\mathcal{P}$ that*

$$DJ_0 = 0, \quad DJ_n = J_{n-1}D, \quad n \geq 1. \tag{3.21}$$

For $\rho = \{\rho_n\}$, denote $\rho^+ \equiv \{\rho_{n+1}\}$. Then

$$DT_\rho = T_{\rho^+} D, \tag{3.22}$$

where T_ρ is defined by (3.12).

Proof. Noting that Hermite polynomials $\{h_n(x)\}$ satisfy recursive relation $h_n'(x) = nh_{n-1}(x)$, the proof is straightforward. ∎

Lemma 3.12 *Define*

$$Q \equiv (I - L)^{-1/2} \equiv \sum_{n=0}^{\infty} (1+n)^{-1/2} J_n. \tag{3.23}$$

Then we have

$$Q = \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-t} T_t dt. \tag{3.24}$$

Proof. Consider its projection onto $\mathcal{H}_n$, eq. (3.24) reduces to the following identity:

$$(n+1)^{-1/2} = \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-(n+1)t} dt. \qquad ∎$$

In the following proof, C_p as well as $\tilde{C}_p$ stands for constant which depends only on p but may be different in different occasions.

Proposition 3.13 *For all $p \in (1, \infty)$, $\exists C_p > 0$, such that for all $F \in \mathcal{P}$,*

$$\|QF\|_{1,p} \leq C_p \|F\|_p. \tag{3.25}$$

Proof. Consider a numerical model. For $0 \leq \theta < \pi/2$, make substitution $t = |\log\cos\theta| = -\log\cos\theta$, $e^{-t} = \cos\theta$. Then $T_t = e^{-tN} = (\cos\theta)^N$,

$$Q = \pi^{-1/2} \int_0^{\pi/2} |\log\cos\theta|^{-1/2} (\cos\theta)^N \sin\theta d\theta. \tag{3.26}$$

For $x, y \in \mathbb{R}^\infty$, $F \in L^p$, define

$$R_\theta F(x,y) = F(x\cos\theta + y\sin\theta).$$

It follows from eq. (3.9) that, $\forall F \in \mathcal{P}$,

$$(T_t F)(x) = \mathbb{E}^y[R_\theta F(x,y)]. \tag{3.27}$$

where $\mathbb{E}^y$ stands for integration with respect to $\gamma^\infty(dy)$. Denote by D_h^y the derivation with respect to y. Then for $h \in l^2$,

$$D_h^y R_\theta F(x,y) = \sin\theta R_\theta D_h F(x,y),$$

hence

$$\begin{aligned} \mathbb{E}^y[D_h^y R_\theta F] &= \sin\theta \mathbb{E}^y[R_\theta D_h F] \\ &= \sin\theta (\cos\theta)^N D_h F \\ &= \sum_n \sin\theta \cos^n\theta J_n D_h F. \end{aligned} \tag{3.28}$$

Denote

$$\varphi(\theta) \equiv \frac{1}{2} |\pi \log\cos\theta|^{-1/2} \cos\theta \operatorname{sgn}\theta. \tag{3.29}$$

The order of φ near point 0 being θ^{-1}, it has the form

$$\varphi(\theta) = \left(\frac{1}{2\sqrt{\pi}\theta} + r(\theta) \right) \operatorname{sgn}\theta,$$

where $r(\theta)$ is some bounded function. Define

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \varphi(\theta) R_\theta F d\theta &\equiv \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{\pi/2} (\varphi(\theta) R_\theta F + \varphi(-\theta) R_{-\theta} F) d\theta \\ &= \lim_{\varepsilon \downarrow 0} \int_\varepsilon^{\pi/2} \frac{(R_\theta F - R_{-\theta} F)}{2\sqrt{\pi} |\log\cos\theta|} \cos\theta d\theta. \end{aligned} \tag{3.30}$$

We need to prove the existence of above limit.

By rotational invariance of Gaussian measure and eq. (3.27),

$$\mathbb{E}^x \mathbb{E}^y [|R_\theta F(x,y)|^p] = \|F\|_p^p < \infty$$

being a constant independent of θ, we have

$$\int_{-\pi/2}^{\pi/2} |R_\theta F(x,y)|^p d\theta < \infty \quad \text{a.e. } (\gamma^\infty \times \gamma^\infty).$$

However, since

$$\int_{-\epsilon}^{\epsilon} \varphi(\theta')R_{\theta+\theta'}F d\theta' \approx \int_0^\epsilon \frac{R_{\theta+\theta'}F - R_{\theta-\theta'}F}{\theta'} d\theta',$$

by the boundedness of Hilbert transformation (3.20) in $L^p(\mathbb{R})$, $\exists C_p > 0$ such that

$$\int_{-\pi/2}^{\pi/2} \left| \int_{-\pi/2}^{\pi/2} \varphi(\theta')R_{\theta+\theta'}F d\theta' \right|^p d\theta \le C_p \int_{-\pi/2}^{\pi/2} |R_\theta F|^p d\theta.$$

By rotational invariance we then have

$$\begin{aligned}
&\mathbb{E}^x\mathbb{E}^y\left[\left| \int_{-\pi/2}^{\pi/2} \varphi(\theta)R_\theta F(x,y)d\theta \right|^p \right] \\
&= \mathbb{E}^x\mathbb{E}^y\left[\left| \int_{-\pi/2}^{\pi/2} \varphi(\theta')R_{\theta+\theta'}F(x,y)d\theta' \right|^p \right] \\
&\le C_p\mathbb{E}^x\mathbb{E}^y[|R_\theta F(x,y)|^p] \\
&= C_p\|F\|_p^p, \qquad (3.31)
\end{aligned}$$

proving the existence of limit in (3.30) in the space $L^p(\gamma^\infty \times \gamma^\infty)$.

Let $h \in l^2$. By eqs. (2.24) and (3.28) we have

$$\begin{aligned}
&\mathbb{E}^y\left[W_h \int_{-\pi/2}^{\pi/2} \varphi(\theta)R_\theta F d\theta\right] = \mathbb{E}^y\left[D_h^y \int_{-\pi/2}^{\pi/2} \varphi(\theta)R_\theta F d\theta\right] \\
&= \int_{-\pi/2}^{\pi/2} \varphi(\theta)\mathbb{E}^y[D_h^y R_\theta F]d\theta \\
&= \sum_n \left(\int_{-\pi/2}^{\pi/2} \varphi(\theta)\sin\theta\cos^n\theta d\theta \right) J_n D_h F \\
&= \sum_n (n+2)^{-1/2} J_n D_h F = D_h QF,
\end{aligned}$$

where the last equality follows from Lemma 3.11. Since $D_h QF = (DQF, h)_H$, in view of the isomorphism from l^2 to $\mathcal{H}_1$, it follows that for a.e. $x\,[\gamma^\infty]$,

$$DQF(x) \simeq J_1^y\left[\int_{-\pi/2}^{\pi/2} \varphi(\theta)R_\theta F(x,y)d\theta \right],$$

where J_1^y stands for the projection onto $\mathcal{H}_1$ with respect to variable y. J_1^y being bounded in L^p (cf. Corollary 3.7), $\exists C_p > 0$ such that

$$\|DQF(x)\|_H^p \le C_p\mathbb{E}^y\left[\left| \int_{-\pi/2}^{\pi/2} \varphi(\theta)R_\theta F(x,y)d\theta \right|^p \right].$$

By integration with respect to x and in view of eq. (3.31), we then have

$$\|DQF\|_p \le C_p\|F\|_p. \qquad (3.32)$$

Finally, by Proposition 3.10, Q extends to a bounded operator in L^p, hence $\exists C_p > 0$ such that $\|QF\|_p \le C_p\|F\|_p$, combining with (3.32) yields (3.25). ∎

By duality, we obtain an inequality in the opposite direction.

Proposition 3.14 *For all $p \in (1,\infty)$, $\exists \tilde{C}_p > 0$ such that for all $F \in \mathcal{P}$,*

$$\|F\|_p \le \tilde{C}_p\|QF\|_{1,p}. \qquad (3.33)$$

Proof. Since $Q^{-2} = I - L = I + \delta D$, it follows that for $F, G \in \mathcal{P}$,

$$\begin{aligned}
(F,G) &= (Q^{-1}QF, Q^{-1}QG) = (Q^{-2}QF, QG) \\
&= (QF,QG) + (\delta DQF, QG) \\
&= (QF,QG) + (DQF, DQG).
\end{aligned}$$

Therefore, $\exists C_{pq} > 0$, for $p^{-1} + q^{-1} = 1$,

$$\begin{aligned}
|(F,G)| &\le \|QF\|_p\|QG\|_q + \|DQF\|_p\|DQG\|_q \\
&\le C_{pq}\|QF\|_{1,p}\|QG\|_{1,q}.
\end{aligned}$$

But in view of (3.25), it holds that

$$\begin{aligned}
\|F\|_p &= \sup\{|(F,G)| : \|G\|_q \le 1\} \\
&\le \sup\{|(F,G)| : \|QG\|_{1,q} \le C_q\} \\
&\le C_{pq}C_q\|QF\|_{1,p},
\end{aligned}$$

hence (3.33) follows immediately. ∎

Remark. Since $Q\mathcal{P} = \mathcal{P}$ is dense in $\mathbb{D}_1^p$ as well as L^p, it follows from Propositions 3.13 and 3.14 that $Q : L^p \longrightarrow \mathbb{D}_1^p$ is an isomorphism of Banach spaces.

We shall extend these inequalities to higher derivatives. For notational simplicity, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms in some linear space and there exists a constant c such that $\|\cdot\|_1 \le c\|\cdot\|_2$, we simply write $\|\cdot\|_1 \preceq \|\cdot\|_2$, if $\|\cdot\|_1 \preceq \|\cdot\|_2$ and $\|\cdot\|_2 \preceq \|\cdot\|_1$, we write $\|\cdot\|_1 \sim \|\cdot\|_2$.

Theorem 3.15 (Meyer's inequalities) *For $p \in (1,\infty)$, $k \in \mathbb{N}$, $\exists C_{k,p} > 0$ and $\tilde{C}_{k,p} > 0$, such that $\forall F \in \mathcal{P}$,*

$$\tilde{C}_{k,p}\|F\|_p \le \|Q^k F\|_{k,p} \le C_{k,p}\|F\|_p. \qquad (3.34)$$

Proof. We prove the theorem by induction. Define an operator

$$\Gamma : \mathbb{D}_1^p(E) \longrightarrow L^p(E) \oplus L^p(H \otimes E)$$

by

$$\Gamma F \equiv \begin{pmatrix} F \\ DF \end{pmatrix}.$$

Obviously, $\forall k \geq 1$ we have $D\Gamma^k = \Gamma^k D$, moreover, by (3.25) and (3.33) we know that $\Gamma Q : L^p(E) \to L^p(E) \oplus L^p(H \otimes E)$ is a bounded operator with bounded inverse. By definition,

$$\|\Gamma F\|_p \sim \|F\|_{1,p}.$$

By induction on k we can prove that

$$\begin{aligned} \|\Gamma^k F\|_p &\sim \|F\|_{k-1,p} + \|DF\|_{k-1,p} \\ &\sim \|F\|_{k,p}. \end{aligned} \tag{3.35}$$

Define

$$M = \left(\frac{2-\mathcal{L}}{1-\mathcal{L}}\right)^{1/2} = \sum_n \left(\frac{2+n}{1+n}\right)^{1/2} J_n.$$

Since $\varphi(x) = \left(\frac{1+2x}{1-x}\right)^{1/2}$ and $\varphi^{-1}(x)$ are analytic near point 0, letting $\rho_n = \left(\frac{n+2}{n+1}\right)^{1/2} = \left(\frac{1+2/n}{1+1/n}\right)^{1/2}$, it follows from Theorem 3.6 that M and M^{-1} extend to bounded operators in $L^p(E)$. If for $k \geq 1$ denote

$$A_k \equiv \begin{pmatrix} I & 0 \\ 0 & M^{-k} \end{pmatrix},$$

then A_k and A_k^{-1} are bounded in $L^p(E) \oplus L^p(H \otimes E)$ and it holds that

$$\Gamma Q^k = Q^k A_k \Gamma.$$

Denote $B_k \doteq \Gamma^k Q^k$. Then $B_1 = \Gamma Q$ is bounded and

$$\begin{aligned} B_k &= \Gamma^{k-1} \Gamma Q^{k-1} Q \\ &= \Gamma^{k-1} Q^{k-1} A_{k-1} B_1 \\ &= B_{k-1} A_{k-1} B_1. \end{aligned}$$

By induction on k we see that both B_k and B_k^{-1} are bounded, hence by (3.35) we have

$$\begin{aligned} \|Q^k F\|_{k,p} &\sim \|\Gamma^k Q^k F\|_p \\ &= \|B_k F\|_p \\ &\sim \|F\|_p. \end{aligned}$$

This is just Meyer's inequalities (3.34). ∎

3.4 Meyer-Watanabe's generalized functionals

S.Watanabe[1] introduced the following Sobolev spaces:

Definition 3.16 For $p \in (1,\infty)$ and $s \in \mathbb{R}$, define a family of norms on $\mathcal{P}(E)$:

$$\|F\|^{\sim}_{s,p} \equiv \|(I-\mathcal{L})^{s/2} F\|_p. \tag{3.36}$$

Denote by $\widetilde{\mathbb{D}}^p_s(E)$ the Banach space obtained by completion of $\mathcal{P}(E)$ with respect to norm $\|\cdot\|^{\sim}_{s,p}$.

Remark. When $s \in \mathbb{N}$, by Meyer's inequalities we know that $\|\cdot\|^{\sim}_{s,p} \sim \|\cdot\|_{s,p}$; therefore, $\widetilde{\mathbb{D}}^p_s(E) = \mathbb{D}^p_s(E)$. Hereafter we will omit the notation $\sim$ in $\widetilde{\mathbb{D}}^p_s(E)$. The monotonicity and consistency of this family of norms are obvious. Hence, if $p \leq p'$ and $s \leq s'$, then

$$\mathbb{D}^{p'}_{s'}(E) \subset \mathbb{D}^p_s(E).$$

Let $p^{-1} + q^{-1} = 1$. If we identify $L^2(E)$ with its dual space $L^2(E)^*$, then $L^p(E)^* = L^q(E)$. For all $F, G \in \mathcal{P}(E)$, we have

$$\mathbb{E}[(F,G)] = \mathbb{E}[((I-\mathcal{L})^{-s/2}F, (I-\mathcal{L})^{s/2}G)].$$

Therefore

$$\|F\|^{\sim}_{-s,q} = \sup\{|\mathbb{E}[(F,G)]| : \|G\|^{\sim}_{s,p} \leq 1\}, \tag{3.37}$$

which means that $\|\cdot\|^{\sim}_{-s,q}$ is the dual norm of $\|\cdot\|^{\sim}_{s,p}$; hence

$$\mathbb{D}^p_s(E)^* = \mathbb{D}^q_{-s}(E). \tag{3.38}$$

It follows that, for $1 < p \leq q < \infty, 0 \leq r \leq s < \infty$, we have

$$\begin{array}{ccccccccc} \mathbb{D}^p_s(E) & \subset & \mathbb{D}^p_r(E) & \subset & L^p(E) & \subset & \mathbb{D}^p_{-r}(E) & \subset & \mathbb{D}^p_{-s}(E) \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \mathbb{D}^q_s(E) & \subset & \mathbb{D}^q_r(E) & \subset & L^q(E) & \subset & \mathbb{D}^q_{-r}(E) & \subset & \mathbb{D}^q_{-s}(E) \end{array}$$

Note that when $s < 0$, elements in $\mathbb{D}^p_s(E)$ may not belong to $L^p(E)$. Define

$$\mathbb{D}^\infty(E) \equiv \bigcap_{s>0} \bigcap_{1<p<\infty} \mathbb{D}^p_s(E), \tag{3.39}$$

equipped with projective limit topology; and

$$\mathbb{D}^{-\infty}(E) \equiv \bigcup_{s>0} \bigcup_{1<p<\infty} \mathbb{D}^p_{-s}(E), \tag{3.40}$$

equipped with inductive limit topology. The definition of $\mathbb{D}^\infty(E)$ is consistent with that in (2.41). $\mathbb{D}^{-\infty}(E)$ is its dual space, whose elements are naturally called *generalized functionals*.

As consequences of Meyer's inequalities, we have

Proposition 3.17 *The operator D uniquely extends to an operator from $\mathbb{D}^{-\infty}(E)$ into $\mathbb{D}^{-\infty}(H \otimes E)$ such that $\forall p \in (1,\infty)$ and $s \in \mathbb{R}$,*

$$D : \mathbb{D}^p_{s+1}(E) \longrightarrow \mathbb{D}^p_s(H \otimes E)$$

is continuous. Especially,

$$D : \mathbb{D}^{\infty}(E) \longrightarrow \mathbb{D}^{\infty}(H \otimes E)$$

is continuous.

As for the adjoint operator, $\delta = D^*$, we have

Proposition 3.18 *The operator δ uniquely extends to an operator from $\mathbb{D}^{-\infty}(H \otimes E)$ into $\mathbb{D}^{-\infty}(E)$ such that $\forall p \in (1, \infty)$ and $s \in \mathbb{R}$,*

$$\delta : \mathbb{D}^p_{s+1}(H \otimes E) \longrightarrow \mathbb{D}^p_s(E)$$

is continuous. Especially,

$$\delta : \mathbb{D}^{\infty}(H \otimes E) \longrightarrow \mathbb{D}^{\infty}(E)$$

is continuous.

In view of $\mathcal{L} = -\delta D$, we have

Proposition 3.19 *The operator $\mathcal{L}$ uniquely extends to an operator from $\mathbb{D}^{-\infty}(E)$ into $\mathbb{D}^{-\infty}(E)$ such that $\forall p \in (1, \infty)$ and $s \in \mathbb{R}$,*

$$\mathcal{L} : \mathbb{D}^p_{s+2}(E) \longrightarrow \mathbb{D}^p_s(E)$$

is continuous. Especially

$$\mathcal{L} : \mathbb{D}^{\infty}(E) \longrightarrow \mathbb{D}^{\infty}(E)$$

is continuous.

Proposition 3.20 *Let E_1, E_2 be real separable Hilbert spaces. If $p, q \in (1, \infty), k \in \mathbb{N}$ and $p^{-1} + q^{-1} = r^{-1} < 1$, then $\exists C = C(p, q, k) > 0$ such that for all $F \in \mathcal{P}(E_1), G \in \mathcal{P}(E_2)$,*

$$\|F \otimes G\|^{\sim}_{k,r} \leq C \|F\|^{\sim}_{k,p} \|G\|^{\sim}_{k,q}. \tag{3.41}$$

Therefore the map

$$(F, G) \longmapsto F \otimes G$$

extends to a continuous bilinear map from $\mathbb{D}^p_k(E_1) \times \mathbb{D}^q_k(E_2)$ to $\mathbb{D}^r_k(E_1 \otimes E_2)$. Especially, if $E_1 = E_2 = \mathbb{R}, F, G \in \mathbb{D}^{\infty}$, then $FG \in \mathbb{D}^{\infty}$, hence $\mathbb{D}^{\infty}$ is a topological algebra. More generally, $\forall k \in \mathbb{N}$, the $\mathbb{D}^{\infty}_k$ defined by (2.40) is a topological algebra (for $k = 0$, $\mathbb{D}^{\infty}_0 = L^{\infty-}$).

Proof. It is easy to verify that

$$D(F \otimes G) = (DF) \otimes G + F \otimes (DG),$$

and for general $k \in \mathbb{N}$,

$$D^k(F \otimes G) = \sum_{j=0}^{k} \binom{k}{j} (D^j F) \otimes (D^{k-j} G). \tag{3.42}$$

By Meyer's inequalities, we have

$$\begin{aligned} \|F \otimes G\|^{\sim}_{k,r} &\leq \sum_{j=0}^{k} \|D^j (F \otimes G)\|_r \\ &\leq \Big(\sum_{j=0}^{k} \|D^j F\|_p \Big) \Big(\sum_{j=0}^{k} \|D^j G\|_q \Big) \\ &\leq \|F\|^{\sim}_{k,p} \|G\|^{\sim}_{k,q}, \end{aligned}$$

(3.41) is proved. ∎

By duality, we have

Corollary 3.21 *Under the conditions of Proposition 3.20, $\exists \tilde{C} = \tilde{C}(p, q, k) > 0$ such that*

$$\|F \otimes G\|^{\sim}_{-k,r} \leq \tilde{C} \|F\|^{\sim}_{k,p} \|G\|^{\sim}_{-k,q}. \tag{3.43}$$

Especially, $\mathbb{D}^{\infty}(E)$ and $\mathbb{D}^{-\infty}(E)$ are $\mathbb{D}^{\infty}$-modules.

The differentiation formulas (2.25), (2.26), (2.34) and (3.2) etc. can be extended to suitable Sobolev spaces. In particular, they still hold if we substitute $\mathcal{S}_M$ and $\mathcal{S}_M(E)$ by $\mathbb{D}^{\infty}$ and $\mathbb{D}^{\infty}(E)$ respectively.

Proposition 3.22 *If f is a smooth functional on $\mathbb{R}^n$, $\varphi_1, \cdots, \varphi_n \in \mathbb{D}^{\infty}$, then $F = f(\varphi_1, \cdots, \varphi_n) \in \mathbb{D}^{\infty}$ and*

$$DF = \sum_{j=1}^{n} \partial_j f(\varphi_1, \cdots, \varphi_n) D\varphi_j, \tag{3.44}$$

$$\begin{aligned} \mathcal{L}F = &\sum_{j=1}^{n} \partial_j f(\varphi_1, \cdots, \varphi_n) \mathcal{L}\varphi_j \\ &+ \sum_{j,k=1}^{n} \partial_j \partial_k f(\varphi_1, \cdots, \varphi_n) (D\varphi_j, D\varphi_k)_H. \end{aligned} \tag{3.45}$$

Especially, if $F, G \in \mathbb{D}^{\infty}$, then

$$\mathcal{L}(FG) = F\mathcal{L}G + G\mathcal{L}F + 2(DF, DG)_H. \tag{3.46}$$

If $F, G \in \mathbb{D}^{\infty}, V \in \mathbb{D}^{\infty}(H)$, then

$$\mathbb{E}[(DF, V)_H] = \mathbb{E}[F\delta V], \tag{3.47}$$

$$\delta(FV) = F\delta V - (DF, V)_H, \tag{3.48}$$

$$\delta(FDG) = -F\mathcal{L}G - (DF, DG)_H, \tag{3.49}$$

$$\mathbb{E}[F\mathcal{L}G] = -\mathbb{E}[(DF, DG)_H] = \mathbb{E}[G\mathcal{L}F]. \tag{3.50}$$

The proofs are left to the reader. Moreover, eq (3.47) can be extended to the case of generalized functionals. For example, for $s \in \mathbb{R}, F \in \mathbb{D}^p_{s+1}, V \in \mathbb{D}^q_{-s}(H)$ $(p^{-1} + q^{-1} = 1)$, we have

$$\langle DF, V \rangle = \langle F, \delta V \rangle, \tag{3.51}$$

where $\langle\cdot,\cdot\rangle$ on two sides are canonical bilinear forms on $\mathbb{D}_r^p(H)\times\mathbb{D}_{-s}^q(H)$ and $\mathbb{D}_{s-1}^p\times\mathbb{D}_{-r-1}^q$ respectively. It is notable that, the canonical bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{D}^{-\infty}\times\mathbb{D}^{\infty}$ is a natural extension of expectation. Indeed, if $F,G\in\mathbb{D}^\infty$, then $\langle F,G\rangle=\mathbb{E}[FG]$. Especially, if $F\in\mathbb{D}^{-\infty}, G=1$, we call $\langle F,1\rangle$ the generalized expectation of F.

Finally, we point out a remarkable property of OU semigroup $\{T_t, t>0\}$ which plays the role of "modifier" in finite dimensional spaces.

Proposition 3.23 *For $t>0, p\in(1,\infty)$ and any real numbers $r\le s$, the operator*

$$T_t:\mathbb{D}_r^p(E)\longrightarrow\mathbb{D}_s^p(E)$$

is continuous. For $F\in\mathbb{D}_r^p(E)$, T_tF converges to F in $\mathbb{D}_r^p(E)$ when $t\downarrow 0$.

Proof. We may assume that $E=\mathbb{R}$. Since T_t and Q commute, it suffices to consider the case $s=0$ (hence $r\le 0$). It follows from (3.16) that, $\forall n_0\in\mathbb{N}$, $\exists c=c(p,n_0)$ so that $\forall F\in L^p, n\ge n_0$,

$$\|T_tJ_nF\|_p\le ce^{-nt}\|F\|_p.$$

Therefore,

$$\|T_tJ_n\|_{\mathcal{L}(L^p)}\le ce^{-nt},$$
$$\|Q^rT_tJ_n\|_{\mathcal{L}(L^p)}\le ce^{-nt}(1+n)^{-r/2}.$$

Since

$$\sum_{n=n_0}^{\infty}\|Q^rT_tJ_n\|_{\mathcal{L}(L^p)}<\infty,$$

it follows that $Q^rT_t\in\mathcal{L}(L^p)$. $Q^r: L^p\to\mathbb{D}_r^p$ being isomorphism of Banach spaces, hence $T_t:\mathbb{D}_r^p\to L^p$ is continuous. The last assertion of proposition follows from commutativity of T_t and $Q^{\pm r}$. ∎

Corollary 3.24 *If $F\in L^{\infty-}(\Omega;E), t>0$, then $T_tF\in\mathbb{D}^\infty(E)$. Especially, if F is a bounded measurable function, then $T_tF\in\mathbb{D}^\infty$.*

§4. Densities of non-degenerate functionals

One of the most important applications of Malliavin calculus lies in the investigation of existence, smoothness as well as many other properties of densities of Brownian functionals. Let $(\Omega,\mathcal{F},\mu;H)$ be a Gaussian probability space, F be an $\mathbb{R}^m$-valued functional (i.e. m-dimensional random vector); its distribution $\mu\circ F^{-1}$ is a probability measure on $\mathcal{B}(\mathbb{R}^m)$. Malliavin calculus provides us an extremely effective method to investigate the conditions under which $\mu\circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure λ^m on $\mathbb{R}^m$ and properties of its density (i.e. the Radon-Nikodym derivative of $\mu\circ F^{-1}$ with respect to λ^m).

4.1 Malliavin covariance matrices, some lemmas

Definition 4.1 *For $F=(F_1,\cdots,F_m)\in\mathbb{D}_1^1(\mathbb{R}^m)$, define*

$$\sigma_{ij}\equiv(DF_i,DF_j)_H,\quad 1\le i,j\le m. \tag{4.1}$$

the matrix $\Sigma(\omega)=(\sigma_{ij}(\omega))_{1\le i,j\le m}$ is called Malliavin covariance matrix. If $\det\Sigma(\omega)>0$ a.s. and

$$[\det\Sigma(\omega)]^{-1}\in L^{\infty-}, \tag{4.2}$$

then Σ (or F itself) is called non-degenerate in the sense of Malliavin (non-degenerate for short).

Lemma 4.2 *If $F\in\mathbb{D}_2^\infty(\mathbb{R}^m)$, then its Malliavin covariance matrix $\Sigma\in\mathbb{D}_1^\infty(\mathbb{R}^m\otimes\mathbb{R}^m)$; if F is non-degenerate, then the inverse matrix $\Sigma^{-1}(\omega)\equiv(\gamma_{ij}(\omega))_{1\le i,j\le m}$ exists a.s., moreover, $\Sigma^{-1}\in\mathbb{D}_1^\infty(\mathbb{R}^m\otimes\mathbb{R}^m)$.*

Proof. For $i,j=1,\cdots,m$, we have $|\sigma_{ij}|\le\|DF_i\|\|DF_j\|$, hence $\sigma_{ij}\in L^{\infty-}$. By a straightforward computation using definition (2.22) we have

$$(D\sigma_{ij},h)_H=(D^2F_i,h\otimes DF_j)_{H\otimes H}+(D^2F_j,h\otimes DF_i)_{H\otimes H},$$
$$\forall h\in H. \tag{4.3}$$

Since $F_i,F_j\in\mathbb{D}_2^\infty$, $D\sigma_{ij}\in L^{\infty-}$, hence $\sigma_{ij}\in\mathbb{D}_1^\infty$. By differentiation of identity

$$\gamma_{ij}=\sum_{k,l=1}^m\gamma_{ik}\sigma_{kl}\gamma_{lj},\quad 1\le i,j\le m,$$

we have

$$D\gamma_{ij}=-\sum_{k,l=1}^m\gamma_{ik}\gamma_{lj}D\sigma_{kl},\quad 1\le i,j\le m. \tag{4.4}$$

Since γ_{ij} is expressed as $(\det\Sigma)^{-1}$ times a polynomial of elements in Σ, $\gamma_{ij}\in L^{\infty-}$. It follows from eq. (4.4) that $D\gamma_{ij}\in L^{\infty-}$, therefore, $\gamma_{ij}\in\mathbb{D}_1^\infty$. ∎

Remark. By the same reason, if $F\in\mathbb{D}^\infty(\mathbb{R}^m)$, then its covariance matrix $\Sigma\in\mathbb{D}^\infty(\mathbb{R}^m\otimes\mathbb{R}^m)$; if, moreover, F is non-degenerate, then $\Sigma^{-1}\in\mathbb{D}^\infty(\mathbb{R}^m\otimes\mathbb{R}^m)$.

In the proof of existence of densities of non-degenerate functionals, the key tools are formula of integration by parts (i.e. adjointness of D and δ as well as selfadjointness of OU operator $\mathcal{L}$) and the following lemma in harmonic analysis. We firstly prove an elementary inequality in finite dimensional spaces.

Lemma 4.3 (Gagliardo-Nirenberg inequality) *If $m>1$, and $m^*=m/(m-1)$, then $\forall\varphi\in C_0^\infty(\mathbb{R}^m)$,*

$$\|\varphi\|_{m^*}\le\frac{1}{m}\sum_{j=1}^m\|\partial_j\varphi\|_1. \tag{4.5}$$

Proof. Denoting $x = (x_1, \cdots, x_m)$, for $j = 1, \cdots, m$ we have

$$\varphi(x) = \int_{-\infty}^{x_j} \partial_j \varphi(x) dx_j,$$

hence

$$|\varphi(x)| \le \int_{-\infty}^{\infty} |\partial_j \varphi(x)| dx_j,$$

$$|\varphi(x)|^{m^*} \le \prod_{j=1}^{m} \left(\int_{-\infty}^{\infty} |\partial_j \varphi(x)| dx_j \right)^{1/(m-1)}.$$

Integrasing the last inequality successively for variables $x_1, \cdots, x_m$, noting that $\int |\partial_j \varphi| dx_j$ is independent of x_j and using the following extended Hölder inequality for other $m-1$ factors,

$$\left| \int \left(\prod_{j=1}^{n} f_j \right) \right| \le \prod_{j=1}^{n} \|f_j\|_n, \tag{4.6}$$

we obtain that

$$\int_{\mathbb{R}^m} |\varphi(x)|^{m^*} dx \le \prod_{j=1}^{m} \|\partial_j \varphi\|_1^{1/(m-1)}.$$

Therefore,

$$\|\varphi\|_{m^*} \le \prod_{j=1}^{m} \|\partial_j \varphi\|_1^{1/m} \le \frac{1}{m} \sum_{j=1}^{m} \|\partial_j \varphi\|_1. \qquad \blacksquare$$

Following is a key lemma in harmonic analysis:

Lemma 4.4 *Let ν be a finite measure on $\mathcal{B}(\mathbb{R}^m)$, if for $j = 1, \cdots, m$, there exists a constant c_j such that $\forall \varphi \in C_0^\infty(\mathbb{R}^m)$,*

$$\left| \int_{\mathbb{R}^m} \partial_j \varphi(x) \nu(dx) \right| \le c_j \|\varphi\|_\infty, \tag{4.7}$$

then ν is absolutely continuous with respect to Lebesgue measure. When $m > 1$, it has density $\rho \in L^{m^*}(\mathbb{R}^m)$, where $m^* = m/(m-1)$.

Proof. In the case $m = 1$, taking any interval (a, b), let φ be the distribution function of uniformly distributed random variable on (a, b). Choosing a sequence $\varphi_n \in C_0^\infty(\mathbb{R})$ such that $\varphi_n \longrightarrow \varphi$ and $\varphi_n' \longrightarrow \varphi'$. By (4.7) we have $\nu([a, b]) \le c_1(b-a)$ which means that ν is absolutely continuous with respect to Lebesgue measure.

In the case $m > 1$, take a modifier $\psi_\epsilon \in C_0^\infty(\mathbb{R}^m)$ and making involution

$$\rho_\epsilon(x) \equiv \int_{\mathbb{R}^m} \psi_\epsilon(x - y) \nu(dy), \quad \epsilon > 0.$$

Then $\rho_\epsilon \in C_b^\infty(\mathbb{R}^m)$ and $\forall \varphi \in C_0^\infty(\mathbb{R}^m)$. We have

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^m} \rho_\epsilon(x) \varphi(x) dx = \int_{\mathbb{R}^m} \varphi(x) \nu(dx) \tag{4.8}$$

(by some technical treatment we may assume that $\rho_\epsilon \in C_0^\infty(\mathbb{R}^m)$). Hence for sufficiently small ϵ we have

$$\left| \int_{\mathbb{R}^m} \partial_j \rho_\epsilon(x) \varphi(x) dx \right| = \left| \int_{\mathbb{R}^m} \rho_\epsilon(x) \partial_j \varphi(x) dx \right|$$
$$\le c_j \|\varphi\|_\infty, \quad j = 1, \cdots, m.$$

It follows that

$$\|\partial_j \rho_\epsilon\|_1 \le c_j, \quad j = 1, \cdots, m.$$

By inequality (4.5), $\exists c > 0$ such that

$$\|\rho_\epsilon\|_{m^*} \le \frac{1}{m} \sum_{j=1}^{m} \|\partial_j \rho_\epsilon\|_1 \le c.$$

Since the bounded sets in L^{m^*} are relatively weakly compact, there exists a subsequence $\{\rho_{\epsilon_k}\}$ in L^{m^*} weakly converging to some function ρ. By eq. (4.8) we have

$$\int_{\mathbb{R}^m} \rho(x) \varphi(x) dx = \int_{\mathbb{R}^m} \varphi(x) \nu(dx), \quad \varphi \in C_0^\infty(\mathbb{R}^m),$$

hence $\nu(dx) = \rho(x) dx$. $\blacksquare$

4.2 Existence of densities

Firstly we discuss the existence and continuity of densities for functional $F \in \mathbb{D}_2^\infty(\mathbb{R}^m)$.

Theorem 4.5 *Let $F = (F_1, \cdots, F_m) \in \mathbb{D}_2^\infty(\mathbb{R}^m)$ be non-degenerate. Then for $i = 1, \cdots, m$, $\exists \Phi_i \in L^{\infty-}$ such that $\forall \varphi \in C_0^\infty(\mathbb{R}^m)$,*

$$\mathbb{E}[\partial_i \varphi \circ F] = \mathbb{E}[\Phi_i \cdot (\varphi \circ F)], \tag{4.9}$$

hence F has a density ρ. In the case $m > 1$, $\rho \in L^{m^*}(\mathbb{R}^m)$, where $m^* = m/(m-1)$.

Proof. By lemma 4.2, the covariance matrix $\Sigma = (\sigma_{ij})$ of F and its inverse $\Sigma^{-1} = (\gamma_{ij})$ belong to $\mathbb{D}_1^\infty(\mathbb{R}^m \otimes \mathbb{R}^m)$. For $i = 1, \cdots, m$, let

$$Z_i \equiv \sum_{j=1}^{m} \gamma_{ij} DF_j, \quad \Phi_i = \delta Z_i.$$

Then $Z_i \in \mathbb{D}_1^\infty(H)$. Using eq. (3.49) (evidently it can be extended to the case of $\mathbb{D}_2^\infty$) we have

$$\Phi_i = -\sum_{j=1}^m (\gamma_{ij} LF_j + (D\gamma_{ij}, DF_j)_H), \tag{4.10}$$

hence $\Phi_i \in L^{\infty-}$. It follows from eq. (3.44) (it also holds for the case of $\mathbb{D}_2^\infty$) that

$$D(\varphi \circ F) = \sum_{i=1}^m (\partial_i \varphi \circ F) DF_i \in \mathbb{D}_1^\infty(H),$$

and therefore

$$(D(\varphi \circ F), DF_j)_H = \sum_{i=1}^m (\partial_i \varphi \circ F)\sigma_{ij}, \quad j = 1, \cdots, m. \tag{4.11}$$

Σ being non-degenerate, by solving eq. (4.11) we obtain that

$$\begin{aligned}\partial_i \varphi \circ F &= \sum_{j=1}^m \gamma_{ij}(D(\varphi \circ F), DF_j)_H \\ &= (D(\varphi \circ F), Z_i)_H, \quad i = 1, \cdots, m.\end{aligned} \tag{4.12}$$

Using formula (3.47) we have

$$\begin{aligned}\mathbb{E}[\partial_i \varphi \circ F] &= \mathbb{E}[(D(\varphi \circ F), Z_i)_H] \\ &= \mathbb{E}[(\varphi \circ F)\delta Z_i] \\ &= \mathbb{E}[(\varphi \circ F)\Phi_i], \quad i = 1, \cdots, m.\end{aligned}$$

(4.9) is proved. Denote by $\nu_F \equiv \mu \circ F^{-1}$ the distribution of F. Then

$$\begin{aligned}\left| \int_{\mathbb{R}^m} \partial_i \varphi(x) \nu_F(dx) \right| &= |\mathbb{E}[\partial_i \varphi \circ F]| \\ &= |\mathbb{E}[(\varphi \circ F)\Phi_i]| \\ &\le \mathbb{E}[|\Phi_i|] \cdot |\varphi|_\infty.\end{aligned}$$

Thus the theorem follows from Lemma 4.4. ∎

Remark. If $F \in \mathbb{D}_2^\infty(\mathbb{R}^m)$ and $(\det \Sigma)^{-1} \in L^p$ for $p > 4m$, then by Hölder inequality we can show that $\Phi_i \in L^r$ for $r = p/4m$.

Let $\beta < 1$. Define

$$\mathcal{H}^\beta(\mathbb{R}^m) \equiv \left\{ \varphi \in C(\mathbb{R}^m) : \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{\|x - y\|^\beta} < \infty \right\} \tag{4.13}$$

to be the space of functions of β-Hölder continuous. Then we have

Theorem 4.6 *Under the assumption of Theorem 4.5, for all $\beta < 1$ and $r > 0$, F has density ρ satisfying that*

$$\rho^r \in \mathcal{H}^\beta(\mathbb{R}^m). \tag{4.14}$$

Proof. For $i = 1, \cdots, m$, let $q_i(x) \equiv \mathbb{E}[\Phi_i | F = x]$, where $\Phi_i \in L^{\infty-}$ is given by Theorem 4.5. Then $\forall \varphi \in C_0^\infty(\mathbb{R}^m)$ we have

$$\begin{aligned}\int_{\mathbb{R}^m} \partial_i \varphi(x)\rho(x)dx &= \mathbb{E}[\partial_i \varphi \circ F] \\ &= \mathbb{E}[(\varphi \circ F)\Phi_i] \\ &= \mathbb{E}[(\varphi \circ F)\mathbb{E}[\Phi_i | F]] \\ &= \int_{\mathbb{R}^m} \varphi(x) q_i(x) \rho(x) dx.\end{aligned} \tag{4.15}$$

It means that, in the sense of generalized derivative,

$$\partial_i \rho = -q_i \rho, \quad i = 1, \cdots, m.$$

Denote $\|\nabla \rho\| \equiv (\sum_{i=1}^m (\partial_i \rho)^2)^{1/2}$. Then $\forall p > 1$,

$$\rho^{-p}\|\nabla \rho\|^p = \left(\sum_{i=1}^m q_i^2 \right)^{p/2} \le m^p \sum_{i=1}^m |q_i|^p.$$

Integrating with respect to measure $\rho(x)dx$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^m} \|\nabla \rho\|^p \rho^{1-p} dx &\le m^p \sum_{i=1}^m \mathbb{E}[|\mathbb{E}[\Phi_i | F]|^p] \\ &\le m^p \sum_{i=1}^m \|\Phi_i\|_p^p < \infty.\end{aligned}$$

Since $\|\nabla \rho^{1/p}\| = \frac{1}{p}\rho^{1/p-1}\|\nabla \rho\|$, we have

$$\|\nabla \rho^{1/p}\|_p^p = p^{-p} \int_{\mathbb{R}^m} \|\nabla \rho\|^p \rho^{1-p} dx < \infty,$$

which means that $\rho^{1/p}$ belongs to Sobolev space $W^{1,p}(\mathbb{R}^m)$. But by Sobolev embedding theorem we have

$$\bigcap_{p > 1} W^{1,p}(\mathbb{R}^m) \subset \bigcap_{\beta < 1} \mathcal{H}^\beta(\mathbb{R}^m), \tag{4.16}$$

which completes the proof. ∎

If we only claim the existence of density, then conditions can be weakened enormously. Bouleau - Hirsch[1,2] obtained the following result:

Theorem 4.7 *Let $p > 1$, $F \in \mathbb{D}_1^p(\mathbb{R}^m)$. If its covariance matrix Σ is invertible a.s., then F has a density.*

The proof is rather complicate, cf. Bouleau - Hirsch[2] or Nualart[1]. Here we give a proof in the special case of $m = 1$ (in that case it even holds for $p = 1$).

Proof. Let $F \in \mathbb{D}_1^1$, we may assume that F is bounded, for instance $|F| < 1$. To prove the absolute continuity, it suffices to prove that for any Borel measurable function $g : (-1, 1) \longrightarrow [0, 1]$, we have $\mathbb{E}[g \circ F] = 0$ whenever $\int_{-1}^1 g(y)dy = 0$.

Choose a sequence of functions $\{g_n\} \subset C_b^1(-1, 1)$ such that

$$\lim_{n\to\infty} g_n(y) = g(y) \quad \text{a.e. } [\mu \circ F^{-1} + \lambda^1].$$

Denote

$$\psi_n(y) = \int_{-1}^y g_n(x)dx, \quad \psi(y) = \int_{-1}^y g(x)dx.$$

By the rule of differentiation for composed functions, $\psi_n \circ F \in \mathbb{D}_1^1$ and $D[\psi_n(F)] = g_n(F)DF$. Since $g_n \longrightarrow g$ a.e. $[\lambda^1]$, we have

$$\lim_{n\to\infty} \psi_n \circ F = \psi \circ F, \qquad \text{a.s. and in } L^1.$$

Since $g_n \longrightarrow g$ a.e. $[\mu \circ F^{-1}]$, we also have

$$\lim_{n\to\infty} D[\psi_n(F)] = g(F)DF, \qquad \text{a.s. and in } L^1(H).$$

From the fact that $\psi \circ F = 0$ a.s. and the closedness of operator D we deduce that $g(F)DF = 0$ a.s.. But $\|DF\|_H > 0$ a.s., it follows that $g \circ F = 0$ a.s.. ∎

Remark. We see from the proof that, even if Σ is not invertible a.s., $\mu \circ F^{-1}$ is still absolutely continuous on $\{\det \Sigma > 0\}$, that is, for any Borel set B with Lebesgue measure 0, $\mu\{F \in B, \det \Sigma > 0\} = 0$. Some other results for one-dimensional case can be found in Yan[3].

4.3 Smoothness of densities

We shall prove that the functional F is infinitely differentiable under the hypothesis that $F \in \mathbb{D}^\infty(\mathbb{R}^m)$. Here we follow the succinct and inspiring method due to S Watanabe[1], namely to give a rigorous meaning to the composition of Schwartz distributions with functionals. Note that $p(x)$, the density of F, can be formally expressed as $\mathbb{E}[\delta_x \circ F]$, i.e. the expectation of composition of Dirac δ function with F. Evidently, $\delta_x \circ F$ cannot be a functional in usual sense. However, we know that if $\varphi \in \mathcal{S}(\mathbb{R}^m)$, the composite functional $\varphi \circ F \in \mathbb{D}^\infty$. For fixed F, the map: $\varphi \to \varphi \circ F$ is a linear map from $\mathcal{S}(\mathbb{R}^m)$ into $\mathbb{D}^\infty$. If we can extend it to a linear and (in some sense) continuous map from $\mathcal{S}^*(\mathbb{R}^m)$ into $\mathbb{D}^{-\infty}$, then $\delta_x \circ F$ can be interpreted as a generalized functional.

Consider the Schwartz tempered distribution space $\mathcal{S}^*(\mathbb{R}^m)$. For the sake of convenience, we introduce a family of norms which is slightly different from but equivalent to that in the example of §3 of Chapter I:

$$\|f\|_{2k}^\sim = \|(1 + |x|^2 - \Delta)^k f\|_\infty, \quad k \in \mathbb{Z}. \tag{4.17}$$

Denote by $\mathcal{T}_{2k}$ the Banach space of completion of $\mathcal{S}(\mathbb{R}^m)$ with respect to norm $\|\cdot\|_{2k}^\sim$. Then we still have (cf. Reed-Simon[1])

$$\mathcal{S}(\mathbb{R}^m) = \varprojlim_k \mathcal{T}_{2k},$$

$$\mathcal{S}^*(\mathbb{R}^m) = \varinjlim_k \mathcal{T}_{-2k}.$$

Lemma 4.8 *Let $\delta_y, y \in \mathbb{R}^m$, be the Dirac δ function, $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_0^m$, $|\alpha| \equiv \sum_{j=1}^m \alpha_j$, $\partial_\alpha \equiv \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$. If $n > m/2$, then $\delta_y \in \mathcal{T}_{-2n}$; if $|\alpha| \le 2k$, then $\partial_\alpha \delta_y \in \mathcal{T}_{-2n-2k}$ and the map*

$$\mathbb{R}^m \ni y \mapsto \delta_y \in \mathcal{T}_{-2n-2k}$$

is $2k$-time continuously differentiable.

Proof. Since the Fourier transformation of δ_y is $e^{-i(\xi \cdot y)}$, it follows that

$$\mathcal{F}[(1 - \Delta)^{-n}\delta_y](\xi) = (1 + |\xi|^2)^{-n} e^{-i(\xi \cdot y)}.$$

By inverse transformation we have

$$[(1 - \Delta)^{-n}\delta_y](x) = \left(\frac{1}{2\pi}\right)^m \int_{\mathbb{R}^m} \frac{e^{i(\xi, x-y)}}{(1 + |\xi|^2)^n} d\xi.$$

If $n > m/2$, then

$$\|(1 + |x|^2 - \Delta)^{-n}\delta_y\|_\infty \le \|(1 - \Delta)^{-n}\delta_y\|_\infty < \infty,$$

hence $\delta_y \in \mathcal{T}_{-2n}$ and the map $y \mapsto (1 + |x|^2 - \Delta)^{-n}\delta_y \in \mathcal{T}_0$ is continuous. If $|\alpha| \le 2k$ and $\varphi \in \mathcal{T}_{2n+2k}$, then $\partial_\alpha \varphi \in \mathcal{T}_{2n}$ and

$$(\partial_\alpha \delta_y, \varphi) = (-1)^{|\alpha|} \partial_\alpha \varphi(y),$$

hence the map $y \mapsto \delta_y \in \mathcal{T}_{-2n-2k}$ is $2k$-time continuously differentiable. ∎

Theorem 4.9 (S. Watanabe) *If $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ is non-degenerate, then $\forall p \in (1, \infty)$ and $n \in \mathbb{N}_0$, $\exists c = c(p, n) > 0, \forall \varphi \in \mathcal{S}(\mathbb{R}^m)$ we have*

$$\|\varphi \circ F\|_{-2n,p} \le c\|\varphi\|_{-2n}^\sim, \tag{4.18}$$

hence the map $\varphi \mapsto \varphi \circ F$ extends uniquely to a linear map from $\mathcal{S}^*(\mathbb{R}^m)$ into $\mathbb{D}^{-\infty}$ and it is continuous when restricted to $\mathcal{T}_{-2n}$ as a map into $\mathbb{D}^p_{-2n}$.

Proof. Similar to the proof of Theorem 4.5, we obtain eqs. (4.11) and (4.12) except that $\varphi \in \mathcal{S}(\mathbb{R}^m)$ and all functionals are in $\mathbb{D}^\infty$.

For $G \in \mathbb{D}^\infty, i = 1, \cdots, m$, let

$$\Phi_i(G) \equiv \sum_{j=1}^m \delta(\gamma_{ij} G DF_j).$$

By eq. (3.49) we have

$$
\begin{aligned}
\Phi_i(G) &= -\sum_{j=1}^{m}\{\gamma_{ij}G\mathcal{L}F_j + (D(\gamma_{ij}G), DF_j)_H\} \\
&= -\sum_{j=1}^{m}\{\gamma_{ij}\mathcal{L}F_j + (D\gamma_{ij}, DF_j)_H\}G - \Big(\sum_{j=1}^{m}\gamma_{ij}DF_j, DG\Big)_H \\
&= \Psi_0 G - (\Psi_1, DG)_H,
\end{aligned}
$$

where $\Psi_0 \in \mathbb{D}^{\infty}, \Psi_1 \in \mathbb{D}^{\infty}(H)$. By eq. (4.12) and formula of integration by parts (3.47) we have

$$
\begin{aligned}
\mathbb{E}[G(\partial_i\varphi\circ F)] &= \sum_{j=1}^{m}\mathbb{E}[(\gamma_{ij}GDF_j, D(\varphi\circ F))_H] \\
&= \sum_{j=1}^{m}\mathbb{E}[(\varphi\circ F)\delta(\gamma_{ij}GDF_j)] \\
&= \mathbb{E}[(\varphi\circ F)\Phi_i(G)], \qquad i = 1,\cdots,m.
\end{aligned}
$$

Replacing φ by $\partial_i\varphi$, G by $\Phi_i(G)$, following the same procedure, we can replace i by any multi-index $\alpha = (\alpha_1,\cdots,\alpha_n)$ and obtain

$$
\mathbb{E}[G(\partial_\alpha\varphi\circ F)] = \mathbb{E}[(\varphi\circ F)\Phi_\alpha(G)]. \tag{4.19}
$$

When $|\alpha| = 2n$, $\Phi_\alpha(G)$ has form

$$
\Phi_\alpha(G) = \Psi_0 G + (\Psi_1, DG)_H + \cdots + (\Psi_{2n}, D^{2n}G)_{H^{\otimes 2n}}, \tag{4.20}
$$

where $\Psi_k \in \mathbb{D}^{\infty}(H^{\otimes k})$ is some polynomial of γ_{ij}, F_j $(1 \le i,j \le m)$ and their derivatives. Especially, for $A = 1 + |x|^2 - \Delta$ we have

$$
\mathbb{E}[G(A^n\varphi\circ F)] = \mathbb{E}[(\varphi\circ F)\Phi_A(G)],
$$

where $\Phi_A(G)$ still has form (4.20). Hence

$$
\begin{aligned}
|\mathbb{E}[G(\varphi\circ F)]| &= |\mathbb{E}[G(A^nA^{-n}\varphi\circ F)]| \\
&= |\mathbb{E}[(A^{-n}\varphi\circ F)\Phi_A(G)]| \\
&\le \|A^{-n}\varphi\|_\infty\|\Phi_A(G)\|_1 \\
&= \|\Phi_A(G)\|_1\|\varphi\|^{\sim}_{-2n}.
\end{aligned}
$$

If $p^{-1} + q^{-1} = 1$, then

$$
\begin{aligned}
\|\varphi\circ F\|_{-2n,p} &= \sup\{|\mathbb{E}[G(\varphi\circ F)]| : \|G\|_{2n,q} \le 1\} \\
&\le \|\varphi\|^{\sim}_{-2n}\sup\{\|\Phi_A(G)\|_1 : \|G\|_{2n,q} \le 1\}.
\end{aligned}
$$

It follows from (4.20) that

$$
c(p,n) \equiv \sup\{\|\Phi_A(G)\|_1 : \|G\|_{2n,q} \le 1\} < \infty. \qquad ∎
$$

The above theorem defines a "generalized composition" of distribution $T \in \mathcal{S}'(\mathbb{R}^m)$ with non-degenerate functional $F \in \mathbb{D}^{\infty}(\mathbb{R}^m)$: $T\circ F$. Watanabe[1] called the map $T \mapsto T\circ F$ *pullback* of T under the map $F : \Omega \longrightarrow \mathbb{R}^m$, while Malliavin[5] referred to it as *lifting up* of T by F because it lifts up distributions on finite dimensional spaces to generalized functionals on infinite dimensional spaces. As we shall see, its dual map is nothing but the conditional expectation given F (as a random vector), $\mathbb{E}^F$. So we may write

$$
(\mathbb{E}^F)^* : \mathcal{S}'(\mathbb{R}^m) \longrightarrow \mathbb{D}^{-\infty} \tag{4.21}
$$

for $(\mathbb{E}^F)^*T \equiv T\circ F$. In particular, if $T = \delta_x$, then

$$
\rho_F(x) \equiv \langle\delta_x\circ F, 1\rangle \tag{4.22}
$$

is just the density of functional F. Therefore, we have the following important result:

Theorem 4.10 *If $F \in \mathbb{D}^{\infty}(\mathbb{R}^m)$ is non-degenerate, then F has density $\rho_F(x)$ which is infinitely differentiable.*

Proof. By Theorem 4.9 and Lemma 4.8, if $n > m/2$, then for $p \in (1,\infty), k \in \mathbb{N}_0$, the map $\mathbb{R}^m \ni x \mapsto \delta_x\circ F \in \mathbb{D}^p_{-2n-2k}$ is $2k$-time continuously differentiable. Since k is arbitrary, it follows that ρ_F defined by (4.22) belongs to $C^\infty(\mathbb{R}^m)$.

Consider $\mathbb{E}^F$, the dual map of (4.21) which is a continuous map from $\mathbb{D}^q_{2n}$ into $\mathcal{T}_{2n}$ and satisfies that $\rho_F(x) = \langle\delta_x, \mathbb{E}^F 1\rangle = (\mathbb{E}^F 1)(x)$. For all $\varphi \in \mathcal{S}(\mathbb{R}^m)$ we have

$$
\begin{aligned}
\mathbb{E}[\varphi\circ F] &= \langle\varphi\circ F, 1\rangle = \langle\varphi, \mathbb{E}^F 1\rangle \\
&= \int_{\mathbb{R}^m}\varphi(x)\rho_F(x)dx.
\end{aligned}
$$

This shows that ρ_F defined by (4.22) is just the density of F. ∎

Replacing 1 by $G \in \mathbb{D}^q_{2n+2k}$ in the proof of Theorem 4.10, we obtain

$$
\begin{aligned}
\mathbb{E}[(\varphi\circ F)G] &= \langle\varphi\circ F, G\rangle = \langle\varphi, \mathbb{E}^F G\rangle \\
&= \int_{\mathbb{R}^m}\varphi(x)\langle\delta_x\circ F, G\rangle dx
\end{aligned}
$$

Therefore, it holds on the set $\{x : \rho_F(x) > 0\}$ that

$$
\langle\delta_x\circ F, G\rangle = \rho_F(x)\mathbb{E}[G|F = x], \quad \text{a.e.}. \tag{4.23}
$$

As a consequence, we obtain the following important result on regularity of conditional expectations:

Corollary 4.11 *If $n > m/2, p \in (1,\infty), k \in \mathbb{N}_0, G \in \mathbb{D}^p_{2n+2k}$, then, on the set $\{x : \rho_F(x) > 0\}$, the conditional expectation $\mathbb{E}[G|F = x]$ has C^{2k}-modification. In particular, if $G \in \mathbb{D}^\infty$, then $\mathbb{E}[G|F = x]$ has C^∞-modification.*

4.4 Examples

The typical example of applications is a probabilistic proof of Hörmander's theorem on hypoellipticity of partial differential operators which we shall discuss in next chapter. Here we only give two simple examples.

Example 1 (Donsker δ function). Let $H = L^2(\mathbb{R}_+; \mathbb{R}^d), \{W(t), t \in \mathbb{R}_+\}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\mathcal{F}^W$ be the completed σ-algebra generated by it. For $h \in H$, define

$$W_h(\omega) \equiv \sum_{j=1}^d \int_0^\infty h_j(t) dW_j(t).$$

Then $(\Omega, \mathcal{F}^W, \mathbb{P}; H)$ is an irreducible Gaussian probability space. For fixed $t \in \mathbb{R}_+, W(t)$ is an $\mathbb{R}^d$-valued polynomial functional. If we denote the base in $\mathbb{R}^d$ by $\{e_1, \cdots, e_d\}$, then

$$DW_j(t) = 1_{[0,t]} e_j, \quad j = 1, \cdots, d.$$

Hence $W(t) \in \mathbb{D}^\infty(\mathbb{R}^d)$ and

$$\sigma_{ij}(t) \equiv (DW_i(t), DW_j(t))_H = \delta_{ij} t, \qquad i, j = 1, \cdots, d.$$

For $t > 0$, $W(t)$ being non-degenerate, it has C^∞ density

$$\begin{aligned} p_t(x) &= (\delta_x(W(t)), 1) \\ &= (2\pi t)^{-1/2} \exp\{-|x|^2/2t\}, \end{aligned}$$

where $\delta_x(W(t)) \in \mathbb{D}^{-\infty}$ is called Donsker δ function. For $F \in \mathbb{D}^\infty$, the conditional expectation

$$\mathbb{E}[F|W(t) = x] = p_t(x)^{-1}(\delta_x(W(t)), F)$$

has C^∞-modification.

As we know (for example, cf. Ikeda - Watanabe[3]), the fundamental solution of the following parabolic equation

$$\partial_t u = \frac{1}{2}\Delta u + v \cdot u$$

can be expressed informally as

$$p(t,x,y) = \mathbb{E}\left[\delta_y(x + W(t)) \exp\left\{\int_0^t v(x + W(s))ds\right\}\right]. \tag{4.24}$$

If $v \in C^{2n}(\mathbb{R}^d), n \geq [d/2] + 1$, has polynomially bounded derivatives and satisfies that

$$\overline{\lim_{|x|\to\infty}} v(x)/|x|^2 = a < \infty,$$

then in the case $a < 1/2t$, we can prove that $\exists p > 1$ such that

$$G = \exp\left\{\int_0^t v(x + W(s))ds\right\} \in \mathbb{D}^p_{2n}.$$

Hence, $p(t,x,\cdot) \in C^{2k}(\mathbb{R}^d)$ with $k = n - 1 - [d/2]$.

Before giving next example, we prove a useful proposition.

Proposition 4.12 *Let $(\Omega, \mathcal{F}, \mu; H)$ be an irreducible Gaussian probability space. If $\varphi_1, \cdots, \varphi_n \in \mathbb{D}^2_1$, f is a Lipschitz function on $\mathbb{R}^n$ with Lipschitz constant K, then $F \equiv f(\varphi_1, \cdots, \varphi_n) \in \mathbb{D}^2_1$, and*

$$DF = \sum_{j=1}^n G_j D\varphi_j, \tag{4.25}$$

where r.v. $G = (G_1, \cdots, G_n)$ satisfies that $|G| \leq K$.

Proof. Firstly suppose that $f \in C^1_b(\mathbb{R}^n)$. By differential rule (2.26) and approximation by smooth functionals we know that $F \in \mathbb{D}^2_1$ and eq. (4.25) holds for $G_j = \partial_j f(\varphi_1, \cdots, \varphi_n)$ $(1 \leq j \leq n)$.

Now suppose that f is a Lipschitz function with constant K. Then there exists a sequence $\{f_m\}$ of C^∞ functions converges uniformly to f with $|\nabla f_m| \leq K$. Therefore

$$F_m \equiv f_m(\varphi_1, \cdots, \varphi_n) @ > L^2(\Omega) >> F.$$

$\{DF_m\}$ being uniformly bounded in $L^2(\Omega; H)$, it has a subsequence $\{DF_{m_k}\}$ which converges weakly in $L^2(\Omega; H)$, that is, $\{F_{m_k}\}$ converges weakly in $\mathbb{D}^2_1$. Since $\mathbb{D}^2_1$ is weakly sequentially complete, and F_m converges to F in $L^2(\Omega)$, it follows that the weak limit of $\{F_{m_k}\}$ in $\mathbb{D}^2_1$ is F, hence $F \in \mathbb{D}^2_1$.

On the other hand, $\{\nabla f_{m_k}(\varphi_1, \cdots, \varphi_n)\}$ being uniformly bounded, it has a subsequence which weakly converges in $L^2(\Omega; \mathbb{R}^n)$ to some G, thus eq. (4.25) holds. ∎

Example 2 (density of maximum of a continuous process). Let $\{X(t), 0 \leq t \leq 1\}$ be a continuous stochastic process on Gaussian probability space $(\Omega, \mathcal{F}, \mu; H)$ satisfying:

1° $\mathbb{E}[\sup_{0 \leq t \leq 1} X(t)^2] < \infty$;

2° $X(t) \in \mathbb{D}^2_1, 0 \leq t \leq 1$;

3° H-valued process $\{DX(t), 0 \leq t \leq 1\}$ has continuous modification and

$$\mathbb{E}\left[\sup_{0\leq t\leq 1} \|DX(t)\|^2_H\right] < \infty.$$

Then the random variable $G = \sup_{0\le t\le 1} X(t) \in \mathbb{D}_1^2$. In fact, let $\{r_k\}$ be all rational numbers in [0,1] and for $n \in \mathbb{N}$ put the first n numbers in order as: $t_1 < t_2 < \cdots < t_n$. Since $\max\{x_1, \cdots, x_n\}$ is a Lipschitz function, so

$$G_n = \max\{X(t_1), \cdots, X(t_n)\} \in \mathbb{D}_1^2.$$

Let $n \to \infty$. Then G_n converges in $L^2(\Omega)$ to G and $\{DG_n\}$ are uniformly bounded in $L^2(\Omega; H)$, hence $G \in \mathbb{D}_1^2$.

It follows from Theorem 4.7 that, if $\|DX(t)\|_H \neq 0$ on the set $\{t \in [0,1] : X(t) = G\}$, then G has a density. The parameter set [0,1] can be replaced by any compact metric space as we see in the proof. As an example, Florit - Nualart[1] proved that the maximum of a Brownian sheet has a C^∞ density.

Chapter III

Stochastic Calculus of Variation for Wiener Functionals

The most important Gaussian probability space in applications is the classical Wiener space. The most important Wiener functionals are so-called *Itô functionals*, namely the Itô integrals and solutions of Itô stochastic differential equations. In this chapter, we make an exposition for the theory of stochastic calculus of variation for Wiener functionals and its applications to regularities of fundamental solutions for parabolic partial differential equations, especially to the probabilistic proof of Hörmander's theorem on hypoellipticity of partial differential operators. Moreover, we introduce two important branches in this area which were developed very recently: the quasi sure analysis and the anticipating stochastic calculus.

§1. Differential calculus of Itô functionals and regularity of heat kernels

1.1 Skorohod integrals

In this paragraph we always assume that $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$, $\Omega = C_0(\mathbb{R}_+; \mathbb{R}^d)$, the Fréchet space of all $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}_+$ which vanish at 0 equipped with topology of uniform convergence on bounded intervals, μ is the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$. For $t \in \mathbb{R}_+, \omega \in \Omega$, let $W_t(\omega) = \omega(t)$. Then $\{W_t, t \in \mathbb{R}_+\}$ is a d-dimensional Brownian motion on $(\Omega, \mathcal{B}(\Omega), \mu)$. Let $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$ be the natural filtration generated by this Brownian motion, $\mathcal{F} = \mathcal{F}_\infty$ be the μ-completion of σ-algebra $\mathcal{B}(\Omega)$. For $h \in H$, let

$$W_h = \int_0^\infty h(t) \cdot dW_t = \sum_{j=1}^d \int_0^\infty h_j(t) dW_t^j.$$

Then $(\Omega, \mathcal{F}, \mu; H)$ is an irreducible Gaussian probability space. For $h \in H$, denote $\overline{h}(t) \equiv \int_0^t h(s) ds$. Then $\overline{H} \equiv \{\overline{h} : h \in H\} \subset \Omega$ is the Cameron-Martin subspace.

Let E be a separable Hilbert space. Then any E-valued functional $F \in$

$L^2(\Omega; E)$ has a unique decomposition:

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{1.1}$$

where $f_n \in H^{\widehat{\otimes} n} \otimes E (n \geq 1)$. Note that $H^{\otimes n} \cong L^2(\mathbb{R}_+^n; (\mathbb{R}^d)^{\otimes n})$, hence $f_n \in \widehat{L^2}(\mathbb{R}_+^n; (\mathbb{R}^d)^{\otimes n} \otimes E)$, that is, the subspace of symmetric functions in $L^2(\mathbb{R}_+^n)$. Denote $\Delta_n \equiv \{(t_1, \cdots, t_n) \in \mathbb{R}_+^n : t_1 \leq t_2 \leq \ldots \leq t_n\}$. Then

$$I_n(f_n) = n! \int_{\Delta_n} f_n(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} \tag{1.2}$$

is an n-fold iterated Itô integral.

If $E = L^2(\mathbb{R}_+; \mathbb{R}^m)$, then $L^2(\Omega; E) \cong L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^m)$, hence $X \in L^2(\Omega; E)$ is equivalent to an $\mathbb{R}^m$ valued stochastic process having the following decomposition:

$$X_t = \mathbb{E}[X_t] + \sum_{n=1}^{\infty} I_n(f_{n+1}(\cdot, t)), \quad t \in \mathbb{R}_+, \tag{1.3}$$

where $f_{n+1} \in L^2(\mathbb{R}_+^{n+1}; (\mathbb{R}^d)^{\otimes n} \otimes \mathbb{R}^m)$, for fixed $t \in \mathbb{R}_+$, $f_{n+1}(\cdot, t) \in \widehat{L^2}(\mathbb{R}_+^n; (\mathbb{R}^d)^{\otimes n} \otimes \mathbb{R}^m)$.

If E is any separable Hilbert space, $X \in L^2(\Omega; H \otimes E) \cong L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^d \otimes E)$ and $X \in D(\delta)$, then we denote δX, the divergence of X, by

$$\delta X \equiv \int_0^{\infty} X_t dW_t, \tag{1.4}$$

and call it the *Skorohod integral* of X.

Definition 1.1 Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^m$ be a stochastic process, if $\forall t \in \mathbb{R}_+$, $X|_{[0,t] \times \Omega}$, the restriction of X in $[0, t] \times \Omega$, is $\mathcal{B}[0, t] \times \mathcal{F}_t$ measurable, then X is called a *progressive process*.

Any progressive process is adapted and measurable. Conversely, any adapted and measurable process has a progressive modification (for instance, cf. Dellacherie - Meyer[1]). In the sequel, when we speak of adapted and measurable process, we always mean its progressive modification. We shall prove that, when X is a progressive process, its Skorohod integral coincides with its Itô integral.

Lemma 1.2 *If $F \in L^2(\Omega; E)$ has decomposition (1.1), then $\forall t \in \mathbb{R}_+$,*

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n 1_{[0,t]}^{\otimes n}), \quad \text{a.s.}. \tag{1.5}$$

Proof. We may assume that $F = I_n(f_n), n \geq 1$. Denote $\Delta_n(t) \equiv \{(t_1, \cdots, t_n) \in \Delta_n : t_n \leq t\}$. Then by eq. (1.2) we have

$$\begin{aligned} \mathbb{E}[F|\mathcal{F}_t] &= n! \int_{\Delta_n(t)} f_n(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} \\ &= I_n(f_n 1_{[0,t]}^{\otimes n}), \quad \text{a.s.}. \end{aligned}$$

∎

It follows that, if $X \in L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^m)$ has decomposition (1.3), then X is a progressive process if and only if $\forall t \in \mathbb{R}_+, n \in \mathbb{N}$,

$$f_{n+1}(\cdot, t) = f_{n+1}(\cdot, t) 1_{[0,t]}^{\otimes n}, \quad \text{a.e.}. \tag{1.6}$$

Lemma 1.3 *If $F \in \mathbb{D}_1^2(E)$, then $\forall t \in \mathbb{R}_+$, $\mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}_1^2(E)$ and*

$$D_t \mathbb{E}[F|\mathcal{F}_s] = \mathbb{E}[D_t F|\mathcal{F}_s] 1_{[0,s]}(t), \quad \text{a.s.}. \tag{1.7}$$

Proof. By eqs. (II.2.46) and (1.5) we have

$$\begin{aligned} D_t \mathbb{E}[F|\mathcal{F}_s] &= \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t) 1_{[0,s]}^{\otimes(n-1)}) 1_{[0,s]}(t) \\ &= \mathbb{E}[D_t F|\mathcal{F}_s] 1_{[0,s]}(t), \quad \text{a.s.}. \end{aligned}$$

∎

Remark. If F is $\mathcal{F}_s$ measurable, then by eq. (1.7), in the case $t > s$, we have $D_t F = 0$ a.s..

Lemma 1.4 *If $X \in \mathbb{D}_2^2(H \otimes E)$, then $\delta X \in \mathbb{D}_1^2(E)$ and $\forall t \in \mathbb{R}_+$,*

$$D_t(\delta X) = X_t + \int_0^{\infty} D_t X_s dW_s. \tag{1.8}$$

Proof. Suppose that X has decomposition (1.3). Then by eq. (II.2.46) we have

$$D_t X_s = \sum_{n=1}^{\infty} n I_{n-1}(f_{n+1}(\cdot, t, s)).$$

Using eq. (II.2.53) to compute its Skorohod integral, we have

$$\int_0^{\infty} D_t X_s dW_s = \sum_{n=1}^{\infty} n I_n(\tilde{f}_{n+1}(\cdot, t, \cdot)),$$

where $\tilde{f}_{n+1}(\cdot, t, \cdot)$ stands for symmetrization of f_{n+1} with respect to n variables when t fixed. On the other hand, by eq. (II.2.53) we have $\delta X = \sum_{n=1}^{\infty} I_n(\tilde{f}_n)$, hence

$$\begin{aligned} D_t(\delta X) &= \sum_{n=1}^{\infty} n I_{n-1}(\tilde{f}_n(\cdot, t)) \\ &= \sum_{n=0}^{\infty} I_n(\tilde{f}_{n+1}(\cdot, t)) + \sum_{n=1}^{\infty} n I_n(\tilde{f}_{n+1}(\cdot, t)) \\ &= X_t + \int_0^{\infty} D_t X_s dW_s. \end{aligned}$$

∎

Lemma 1.5 *If $X \in \mathbb{D}_3^2(H \otimes E)$, then*

$$\mathcal{L}\delta X = \delta \mathcal{L} X - \delta X. \tag{1.9}$$

Proof. By eq. (1.8) we have

$$\begin{aligned} L\delta X &= -\delta D\delta X \\ &= -\int_0^\infty D_t(\delta X)dW_t \\ &= -\int_0^\infty X_t dW_t - \int_0^\infty \int_0^\infty D_t X_s dW_s dW_t \\ &= -\delta X + \delta L X. \end{aligned}$$ ∎

Lemma 1.6 *If $X \in L^2(\Omega; H \otimes \mathbb{R}^m) \sim L^2(\mathbb{R}_+ \times \Omega; \mathbb{R}^m \otimes \mathbb{R}^d)$ is a progressive process, then $X \in \mathcal{D}(\delta)$ and*

$$\delta X = \int_0^\infty X_t \cdot dW_t$$

coincides with the Itô integral of X.

Proof. Suppose that X has decomposition (1.3) and in which f_{n-1} satisfies eq. (1.6) for all $n \in \mathbb{N}$. Since

$$I_n(f_{n+1}(\cdot,t)) = n! \int_{\Delta_n(t)} f_{n+1}(t_1, \cdots, t_n, t) dW_{t_1} \cdots dW_{t_n},$$

by computing its $(n+1)$-fold iterated the Itô integral we have

$$\begin{aligned} &\int_0^\infty I_n(f_{n+1}(\cdot,t)) \cdot dW_t \\ &= n! \int_{\Delta_{n+1}} f_{n+1}(t_1, \cdots, t_n, t) dW_{t_1} \cdots dW_{t_n} dW_t \\ &= (n+1)! \int_{\Delta_{n+1}} \tilde{f}_{n+1}(t_1, \cdots, t_n, t) dW_{t_1} \cdots dW_{t_n} dW_t \\ &= I_{n+1}(\tilde{f}_{n+1}). \end{aligned}$$

By eqs. (II.2.50) and (1.6) we know that $\|\tilde{f}_{n+1}\|^2 \le \frac{1}{n+1}\|f_{n+1}\|^2$, hence

$$\sum_{n=1}^\infty n! \|\tilde{f}_n\|^2 \le \sum_{n=1}^\infty (n-1)! \|f_n\|^2 = \|X\|^2 < \infty.$$

It follows from Proposition II.2.14 that $X \in \mathcal{D}(\delta)$ and

$$\begin{aligned} \delta X &= \sum_{n=0}^\infty I_{n+1}(\tilde{f}_{n+1}) \\ &= \sum_{n=0}^\infty \int_0^\infty I_n(f_{n+1}(\cdot,t)) \cdot dW_t \\ &= \int_0^\infty X_t \cdot dW_t \quad \text{(Itô integral).} \end{aligned}$$ ∎

As a consequence, we obtain the Clark-Ocone formula (for instance, cf. Clark[1], Haussmann[1], Ocone[1]) for representation of functionals in stochastic integrals.

Theorem 1.7 *If $F \in \mathbb{D}_1^2(E)$, then*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dW_t. \tag{1.10}$$

Proof. Suppose that F has decomposition (1.1). By Lemma 1.3 we have

$$\mathbb{E}[D_t F | \mathcal{F}_t] = \sum_{n=1}^\infty n I_{n-1}(f_n(\cdot,t) 1_{[0,t]}^{\otimes(n-1)}).$$

Since the symmetrization over n variables of $nf_n(\cdot,t)1_{[0,t]}^{\otimes(n-1)}$ is just f_n, computing its Skorohod integral according to eq. (II.2.53), we obtain that

$$\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dW_t = \sum_{n=1}^\infty I_n(f_n) = F - \mathbb{E}[F].$$ ∎

Note that $\{\mathbb{E}[D_t F | \mathcal{F}_t], t \in \mathbb{R}_+\}$ being a progressive process, the stochastic integral in eq. (1.10) coincides with Itô integral. Moreover, if X is progressive, then δX in eqs. (1.8) and (1.9) are understood as Itô integrals, therefore, Lemmas 1.4 and 1.5 can be extended to cases of $X \in \mathbb{D}_1^2(H \otimes E)$ and $X \in \mathbb{D}_2^2(H \otimes E)$ respectively.

Next theorem is an extension of Itô's isometry.

Theorem 1.8 If $X, Y \in \mathbb{D}_1^2(H) \cong L^2(\mathbb{R}_+; \mathbb{D}_1^2)$, then

$$\mathbb{E}[\delta X \delta Y] = \int_0^\infty \mathbb{E}[X_t Y_t] dt + \int_0^\infty \int_0^\infty \mathbb{E}[D_s X_t D_t Y_s] ds dt. \tag{1.11}$$

Proof. By Lemma 1.4 and formula (II.3.47) we have

$$\begin{aligned} \mathbb{E}[\delta X \delta Y] &= \mathbb{E}\Big[\int_0^\infty X_t D_t(\delta Y) dt\Big] \\ &= \mathbb{E}\Big[\int_0^\infty X_t Y_t dt\Big] + \mathbb{E}\Big[\int_0^\infty X_t \Big(\int_0^\infty D_t Y_s dW_s\Big) dt\Big] \\ &= \int_0^\infty \mathbb{E}[X_t Y_t] dt + \int_0^\infty \mathbb{E}\Big[X_t \int_0^\infty D_t Y_s dW_s\Big] dt \\ &= \int_0^\infty \mathbb{E}[X_t Y_t] dt + \int_0^\infty \mathbb{E}\Big[\int_0^\infty (D_s X_t D_t Y_s) ds\Big] dt, \end{aligned}$$

hence eq. (1.11) follows. ∎

Remark. If X and Y are progressive processes, then in case $t > s$, $D_t Y_s = 0$ a.s., while in case $s > t$, $D_s X_t = 0$ a.s., hence the last integral in eq. (1.11) vanishes, we obtain again the Itô isometry.

1.2 Smoothness of solutions to stochastic differential equations

Consider the *Itô stochastic differential equation*

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s) \cdot dW_s, \quad t \geq 0, \tag{1.12}$$

where $x \in \mathbb{R}^m$, $b : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \longrightarrow \mathbb{R}^m \otimes \mathbb{R}^d$ are Borel measurable, satisfying the Lipschitz condition: $\exists K > 0, \forall x, y \in \mathbb{R}^m$,

$$|b(x) - b(y)| + \|\sigma(x) - \sigma(y)\| \leq K|x - y|. \tag{1.13}$$

It follows from the theory of stochastic differential equations (for instance, cf. Ikeda - Watanabe[3] or Huang[4]) that there exists a unique strong solution $X = X(x, t, \omega)$ satisfying that

1° $\forall x \in \mathbb{R}^m$, $X(x, \cdot, \cdot)$ is a diffusion process;

2° for a.a.$\omega[\mu]$, $X(\cdot, \cdot, \omega)$ is continuous in (x, t);

3° for a.a.$\omega[\mu]$, $\forall t \in \mathbb{R}_+$, $X(\cdot, t, \omega)$ is a homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$;

4° for $p \geq 2, T > 0$ and $R > 0, \exists C = C(p, T, R)$ such that

$$\sup_{|x| \leq R} \mathbb{E}\Big[\sup_{0 \leq t \leq T} |X(x, t)|^p\Big] \leq C. \tag{1.14}$$

If functions b and σ are infinitely differentiable with bounded derivatives, then

5° for a.a.$\omega[\mu]$, $\forall t \in \mathbb{R}_+$, $X(\cdot, t, \omega)$ is a C^∞-homeomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$;

6° for $p \geq 2, T > 0, k \in \mathbb{N}$ and $R > 0, \exists C = C(p, T, k, R)$ such that

$$\sup_{|x| \leq R} \mathbb{E}\Big[\max_{|\alpha| \leq k} \sup_{0 \leq t \leq T} |\partial_\alpha X(x, t)|^p\Big] \leq C, \tag{1.15}$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$, $|\alpha| = \sum_{j=1}^m \alpha_m$, $\partial_\alpha = \partial_1^{\alpha_1} \cdots \partial_m^{\alpha_m}$ are partial derivatives with respect to x.

For $t \in \mathbb{R}_+$, let

$$J_t \equiv J(x, t, \omega) = (\partial_j X^i(x, t, \omega))_{1 \leq i, j \leq m} \tag{1.16}$$

be the Jacobian of X with respect to initial value x. Then J_t and its inverse J_t^{-1} respectively satisfy the following Itô stochastic differential equations:

$$J_t = I + \int_0^t A_0^{(1)}(X_s) J_s ds + \sum_{k=1}^d \int_0^t A_k^{(1)}(X_s) J_s \cdot dW_s^k, \tag{1.17}$$

$$\begin{aligned} J_t^{-1} = I - \int_0^t J_s^{-1} \Big\{ A_0^{(1)}(X_s) - \sum_{k=1}^d [A_k^{(1)}(X_s)]^2 \Big\} ds \\ - \sum_{k=1}^d \int_0^t J_s^{-1} A_k^{(1)}(X_s) \cdot dW_s^k, \end{aligned} \tag{1.18}$$

where I is the $m \times m$ identity matrix, $A_0^{(1)}(x) = (\partial_j b^i(x))_{1 \leq i, j \leq m}$, $A_k^{(1)}(x) = (\partial_j \sigma_k^i(x))_{1 \leq i, j \leq m}$, $k = 1, 2, \cdots, d$.

Next we will consider the derivatives of X with respect to "sample points" ω, namely the weak derivatives in the sense of Malliavin.

Theorem 1.9 *If coefficients b and σ in equation (1.12) are C^∞ functions with bounded derivatives of all orders, then its unique solution $X = X(x, t, \cdot) \in \mathbb{D}^\infty(\mathbb{R}^m)$ $(\forall x \in \mathbb{R}^m, t > 0)$, and its covariance matrix $\Sigma_t = \Sigma(x, t, \omega)$ has the expression*

$$\Sigma_t = J_t \Big[\int_0^t J_s^{-1} a(X_s) (J_s^{-1})^* ds \Big] J_t^*, \quad \text{a.s.,} \tag{1.19}$$

where $a = \sigma\sigma^*$, J_t is the Jacobian of X_t with respect to initial value x.

Proof. By estimate (1.14) we have $X_t \in L^{\infty-}(\Omega; \mathbb{R}^m)$ $(t \geq 0)$. Since J_t and J_t^{-1} are solutions of equations (1.17) and (1.18) respectively, it follows that $J_t, J_t^{-1} \in L^{\infty-}(\Omega; \mathbb{R}^m \otimes \mathbb{R}^m)$.

Using Picard's approximation:

$$\begin{aligned} X_t^{(0)} &= x; \\ X_t^{(n+1)} &= x + \int_0^t b(X_s^{(n)}) ds + \int_0^t \sigma(X_s^{(n)}) \cdot dW_s, \quad n \geq 0, \end{aligned}$$

by Proposition II.4.12 and Lemma 1.4 it is easy to prove that $X \in \mathbb{D}_1^2(H \otimes \mathbb{R}^m)$.

X being continuous adapted process, in the case $s > t$, $D_s X_t = 0$ a.s.. In the case $s \leq t$, by Lemma 1.4 and the differential rule for composite functionals we have

$$\begin{aligned} D_s X_t &= \int_s^t D_s b(X_r) dr + \sigma(X_s) + \sum_{k=1}^d \int_s^t D_s \sigma_k(X_r) dW_r^k \\ &= \int_s^t A_0^{(1)}(X_r) D_s X_r dr + \sigma(X_s) \\ &\quad + \sum_{k=1}^d \int_s^t A_k^{(1)}(X_r) D_s X_r dW_r^k, \end{aligned} \tag{1.20}$$

where σ_k stands for k-th column of matrix σ $(k = 1, \cdots, d)$.

On the other hand, by eqs. (1.17) and (1.18) we have

$$J_t J_s^{-1} = I + \int_s^t A_0^{(1)}(X_r) J_r J_s^{-1} dr + \sum_{k=1}^d \int_s^t A_k^{(1)}(X_r) J_r J_s^{-1} dW_r^k,$$

hence

$$\begin{aligned} J_t J_s^{-1} \sigma(X_s) = \sigma(X_s) + \int_s^t A_0^{(1)}(X_r) J_r J_s^{-1} \sigma(X_s) dr \\ + \sum_{k=1}^d \int_s^t A_k^{(1)}(X_r) J_r J_s^{-1} \sigma(X_s) dW_r^k. \end{aligned} \tag{1.21}$$

Comparing eqs. (1.20) and (1.21), by uniqueness of solutions for stochastic differential equations we have

$$D_s X_t = J_t J_s^{-1} \sigma(X_s) 1_{[0,t]}(s), \quad \text{a.s.}. \tag{1.22}$$

Therefore,

$$\begin{aligned} \Sigma_t &= \int_0^\infty (D_s X_t)(D_s X_t)^* ds \\ &= J_t \left[\int_0^t J_s^{-1} a(X_s) (J_s^{-1})^* ds \right] J_t^*, \quad \text{a.s.,} \end{aligned}$$

moreover,

$$\|DX_t\|^2_{H \otimes R^m} = \operatorname{Tr} \Sigma_t \in L^{\infty-}.$$

Replacing (1.12) by (1.20), by a similar procedure we can compute higher order derivatives and prove that $\forall k \in \mathbb{N}, t \in \mathbb{R}_+, \|D^k X_t\| \in L^{\infty-}$, consequently $X_t \in \mathbb{D}^\infty(\mathbb{R}^m)$. ∎

Approximating b and σ by smooth functions and applying Proposition II.4.12 and Theorem II.4.7 we can prove the following theorem (for details cf. Nualart[1] or Bouleau - Hirsch[1,2]):

Theorem 1.10 *If the coefficients b and σ in equation (1.12) satisfy Lipschitz condition (1.13), then its unique solution $X = X(x,t,\cdot) \in \mathbb{D}_1^\infty(\mathbb{R}^m)$ $(\forall x \in \mathbb{R}^m, t > 0)$. Let*

$$\tau = \inf \left\{ t > 0 : \int_0^t 1_{\{\det a(X_s) \neq 0\}} ds > 0 \right\}.$$

Then on the set $\{t > \tau\}$, $\mu \circ X_t^{-1}$ is absolutely continuous with respect to Lebesgue measure.

1.3 Hypoellipticity and Hörmander's conditions

Consider the second order partial differential operator

$$L = \frac{1}{2} \sum_{i,j=1}^m a^{ij}(\cdot) \partial_i \partial_j + \sum_{i=1}^m b^i(\cdot) \partial_i \tag{1.23}$$

and the Cauchy problem for heat equation:

$$\begin{cases} \partial_t u(t,x) = Lu(t,x), \quad t > 0, x \in \mathbb{R}^m; \\ u(0,x) = \varphi(x). \end{cases} \tag{1.24}$$

As we know, if $\varphi \in C_b^2(\mathbb{R}^m)$, then

$$u_\varphi(t,x) \equiv \mathbb{E}[\varphi(X(x,t,\cdot))]$$

is the solution of (1.24). By Theorem II.4.10 we know, if X_t is non-degenerate, i.e.

$$[\det \Sigma_t]^{-1} \in L^{\infty-},$$

then the transition probability $P(t,x,\cdot) = \mu \circ X(x,t,\cdot)^{-1}$ of diffusion process X has C^∞ density

$$p(t,x,y) = \mathbb{E}[\delta_y(X(x,t,\cdot))],$$

which is the fundamental solution of equation (1.24) (so called *heat kernel*).

From theory of partial differential equations we know, if matrix $a(x)$ is uniformly positive definite, i.e. $\exists \eta > 0$ such that $a(\cdot) \geq \eta I$, then the conclusion is true. In 1967, Hörmander[1] obtained a much weaker condition for hypoellipticity of differential operators, namely the well-known Hörmander's condition. To state Hörmander's theorem, we write the operator L in form of vector fields. For notational simplicity, we shall adopt the Einstein's convention: when an index repeatly appears as subscript and superscript, it always means taking sum over this index. Let

$$A_k(\cdot) = \sigma_k^i(\cdot) \partial_i, \quad k = 1, \cdots, d,$$

$$A_0(\cdot) = \left[b^i(\cdot) - \frac{1}{2} \sum_{k=1}^d \sigma_k^j(\cdot) \partial_j \sigma_k^i(\cdot) \right] \partial_i.$$

Then $A_0, A_1, \cdots, A_k$ are C^∞ vector fields on $\mathbb{R}^m$. Noting that

$$\sum_{k=1}^d A_k^2 = a^{ij} \partial_i \partial_j + \sum_{k=1}^d \sigma_k^j [\partial_j \sigma_k^i] \partial_i,$$

we have

$$L = \frac{1}{2} \sum_{k=1}^d A_k^2 + A_0. \tag{1.25}$$

If we put $\tilde{b} = b - \frac{1}{2} \sum_{k=1}^d A_k^{(1)} \sigma_k$, where $A_k^{(1)} \equiv (\partial_j \sigma_k^i)_{1 \leq i,j \leq m}$, then $A_0(\cdot) = \tilde{b}^i(\cdot) \partial_i$. Since Itô equation (1.12) is equivalent to the following *Fisk-Stratonovich equation*

$$X_t = x + \int_0^t \tilde{b}(X_s) ds + \int_0^t \sigma(X_s) \circ dW_s, \quad t \geq 0, \tag{1.26}$$

it can be written in form of vector fields:

$$dX_t = A_0(X_t) dt + A_k(X_t) \circ dW_t^k. \tag{1.27}$$

The last equation means that, $\forall f \in C_b^\infty(\mathbb{R}^m)$, it holds that

$$df(X_t) = (A_0 f)(X_t) dt + (A_k f)(X_t) \circ dW_t^k.$$

In the sequel, for $V \in C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, V is also understood as the C^∞ vector field: $V(\cdot) = V^i(\cdot)\partial_i$. Note that eqs. (1.17) and (1.18) are respectively equivalent to Fisk-Stratonovich equations:

$$dJ_t = \widetilde{A}_0^{(1)}(X_t)J_t dt + A_k^{(1)}(X_t)J_t \circ dW_t^k \tag{1.28}$$

and

$$dJ_t^{-1} = -J_t^{-1}\widetilde{A}_0^{(1)}(X_t)dt - J_t^{-1}A_k^{(1)}(X_t) \circ dW_t^k, \tag{1.29}$$

where $\widetilde{A}_0^{(1)}(x) = (\partial_j \widetilde{b}^i(x))_{1\le i,j\le m}$. By rule of Stratonovich differentiation we have

$$\begin{aligned} d[J_t^{-1}V(X_t)] &= (dJ_t^{-1}) \circ V(X_t) + J_t^{-1} \circ dV(X_t) \\ &= -J_t^{-1}\widetilde{A}_0^{(1)}(X_t)V(X_t)dt - J_t^{-1}A_k^{(1)}(X_t)V(X_t) \circ dW_t^k \\ &\quad + J_t^{-1}(A_0V)(X_t)dt + J_t^{-1}(A_kV)(X_t) \circ dW_t^k. \end{aligned}$$

Since $[\widetilde{A}_0^{(1)}(x)V(x)]^i = V^j(x)\partial_j\widetilde{b}^i(x) = (V\widetilde{b}^i)(x)$, by notations of vector fields we have

$$\widetilde{A}_0^{(1)}(x)V(x) = (VA_0)(x).$$

Similarly,

$$A_k^{(1)}(x)V(x) = (VA_k)(x), \quad k = 1, \cdots, d,$$

hence

$$\begin{aligned} d[J_t^{-1}V(X_t)] &= J_t^{-1}(A_0V - VA_0)(X_t)dt \\ &\quad + J_t^{-1}(A_kV - VA_k)(X_t) \circ dW_t^k \\ &= J_t^{-1}[A_0, V](X_t)dt \\ &\quad + J_t^{-1}[A_k, V](X_t) \circ dW_t^k, \end{aligned} \tag{1.30}$$

where $[\cdot,\cdot]$ stands for the Lie bracket. Combining equations (1.26) and (1.28), the solution $R_t \equiv (X_t, J_t)$ is an $\mathbb{R}^m \times (\mathbb{R}^m \otimes \mathbb{R}^m)$-valued stochastic process with initial value $R_0 = (x, I)$. For any vector field V, we define the function $f_V : \mathbb{R}^m \times (\mathbb{R}^m \otimes \mathbb{R}^m) \to \mathbb{R}^m$ as follows:

$$f_V(r) = J^{-1}V(x), \quad \text{if } r = (x, J). \tag{1.31}$$

Therefore, eq. (1.30) takes form:

$$\begin{cases} df_V(R_t) = f_{[A_0,V]}(R_t)dt + f_{[A_k,V]}(R_t) \circ dW_t^k, \\ f_V(R_0) = V(x). \end{cases} \tag{1.32}$$

In order to translate it into Itô equation, we introduce the following notations:

$$\{A_k, V\} = [A_k, V], \quad k = 1, \cdots, d,$$

$$\{A_0, V\} = [A_0, V] + \frac{1}{2}\sum_{k=1}^d [A_k, [A_k, V]].$$

Replacing V by $[A_j, V]$ in eq. (1.32), we obtain that

$$\begin{aligned} df_{[A_j,V]}(R_t) &= f_{[A_0,[A_j,V]]}(R_t)dt \\ &\quad + f_{[A_k,[A_j,V]]}(R_t) \circ dW_t^k, \quad j = 1, \cdots, d. \end{aligned}$$

Since

$$\begin{aligned} f_{[A_k,V]}(R_t) \circ dW_t^k &= f_{[A_k,V]}(R_t) \cdot dW_t^k \\ &\quad + \frac{1}{2}\sum_{k=1}^d f_{[A_k,[A_k,V]]}(R_t)dt, \end{aligned}$$

it follows that eq. (1.32) is equivalent to Itô equation:

$$\begin{cases} df_V(R_t) = f_{\{A_0,V\}}(R_t)dt + f_{\{A_k,V\}}(R_t) \cdot dW_t^k, \\ f_V(R_0) = V(x). \end{cases} \tag{1.33}$$

Let $\widehat{\mathcal{V}}_n$ and $\mathcal{V}_n$ be the following sets of vector fields:

$$\widehat{\mathcal{V}}_0 \equiv \{A_1, \cdots, A_d\} \quad \text{(it does not include } A_0\text{)},$$

$$\widehat{\mathcal{V}}_n \equiv \{\{A_0, V\}, \{A_k, V\}, V \in \widehat{\mathcal{V}}_{n-1}, k = 1, \cdots, d\}, \quad n \ge 1,$$

$$\mathcal{V}_n \equiv \bigcup_{m=0}^n \widehat{\mathcal{V}}_m, \quad n \in \mathbb{N}_0.$$

The *Hörmander condition* means that

(H): the Lie algebra generated by vector fields $\{A_k, [A_0, A_k], k = 1, \cdots, d\}$ has dimension m at any $x \in \mathbb{R}^m$ (note that A_0 appears only in Lie brackets).

Evidently this condition is equivalent to

(H)$'$: $\forall x \in \mathbb{R}^m, \exists N \in \mathbb{N}_0$ and $V_1, \cdots, V_m \in \mathcal{V}_N$, such that $V_1(x), \cdots, V_m(x)$ are linearly independent,

or equivalently:

(H)$''$: $\forall x \in \mathbb{R}^m, \exists N \in \mathbb{N}_0$ such that

$$\inf_{l \in S}\max_{V \in \mathcal{V}_N}(l, V(x))^2 > 0. \tag{1.34}$$

Here $S = S^{m-1} \equiv \{x \in \mathbb{R}^m : |x| = 1\}$ is the unit sphere in $\mathbb{R}^m$. In fact, since there are only finite vector fields in $\mathcal{V}_n$, we can arrange them as a matrix. If its rank is less than m, then its rows are linearly dependent, hence there exists $l \in S$ such that the left-hand side of (1.34) vanishes; conversely, if it has rank m, then its rows are linearly independent, hence for any $l \in S$, the left-hand side of (1.34) is strictly positive.

The Hörmander's theorem asserts that, under condition (H), L is a *hypoelliptic operator*, that is, for any open set U in $\mathbb{R}^m$ and any distribution $u \in \mathcal{D}'(\mathbb{R}^m)$, if $Lu|_U \in C^\infty(U)$, then $u|_U \in C^\infty(U)$.

From the classical theory of partial differential equations we know, if L is elliptic (i.e. $a(x)$ is positive definite), then it is hypoelliptic. To illustrate that the converse is not true, we consider an example:

Example (Kolmogorov 1934). Let $m = 2, d = 1, A_0(x) = x_1\partial_2, A_1(x) = \partial_1$. Then

$$L = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + x_1\frac{\partial}{\partial x_2}. \tag{1.35}$$

In this case,

$$a(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix},$$

hence L is not elliptic. However,

$$\begin{aligned}[A_1, A_0] &= \partial_1(x_1\partial_2) - x_1\partial_2\partial_1 \\ &= x_1\partial_1\partial_2 + \partial_2 - x_1\partial_2\partial_1 = \partial_2.\end{aligned}$$

(∂_1, ∂_2) being base of tangent space over $\mathbb{R}^2$, condition (H) is satisfied, hence by Hörmander's theorem L is hypoelliptic.

Consider the L-diffusion process $X_t = (X_t^1, X_t^2)$ which is a solution of the following Ito equation:

$$\begin{cases} X_t^1 = x_1 + W_t, \\ X_t^2 = x_2 + \int_0^t X_s^1 ds. \end{cases} \tag{1.36}$$

Obviously, X is a Gaussian process and

$$m_t = \mathbb{E}[X_t] = \begin{pmatrix} x_1 \\ x_2 + x_1 t \end{pmatrix},$$

$$V_t = \operatorname{cov}(X_t) = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix},$$

$$V_t^{-1} = \frac{2}{t^3}\begin{pmatrix} 2t^2 & -3t \\ -3t & 6 \end{pmatrix}, \quad \det V_t = \frac{t^4}{12}.$$

Its transition probability has density

$$\begin{aligned} p(t, x, y) &= (2\pi)^{-1}(\det V_t)^{-1/2}\exp\{-\tfrac{1}{2}(y - m_t, V_t^{-1}(y - m_t))\} \\ &= \frac{\sqrt{3}}{\pi t^2}\exp\left\{-\frac{(x_1 - y_1)^2}{2t} - \frac{6[y_2 - x_2 - (x_1 + y_1)t/2]^2}{t^3}\right\}. \end{aligned} \tag{1.37}$$

This density is C^∞ in y which is the fundamental solution for equation $\partial_t u = Lu$. It follows from eq. (1.36) that

$$J_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad J_t^{-1} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.$$

Computing its Malliavin covariance matrix by eq. (1.19) we obtain

$$\Sigma_t = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix} = V_t,$$

which is obviously non-degenerate.

The key step in the proof of Hörmander's theorem is to prove that, under condition (H), the corresponding Malliavin covariance matrix is non-degenerate.

1.4 A probabilistic proof of Hörmander's theorem

As we mentioned before, the first probabilistic proof of Hörmander's theorem was given by Malliavin[1]. After that, Bismut[1], Kusuoka-Stroock[3] and Ikeda-Watanabe[1] have given proofs in different ways. Here we adopt the much simplified proof given by Norris[1] following an idea of Stroock[3]. Firstly we prove a lemma:

Lemma 1.11 *Let X be a one-dimensional Itô process:*

$$X_t = x + \int_0^t Y_s^0 ds + \sum_{k=1}^d \int_0^t Y_s^k dW_s^k, \quad t \geq 0, \tag{1.38}$$

where Y^0 is also a one-dimensional Itô process:

$$Y_t^0 = y + \int_0^t Z_s^0 ds + \sum_{k=1}^d \int_0^t Z_s^k dW_s^k, \quad t \geq 0, \tag{1.39}$$

where $x, y \in \mathbb{R}$, $Y = (Y^1, \cdots, Y^d)$ and $Z = (Z^1, \cdots, Z^d)$ are d-dimensional progressive processes. If $\exists K > 0$ and a bounded stopping time $\tau > 0$ such that

$$\sup_{0 \leq t \leq \tau}\{|Y_t^0| + |Z_t^0| + |Y_t| + |Z_t|\} \leq K,$$

then $\forall q > 8, \nu < (q - 8)/9$, and sufficiently small $\epsilon > 0$, $\exists c > 0$ such that

$$\mu\left\{\int_0^\tau X_t^2 dt < \epsilon^q, \int_0^\tau (|Y_t^0|^2 + |Y_t|^2)dt \geq \epsilon\right\} \leq c\exp\{-\epsilon^{-\nu}\}. \tag{1.40}$$

Proof. Let $A_t \equiv \int_0^t Y_s^0 ds, M_t \equiv \int_0^t Y_s dW_s, Q_t \equiv \int_0^t A_s Z_s dW_s$, $N_t = \int_0^t X_s Y_s dW_s$,

$$S_1 \equiv \{[N]_\tau < \epsilon_1, \sup_{t \leq \tau}|N_t| \geq \delta_1\},$$

$$S_2 = \{[M]_\tau < \epsilon_2, \sup_{t \leq \tau}|M_t| \geq \delta_2\},$$

$$S_3 = \{[Q]_\tau < \epsilon_3, \sup_{t \leq \tau}|Q_t| \geq \delta_3\},$$

where $[\cdot]$ stands for the quadratic variation of continuous local martingales. By exponential inequality (cf. Remark below) we have

$$\mu(S_i) \leq 2\exp\{-\delta_i^2/2c_i\}, \quad i = 1,2,3.$$

Let $q_1 = (q-\nu)/2, q_2 = (q_1/2-\nu)/2, q_3 = (2q_2-\nu)/2$. Then $q > q_1 > q_2 > q_3 > 1$. Let $\delta_i = \epsilon^{q_i} (i = 1,2,3), \epsilon_1 = c_1\epsilon^q, \epsilon_2 = c_2\epsilon^{q_1/2}, \epsilon_3 = c_3\epsilon^{2q_2}$ (where constants c_1, c_2, c_3 are to be determined). Then

$$\delta_i^2/\epsilon_i \sim \epsilon^{-\nu}, \quad i = 1,2,3.$$

To prove the lemma, it suffices to prove that, if ϵ is sufficiently small, we can choose appropriate constants c_1, c_2, c_3 so that the set in (1.40) is contained in $\cup_{i=1}^3 S_i$. In other words. We claim that: when ϵ is sufficiently small, if $\omega \notin \cup_{i=1}^3 S_i$ and $\int_0^\tau X_t^2 dt < \epsilon^q$. Then $\int_0^\tau (|Y_t^0|^2 + |Y_t|^2)dt < \epsilon$. Let $\tau \leq T$, we prove it in three steps:

1° Let $c_1 = K^2$, then

$$[N]_\tau = \int_0^\tau X_t^2 |Y_t|^2 dt < K^2\epsilon^q = \epsilon_1.$$

Since $\omega \notin S_1$, it follows that $\sup_{t\leq\tau} |N_t| < \delta_1 = \epsilon^{q_1}$. But

$$\sup_{t\leq\tau}\left|\int_0^t X_s Y_s^0 ds\right| \leq \left(T\int_0^\tau (X_s Y_s^0)^2 ds\right)^{1/2} < KT^{1/2}\epsilon^{q/2},$$

hence

$$\sup_{t\leq\tau}\left|\int_0^t X_s dX_s\right| < \epsilon^{q_1} + KT^{1/2}\epsilon^{q/2}.$$

By Itô formula

$$[M]_t = X_t^2 - x^2 - 2\int_0^t X_s dX_s,$$

we know that

$$\int_0^\tau [M]_t dt < \epsilon^q + 2T(\epsilon^{q_1} + KT^{1/2}\epsilon^{q/2}).$$

Since $q_1 < q/2$, if ϵ is sufficiently small, $\exists c_0 > 0$ so that the last expression is less than $c_0\epsilon^{q_1}$. $[M]$ being an increasing process, for $\delta > 0$ it holds that

$$\delta[M]_{\tau-\delta} < \int_{\tau-\delta}^\tau [M]_t dt < c_0\epsilon^{q_1},$$

$$[M]_\tau < [M]_{\tau-\delta} + K^2\delta.$$

Let $\delta = \epsilon^{q_1/2}$. Then $\exists c_2 > 0$ such that $[M]_\tau < c_2\epsilon^{q_1/2}$.

2° Let $\epsilon_2 = c_2\epsilon^{q_1/2}$. Since $\omega \notin S_2$, it follows that $\sup_{t\leq\tau}|M_t| < \delta_2 = \epsilon^{q_2}$. Since $\int_0^\tau X_t^2 dt < \epsilon^q$, we have

$$\lambda^1\{0 \leq t \leq \tau : |X_t| \geq \epsilon^{q/3}\} \leq \epsilon^{q/3},$$

where λ^1 is Lebesgue measure. Now that $X_t = x + A_t + M_t$, so we have

$$\lambda^1\{0 \leq t \leq \tau : |x + A_t| \geq \epsilon^{q/3} + \epsilon^{q_2}\} \leq \epsilon^{q/3},$$

hence $\forall t \in [0,\tau]$, $\exists s \in [0,\tau], |s-t| \leq \epsilon^{q/3}$, such that $|x + A_s| < \epsilon^{q/3} + \epsilon^{q_2}$, therefore,

$$|x + A_t| < |x + A_s| + \left|\int_s^t Y_r^0 dr\right| < (1+K)\epsilon^{q/3} + \epsilon^{q_2}.$$

Especially we have $|x| < (1+K)\epsilon^{q/3} + \epsilon^{q_2}$, hence $\forall t \in [0,\tau]$, $|A_t| < 2((1+K)\epsilon^{q/3} + \epsilon^{q_2})$. Since $q_2 < q/3$, for sufficiently small ϵ, we have $|A_t| < 3\epsilon^{q_2}$.

3° By Itô formula,

$$\begin{aligned}\int_0^\tau (Y_t^0)^2 dt &= \int_0^\tau Y_t^0 dA_t \\ &= Y_\tau^0 A_\tau - \int_0^\tau A_t(Z_t^0 dt + Z_t \cdot dW_t),\end{aligned} \tag{1.41}$$

noting that $|Y_\tau^0 A_\tau| < 3K\epsilon^{q_2}$, $|\int_0^\tau A_t Z_t^0 dt| < 3KT\epsilon^{q_2}$ and that $[Q]_\tau = \int_0^\tau A_t^2 |Z_t|^2 dt < 9K^2T\epsilon^{2q_2}$, letting $c_3 = 9K^2T, \epsilon_3 = c_3\epsilon^{2q_2}$, since $\omega \notin S_3$, we have $\sup_{t\leq\tau}|Q_t| < \delta_3 = \epsilon^{q_3}$. Especially, $|Q_\tau| < \epsilon^{q_3}$, by (1.41) we have

$$\int_0^\tau (Y_t^0)^2 dt < 3K(1+T)\epsilon^{q_2} + \epsilon^{q_3}.$$

Since $q_3 < q_2$, for sufficiently small ϵ, the above expression is less than $2\epsilon^{q_3}$, therefore,

$$\int_0^\tau (|Y_t^0|^2 + |Y_t|^2)dt < 2\epsilon^{q_3} + c_2\epsilon^{q_1/2} < \epsilon. \qquad \blacksquare$$

Remark. Let M be a continuous local martingale with initial value 0, $a > 0, c > 0, t > 0$. Define

$$Z_s = \exp\left\{\frac{c}{a}M_s - \frac{c^2}{2a^2}[M]_s\right\}, \quad s \geq 0.$$

Then Z is a nonnegative supermartingale. By the maximum inequality for supermartingales we have

$$\begin{aligned}\mu\Big\{[M]_t < a; \sup_{0\leq s\leq t}|M_s| \geq c\Big\} &\leq 2\mu\Big\{\sup_{0\leq s\leq t} Z_s \geq \exp(c^2/2a)\Big\} \\ &\leq 2\exp\{-c^2/2a\}.\end{aligned}$$

This is so called exponential inequality for continuous local martingales.

Theorem 1.12 *If the coefficients b and σ in eq. (1.12) are C^∞ functions with bounded derivatives of all orders, and if the operator L given by (1.25) satisfies the condition* (H), *then the unique solution $X = X(x,t,\omega)$ has C^∞ densities $p(t,x,\cdot)$ of transition probabilities, that is, the heat equation $\partial_t u = Lu$ has C^∞ smooth fundamental solution.*

Proof. It suffices to prove that, under the condition $(H)^{\prime\prime}$, $\forall t > 0$, the covariance matrix Σ_t is non-degenerate, namely, $(\det \Sigma_t)^{-1} \in L^{\infty-}$. Note that by Theorem 1.9 and the fact that $(\det J_t)^{-1} \in L^{\infty-}$, it suffices to prove the non-degeneracy of the following matrix:

$$\tilde{\Sigma}_t \equiv \int_0^t J_s^{-1} a(X_s)(J_s^{-1})^* ds.$$

Fix $t > 0$ and $c > 0$. Define

$$\tau_c \equiv \inf\{s \geq 0 : |X_s - x| \vee \|J_s^{-1} - I\| > c^{-1}\} \wedge t.$$

Then τ_c is a stopping time and for $\epsilon \in (0,t)$,

$$\{\tau_c \leq \epsilon\} = \left\{ \sup_{s \leq \epsilon} |X_s - x| \vee \|J_s^{-1} - I\| \geq c^{-1} \right\}.$$

By estimation of solutions of eqs. (1.12) and (1.18) we know that, $\forall p > 1$

$$E\left[\sup_{s\leq\epsilon} |X_s - x|^p \vee \|J_s^{-1} - I\|^p \right] = O(\epsilon^{p/2}),$$

therefore,

$$\tau_c^{-1} \in L^{\infty-}.$$

By condition $(\mathrm{H})^{\prime\prime}$ and the continuous dependence of solutions of eq. (1.33) with respect to initial values, we conclude that $\forall l_0 \in S = S^{m-1}, \exists N \in \mathbb{N}_0, V \in \mathcal{V}_N$, and some neighborhood S_0 of l_0, for sufficiently large c and sufficiently small $\delta > 0$, we have

$$\inf_{l\in S_0,\, s<\tau_c} \inf (l, f_V(R_s))^2 \geq \delta. \tag{1.42}$$

Hence $\forall p > 1$ (denoting τ_c by τ for simplicity),

$$\sup_{l\in S_0} \mu\left\{ \int_0^\tau (l, f_V(R_s))^2 ds < \epsilon \right\} \leq \mu\{\delta\tau < \epsilon\} = o(\epsilon^p). \tag{1.43}$$

Suppose that $V = \{A_{k_j}, \{A_{k_{j-1}}, \cdots \{A_{k_1}, A_{k_0}\} \cdots\}\}$, where $0 \leq j \leq N, 1 \leq k_0 \leq d, 0 \leq k_1, \cdots, k_j \leq d$. Putting $V_0 \equiv A_{k_0}, V_1 \equiv \{A_{k_1}, V_0\}, \cdots, V_j \equiv \{A_{k_j}, V_{j-1}\} = V$, by induction we shall prove that: for $i = j, j-1, \cdots, 0$, it holds that

$$\sup_{l\in S_0} \mu\left\{ \int_0^\tau (l, f_{V_i}(R_s))^2 ds < \epsilon \right\} = o(\epsilon^p), \quad 1 < p < \infty \tag{1.44}$$

In the case $i = j$, it is reduced to (1.43). Assume that (1.44) holds for i, we claim that it holds for $i-1$. Note that for $l \in S$ and any C^∞ vector field V we have

$$\begin{cases} d(l, f_V(R_t)) = (l, f_{\{A_0,V\}}(R_t))dt + (l, f_{\{A_k,V\}}(R_t))dW_t^k, \\ (l, f_V(R_0)) = (l, V(x)). \end{cases} \tag{1.45}$$

By Lemma 1.11 for $q > 8$ and sufficiently small ϵ we have

$$\mu\left\{ \int_0^\tau (l, f_{V_{i-1}}(R_s))^2 ds < \epsilon^q, \int_0^\tau \sum_{k=0}^d (l, f_{\{A_k,V_{i-1}\}}(R_s))^2 ds \geq \epsilon \right\}$$
$$= o(\epsilon^p), \quad 1 < p < \infty.$$

By inductive assumption we know

$$\sup_{l\in S_0} \mu\left\{ \int_0^\tau \sum_{k=0}^d (l, f_{\{A_k,V_{i-1}\}}(R_s))^2 ds < \epsilon \right\} = o(\epsilon^p).$$

Therefore,

$$\sup_{l\in S_0} \mu\left\{ \int_0^\tau (l, f_{V_{i-1}}(R_s))^2 ds < \epsilon^q \right\} = o(\epsilon^p),$$

this proves (1.44). In particular, for $i = 0$ there exists $k \in [1,d]$ so that

$$\sup_{l\in S_0} \mu\left\{ \int_0^\tau (l, f_{A_k}(R_s))^2 ds < \epsilon \right\} = o(\epsilon^p).$$

S being a compact set, we can choose a finite number of neighborhoods to cover it, hence

$$\mu\left\{ \inf_{l\in S} \int_0^\tau \sum_{k=1}^d (l, f_{A_k}(R_s))^2 ds < \epsilon \right\} = o(\epsilon^p), \quad 1 < p < \infty. \tag{1.46}$$

Since $\tau \leq t$, the above equality still holds when τ is replaced by t. However,

$$\inf_{l\in S} \int_0^t \sum_{k=1}^d (l, f_{A_k}(R_s))^2 ds$$
$$= \inf_{l\in S} \int_0^t |J_s^{-1}\sigma(X_s)^* l|^2 ds$$
$$= \inf_{l\in S} (l, \tilde{\Sigma}_t l)$$

which is just the minimum eigenvalue of $\tilde{\Sigma}_t$ (denoted by $\lambda_{\min}$), it follows that $\lambda_{\min}^{-1} \in L^{\infty-}$, hence

$$[\det \tilde{\Sigma}_t]^{-1} \in L^{\infty-}. \qquad ∎$$

§2. Potential theory over Wiener spaces and quasi-sure analysis

Malliavin calculus provided a possibility of extending the potential theory over finite dimensional spaces to infinite dimensional cases. In this paragraph, we assume that (H, X, μ) is an abstract Wiener space, $\mathbb{D}_k^p$ $(k \in \mathbb{N}_0, p \in (1,\infty))$ are the Watanabe-Sobolev spaces of functionals on X. As we know, any function in finite dimensional Sobolev space $W^{k,p}(\mathbb{R}^m)$ has continuous modification whenever $kp > m$. However, since $m = \infty$ for infinite dimensional case, one could not expect that functionals in $\mathbb{D}_k^p$ have continuous modifications. In order to investigate regularity of functionals in Sobolev spaces, Malliavin[2] introduced the notions of (k,p)-capacities and (k,p)-quasi-continuity and initiated the research field of *quasi-sure analysis*.

In the classical probability theory, events of probability zero, namely sets of measure zero, are negligible. Hence we may call it almost sure analysis. However, in the infinite dimensional analysis, some sets of measure zero are not negligible. For instance, let $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ be a non-degenerate functional. According to discussions in §4 of Chapter II, $\nu_y \equiv \delta_y \circ F$ is understood as conditional distribution under the condition $F = y$. Obviously, $\{\nu_y\}_{y \in \mathbb{R}^m}$ are not mutually equivalent and unlikely absolutely continuous with respect to μ, hence μ-null sets are not necessary ν_y-null sets, propositions which hold μ-a.s. need not hold ν_y-a.s.. A finer analysis requires that the negligible sets are "common null sets" for a family of measures which may be singular each other, namely the *slim sets*. This is known as quasi-sure analysis.

In the classical theory of stochastic processes, one investigates path spaces. But in modern theoretical physics, investigation of *loop spaces* on some Riemannian manifolds is required. Since loops can be treated as paths with fixed terminal points, the quasi-sure analysis becomes an important method in the investigation of loop spaces.

2.1 (k,p)-capacities

Let (H, X, μ) be an abstract Wiener space, $\mathcal{F}$ the μ-completion of σ-algebra $\mathcal{B}(X)$.

Definition 2.1 For $k \in \mathbb{N}_0, 1 < p < \infty$ and any open set O in X, define

$$\mathcal{V}_{k,p}^O = \{\varphi \in \mathbb{D}_k^p : \varphi \geq 0, \varphi(x) \geq 1 \ \mu\text{-a.e.} x \in O\}, \tag{2.1}$$

$$C_{k,p}(O) \equiv \inf\{\|\varphi\|_{k,p} : \varphi \in \mathcal{V}_{k,p}^O\}, \tag{2.2}$$

and for any subset A of X, define

$$C_{k,p}(A) \equiv \inf\{C_{k,p}(O) : O \text{is open and} O \supset A\}. \tag{2.3}$$

The set functions $C_{k,p}$ are called *(k,p)-capacities* on X. If $C_{k,p}(A) = 0$, then A is called a set of *(k,p)-capacity zero*, if it vanishes for all $k \in \mathbb{N}_0$ and $1 < p < \infty$, then A is called a *slim set*. Let $\pi(x)$ be some proposition depending on $x \in X$. If all points $x \in X$ at which $\pi(x)$ is false constitute a set of (k,p)-capacity zero (respectively, slim set), then we say that $\pi(x)$ holds *(k,p)-quasi-surely* or *(k,p)-quasi-everywhere* (respectively, *quasi-surely* or *quasi-everywhere*) and is denoted simply by (k,p)-q.s. or (k,p)-q.e. (respectively, q.s. or q.e.).

Let $1 < p, q < \infty, k, l \in \mathbb{N}_0, A, B, A_j, B_j$ be subsets of X, it is easy to prove that capacities have the following properties:

Lemma 2.2 *If O is an open set in X, then there exists a unique element $e_O \in \mathcal{V}_{k,p}^O \subset \mathbb{D}_k^p$ such that*

$$C_{k,p}(O) = \|e_O\|_{k,p}. \tag{2.4}$$

Proof. The operator $Q^k = (I - \mathcal{L})^{-k/2}$ defined by eq. (II.3.23) being a Banach space isomorphism from L^p onto $\mathbb{D}_k^p$, $\mathbb{D}_k^p$ is a uniformly convex space (cf. Appendix B). Since $\mathcal{V}_{k,p}^O$ is a closed convex subset of $\mathbb{D}_k^p$, there exists a unique point $e_O \in \mathcal{V}_{k,p}^O$ at which its norm $\|e_O\|_{k,p}$ attains the infimum $C_{k,p}(O)$.∎

Remark. Since

$$Q^k = (I - \mathcal{L})^{-k/2} = \frac{1}{\Gamma(k/2)} \int_0^\infty t^{k/2-1} e^{-t} T_t dt,$$

it follows that Q^k is positivity preserving. Conversely, if $e_O = Q^k\varphi, \varphi \in L^p$, then $\varphi \geq 0$ a.s.. This can be proved as follows:

Let $B = \{x : \varphi(x) < 0\}, \psi = Q^k 1_B$. Then $\psi \geq 0$ a.s., hence $\forall \lambda \geq 0, e_O + \lambda\psi \in \mathcal{V}_{k,p}^O$. Let $f(\lambda) \equiv \|e_O + \lambda\psi\|_{k,p}^p = \|\varphi + \lambda 1_B\|_p^p$. Since $f(\lambda)$ attains its infimum at $\lambda = 0$, the right-hand derivative

$$f'_+(0) = p \int_B |\varphi|^{p-2}\varphi d\mu \geq 0,$$

it follows that $\mu(B) = 0$.

Proposition 2.3 The capacities have the following properties:

1° If $A \in \mathcal{F}$, then $\mu(A) \leq C_{k,p}^p(A)$.

2° If $k \leq l, p \leq q$, then $C_{k,p}(A) \leq C_{l,q}(A)$.

3° If $A \subset B$, then $C_{k,p}(A) \leq C_{k,p}(B)$.

4° If $A = \bigcup_{j=1}^\infty A_j$, then

$$C_{k,p}(A) \leq \sum_{j=1}^\infty C_{k,p}(A_j). \tag{2.5}$$

$$C_{k,p}^p(A) \leq \sum_{j=1}^\infty C_{k,p}^p(A_j). \tag{2.6}$$

5° (Borel-Cantelli's Lemma)

$$\sum_{j=1}^{\infty} C_{k,p}(A_j) < \infty \Longrightarrow C_{k,p}(\overline{\lim}_j A_j) = 0. \tag{2.7}$$

Proof. Let O be an open set, $0 \le k \le l, 1 < p \le q$. Then by definition,

$$\mu(O)^{1/p} = C_{0,p}(O) \le C_{k,p}(O) \le C_{l,q}(O).$$

By taking their infimums we obtain 1°, 2° and 3°. In the proof of 4° we may assume that $\{A_j\}$ are open sets and the series in (2.5) and (2.6) are convergent. By definition, $\forall j \in \mathbb{N}, \epsilon > 0, \exists \varphi_j \in \mathcal{V}_{k,p}^{A_j}$ such that $\|\varphi_j\|_{k,p} < C_{k,p}(A_j) + \epsilon 2^{-j}$. Let $\varphi \equiv \sum_j \varphi_j$. Then $\varphi \in \mathcal{V}_{k,p}^{A}$ and

$$\|\varphi\|_{k,p} < \sum_j \|\varphi_j\|_{k,p} < \sum_j C_{k,p}(A_j) + \epsilon.$$

Since ϵ is arbitrary, (2.5) follows. To prove (2.6), noting that by Lemma 2.2, $\forall j \in \mathbb{N}, \exists e_{A_j} \in \mathcal{V}_{k,p}^{A_j} \subset \mathbb{D}_k^p$, such that $C_{k,p}(A_j) = \|e_{A_j}\|_{k,p}$. Since $Q^k : L^p \to \mathbb{D}_k^p$ is a Banach space isomorphism, we may assume that $e_{A_j} = Q^k \varphi_j, \varphi_j \in L^p$ and $\varphi_j \ge 0$ $(j \in \mathbb{N})$. Let $\varphi \equiv \sup_j \varphi_j, \psi = Q^k \varphi$. Then $\psi \ge \sup_j e_{A_j}$, hence $\psi \in \mathcal{V}_{k,p}^{A}$ and

$$\begin{aligned} C_{k,p}^p(A) &\le \|\psi\|_{k,p}^p = \|\varphi\|_p^p \\ &\le \sum_j \|\varphi_j\|_p^p = \sum_j \|e_{A_j}\|_{k,p}^p \\ &= \sum_j C_{k,p}^p(A_j). \end{aligned}$$

Implication (2.7) is a simple consequence of countable subadditivity (2.5). ∎

By property 1° we know that sets of (k,p)-capacity zero are μ-null sets, hence (k,p)-q.s. implies μ-a.s., all quasi-sure propositions hold almost surely. By property 4°, countable union of slim sets is again a slim set.

Next lemma is a weaker form of Tchebycheff's inequality.

Lemma 2.4 *If ψ is continuous and $\varphi = Q^k \psi$ $(k \in \mathbb{N}_0)$, then $\forall \epsilon > 0$ and $p \in (1, \infty)$ we have*

$$C_{k,p}\{x : |\varphi(x)| > \epsilon\} \le \epsilon^{-1} \|\varphi\|_{k,p}. \tag{2.8}$$

Proof. We may assume that ψ is bounded. Since

$$Q^k = (I - \mathcal{L})^{-k/2} = \frac{1}{\Gamma(k/2)} \int_0^{\infty} t^{k/2-1} e^{-t} T_t dt,$$

it follows that φ is continuous, hence $\{|\varphi| > \epsilon\}$ is open. Since $\psi = \psi_+ - \psi_-, \varphi = Q^k\psi_+ - Q^k\psi_-$, we have $\{|\varphi| > \epsilon\} \subset \{Q^k\psi_+ > \epsilon\} \cup \{Q^k\psi_- > \epsilon\}$. By (2.6) we

have

$$\begin{aligned} C_{k,p}^p(|\varphi| > \epsilon) &\le C_{k,p}^p(Q^k\psi_+ > \epsilon) + C_{k,p}^p(Q^k\psi_- > \epsilon) \\ &\le \epsilon^{-p}(\|Q^k\psi_+\|_{k,p}^p + \|Q^k\psi_-\|_{k,p}^p) \\ &= \epsilon^{-p}(\|\psi_+\|_p^p + \|\psi_-\|_p^p) \\ &= \epsilon^{-p}\|\varphi\|_{k,p}^p. \end{aligned}$$

(2.8) is proved. ∎

2.2 Quasi-continuous modifications

Definition 2.5 *A functional φ on X is called (k,p)-quasi-continuous if $\forall \epsilon > 0$ there exists an open set O_ϵ such that $C_{k,p}(O_\epsilon) < \epsilon$ and the restriction of φ to O_ϵ^c is continuous; it is called quasi-continuous if it is (k,p)-quasi-continuous for all $k \in \mathbb{N}_0$ and $p \in (1, \infty)$.*

Remark. Let K be a closed set. If O is the largest open set such that $\mu(O \cap K) = 0$, then $\mathrm{ess}(K) = K \setminus O$ is called the *μ-essential part* of K. It is easy to see that $\mu(\mathrm{ess}(K)) = \mu(K)$, and that if φ is continuous on K and $\varphi = 0$ μ-a.e. on K, then φ vanishes everywhere on $\mathrm{ess}(K)$.

By the properties of capacities we know that φ is (k,p)-quasi-continuous if and only if there exists a decreasing sequence of open sets $\{O_n\}$ such that $\lim_{n\to\infty} C_{k,p}(O_n) = 0$, $\forall n$, $\varphi|_{O_n^c}$ is continuous and that $\mathrm{ess}(O_n^c) = O_n^c$. By the tightness of capacities (cf. Theorem 2.12), we may assume that $\{O_n^c\}$ are compact sets. Such a sequence is called a nest. By a diagonalization procedure it is easy to see that φ is quasi-continuous if and only if there exists a nest $\{O_n\}$ of open sets described above such that $\lim_{n\to\infty} C_{n,n}(O_n) = 0$.

Quasi-continuous functionals have the following important properties:

Theorem 2.6 *If φ is (k,p)-quasi-continuous, then*

1° $\varphi = 0$ μ-a.s. $\Longrightarrow \varphi = 0$ (k,p)-q.s.;

2° $\varphi \ge 0$ μ-a.s. $\Longrightarrow \varphi \ge 0$ (k,p)-q.s..

Proof. Let $\{O_n\}$ be a nest of open sets related to φ. Since $\varphi = 0$ μ-a.e. on O_n^c and $O_n^c = \mathrm{ess}(O_n^c)$, it follows that φ vanishes everywhere on O_n^c $(\forall n \in \mathbb{N})$, hence

$$C_{k,p}(x : \varphi(x) \ne 0) \le C_{k,p}(\cap_n O_n) = 0,$$

thus 1° is proved. By applying 1° to $\psi \circ \varphi$, where $\psi \in C_b^\infty(\mathbb{R})$ and for $t < 0, \psi(t) > 0$ while for $t \ge 0, \psi(t) = 0$, we obtain that $\{x : \varphi(x) < 0\} = \{x : \psi(\varphi(x)) \ne 0\}$ proving 2°. ∎

Corollary 2.7 *If φ is quasi-continuous and O is an open set in X, then*

1° $\varphi = 0$ *μ-a.e. on $O \Longrightarrow \varphi = 0$ q.e. on O;*

2° $\varphi \ge 0$ *μ-a.e. on $O \Longrightarrow \varphi \ge 0$ q.e. on O.*

Definition 2.8 Let φ be a functional. If a functional φ^* is (k,p)-quasi-continuous (respectively, quasi-continuous) such that $\varphi^* = \varphi$ μ-a.s., then φ^* is

called the *(k,p)-quasi-continuous modification* (respectively, *quasi-continuous modification*) or *redefinition* of φ.

By Theorem 2.6, the quasi-continuous modification is unique up to q.s. equivalence.

The following is a capacity version of Lusin's theorem:

Theorem 2.9 *If $\varphi \in \mathbb{D}^p_k$, then there exists a (k,p)-quasi-continuous modification φ^* uniquely defined up to a set of (k,p)-capacity zero; if $\varphi \in \mathbb{D}^\infty$, then there exists a quasi-continuous modification φ^* uniquely defined up to a slim set.*

Proof. It suffices to prove the first conclution. Let $Q = (I - C)^{-1/2}$,

$$\mathcal{C} = \{\varphi \in \mathbb{D}^p_k : \varphi = Q^k\psi, \psi \text{ is bounded and continuous}\}.$$

Then $\mathcal{C}$ is dense in $\mathbb{D}^p_k$, hence for $\varphi \in \mathbb{D}^p_k$ there exist $\{\varphi_j\} \subset \mathcal{C}, \varphi_j = Q^k\psi_j, \psi_j$ are bounded and continuous such that

$$\|\varphi_{j-1} - \varphi_j\|_{k,p} + \|\varphi_j - \varphi\|_{k,p} < 4^{-j}, \quad j \ge 1.$$

Let

$$O_n = \bigcup_{j=n+1}^{\infty} \{x : |\varphi_{j+1}(x) - \varphi_j(x)| > 2^{-j}\}, \quad n \ge 0.$$

By Lemma 2.4 we have

$$\begin{aligned} C_{k,p}(O_n) &\le \sum_{j=n+1}^{\infty} C_{k,p}(|\varphi_{j+1} - \varphi_j| > 2^{-j}) \\ &\le \sum_{j=n+1}^{\infty} 2^j \|\varphi_{j+1} - \varphi_j\|_{k,p} < 2^{-n}. \end{aligned}$$

Since on O^c_n,

$$\sum_{j=1}^{\infty} \sup_x |\varphi_{j+1}(x) - \varphi_j(x)| < \infty,$$

so $\{\varphi_j\}$ converges uniformly. Define

$$\varphi^*(x) = \begin{cases} \lim_{j\to\infty}\varphi_j(x), & x \in \bigcup_n O^c_n; \\ 0, & x \in \bigcap_n O_n. \end{cases}$$

It is easy to see that φ^* is the (k,p)-quasi-continuous modification of φ. ∎

Proposition 2.10 (Tchebycheff inequality) *If $\varphi \in \mathbb{D}^p_k, \epsilon > 0$, then*

$$C_{k,p}(x : |\varphi^*(x)| > \epsilon) \le \epsilon^{-1}\|\varphi\|_{k,p}, \tag{2.9}$$

where φ^* is the (k,p)-quasi-continuous modification of φ.

Proof. Let $\varphi = Q^k\psi = Q^k\psi_+ - Q^k\psi_-$, $\psi \in L^p$. Then $\varphi^* = (Q^k\psi_+)^* - (Q^k\psi_-)^*$ (k,p)-q.s., hence except a set of (k,p)-capacity zero we have

$$\{|\varphi^*| > \epsilon\} \subset \{(Q^k\psi_+)^* > \epsilon\} \cup \{(Q^k\psi_-)^* > \epsilon\}.$$

Choose a nest $\{O_m\}$ of open sets such that $\lim_{m\to\infty} C_{k,p}(O_m) = 0$ and $(Q^k\psi_+)^*$ and $(Q^k\psi_-)^*$ are continuous on every set O^c_m. It is easy to see that, $\forall m \in \mathbb{N}, \exists$ open set G_m such that $\{(Q^k\psi_\pm)^* > \epsilon\} \cap O^c_m = G_m \cap O^c_m$. Hence by Definition 2.1 we have

$$\begin{aligned} C^p_{k,p}(|\varphi^*| > \epsilon) &\le C^p_{k,p}((Q^k\psi_+)^* > \epsilon) + C^p_{k,p}((Q^k\psi_-)^* > \epsilon) \\ &\le \epsilon^{-p}(\|Q^k\psi_+\|^p_{k,p} + \|Q^k\psi_-\|^p_{k,p}) + 2C^p_{k,p}(O_m). \end{aligned}$$

Letting $m \to \infty$, we have

$$\begin{aligned} C^p_{k,p}(|\varphi^*| > \epsilon) &\le \epsilon^{-p}(\|\psi_+\|^p_p + \|\psi_-\|^p_p) \\ &= \epsilon^{-p}\|\psi\|^p_p = \epsilon^{-p}\|\varphi\|^p_{k,p}. \end{aligned}$$ ∎

Theorem 2.11 *If $\{\varphi_n\}$ converges in $\mathbb{D}^p_k$ to φ, then there exist a subsequence $\{\varphi_{n_j}\}$ and a nest $\{O_m\}$ of open sets satisfying that $\lim_{m\to\infty} C_{k,p}(O_m) = 0$ such that $\{\varphi^*_{n_j}\}$ converges uniformly to φ^* on every set O^c_m. In particular, $\varphi^*_{n_j} \longrightarrow \varphi^*$, (k,p)-q.s..*

Proof. Without loss of generality we may assume that $\varphi = 0$. By inequality (2.9), $\forall \epsilon > 0$ we have

$$\lim_{n\to\infty} C_{k,p}(|\varphi^*_n| > \epsilon) = 0.$$

Choose a subsequence $\{n_j\}$ so that $\forall j$,

$$C_{k,p}(|\varphi^*_{n_j}| > j^{-1}) < j^{-2},$$

therefore, there exist open sets $\tilde{O}_j \supset \{|\varphi^*_{n_j}| > j^{-1}\}$ such that $C_{k,p}(\tilde{O}_j) < j^{-2}$. Then $O_m = \bigcup_{j=m}^{\infty} \tilde{O}_j$ constitute the desired nest. ∎

2.3 Tightness, continuity and invariance of capacities

The following theorem shows the tightness of capacities.

Theorem 2.12 *There exists a sequence of compact sets in X:*

$$K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$$

so that

$$\lim_{n\to\infty} C_{k,p}(K^c_n) = 0.$$

Proof. By Theorem I.4.18, there exists another Banach space Y such that $\mu(Y) = 1, H \subset Y \subset X$, and the imbedding map $i : Y \to X$ is compact. Let $\varphi(y) = \|y\|_Y$ be defined μ-a.s. on X. By Fernique Theorem (I.4.20), $\exists\lambda > 0$ such that $\mathbb{E}[\exp(\lambda\varphi^2)] < \infty$, hence $\varphi \in L^{\infty-}$. By Corollary II.3.24, $\forall t > 0$, $T_t\varphi \in \mathbb{D}^\infty$. Since

$$(T_t\varphi)(y) = \int \|e^{-t}y + \sqrt{1 - e^{-2t}}z\|_Y \mu(dz),$$

letting $\delta_1 = e^{-t}, \delta_2 = \sqrt{1-e^{-2t}}\int \|z\|_Y \mu(dz)$, we have

$$\delta_1 \|y\|_Y - \delta_2 \le (T_t\varphi)(y) \le \delta_1 \|y\|_Y + \delta_2.$$

$(T_t\varphi)^{-1}([0,n]) \subset \{y : \|y\|_Y \le (n+\delta_2)/\delta_1\}$ being bounded subset of Y, denote by K_n its closure in Y. Then K_n is a compact set in X. Since $T_t\varphi > n$ μ-a.e. on K_n^c, by Theorem 2.6, its (k,p)-quasi-continuous modification $(T_t\varphi)^* > n$ (k,p)-q.e. on K_n^c, hence by Proposition 2.10 we have

$$\begin{aligned} C_{k,p}(K_n^c) &\le C_{k,p}(y : (T_t\varphi)^*(y) > n) \\ &\le n^{-1}\|T_t\varphi\|_{k,p} \to 0. \end{aligned}$$

The proof is complete. ∎

Proposition 2.13 *Let A be a subset of X, define*

$$\tilde{V}^A_{k,p} \equiv \{\varphi \in \mathbb{D}^p_k : \varphi^* \ge 1 \ (k,p)\text{-q.e. on } A\}. \tag{2.10}$$

Then $\tilde{V}^A_{k,p}$ is a closed convex subset of $\mathbb{D}^p_k$ and

$$C_{k,p}(A) = \inf\{\|\varphi\|_{k,p} : \varphi \in \tilde{V}^A_{k,p}\}. \tag{2.11}$$

Therefore, there exists a unique element $e_A \in \tilde{V}^A_{k,p} \subset \mathbb{D}^p_k$ such that

$$C_{k,p}(A) = \|e_A\|_{k,p}. \tag{2.12}$$

e_A is called (k,p)-equilibrium potential of A.

Proof. The closedness of $\tilde{V}^A_{k,p}$ follows from Theorem 2.11. $\mathbb{D}^p_k$ being isomorphic to L^p, it is uniformly convex, hence there exists a unique element e_A so that the right-hand sides of eqs. (2.11) and (2.12) are equal. Firstly we assume that A is open. By Theorem 2.6, if $\varphi \ge 1$ a.e. on A, then $\varphi^* \ge 1$ (k,p)-q.e. on A, therefore,

$$\|e_A\|_{k,p} = C_{k,p}(A) \ (A \text{ is open}). \tag{2.13}$$

Next assume that A is arbitrary and the open set $O \supset A$. Hence

$$\tilde{V}^O_{k,p} \subset \tilde{V}^A_{k,p}.$$

Taking infimum of $\|\varphi\|_{k,p}$, we have $C_{k,p}(O) \ge \|e_A\|_{k,p}$. Again by taking infimum over all open sets O which contain A, we obtain that

$$C_{k,p}(A) \ge \|e_A\|_{k,p}. \tag{2.14}$$

Now we prove the inverse inequality. Let e_A^* be (k,p)-quasi-continuous modification of e_A. Then $\forall \epsilon > 0$, there exists an open set O_ϵ with $C_{k,p}(O_\epsilon) < \epsilon$ such that e_A^* is continuous on O_ϵ^c and $e_A^* \ge 1$ on $O_\epsilon^c \cap A$. Let

$$\Omega_\epsilon \equiv \{x \in O_\epsilon^c : e_A^*(x) > 1-\epsilon\} \cup O_\epsilon.$$

Then Ω_ϵ is open and $\Omega_\epsilon \supset A$. Denoting the equilibrium potential of O_ϵ by e_ϵ, since $e_A + e_\epsilon > 1-\epsilon$ a.e. on Ω_ϵ, it follows that

$$\begin{aligned} C_{k,p}(A) &\le C_{k,p}(\Omega_\epsilon) \\ &< (1-\epsilon)^{-1}\|e_A + e_\epsilon\|_{k,p} \\ &\le (1-\epsilon)^{-1}(\|e_A\|_{k,p} + \epsilon). \end{aligned}$$

Letting $\epsilon \downarrow 0$ we obtain $C_{k,p}(A) \le \|e_A\|_{k,p}$. ∎

By virtue of equilibrium potential, we can prove continuity of the capacities.

Theorem 2.14 *The capacities $C_{k,p}$ have the following properties:*

1° *If a sequence of sets $\{A_n\}$ is non-decreasing, then*

$$C_{k,p}(A_n) \uparrow C_{k,p}(\cup_n A_n); \tag{2.15}$$

2° *If a sequence of compact sets $\{K_n\}$ is non-increasing, then*

$$C_{k,p}(K_n) \downarrow C_{k,p}(\cap_n K_n). \tag{2.16}$$

Proof. 1° Let $A = \bigcup_n A_n$. Denote the equilibrium potential of A_n by e_n. Then $\|e_n\|_{k,p} = C_{k,p}(A_n)$ and $\{e_n\}$ is bounded in $\mathbb{D}^p_k$. By Banach-Saks-Kakutani Theorem (cf. Appendix B), there exists a subsequence $\{e_{n_j}\}$ so that $S_m \equiv \frac{1}{m}\sum_{j=1}^m e_{n_j}$ converges in $\mathbb{D}^p_k$. Denote its limit by e_A. By Theorem 2.11, there exists a subsequence $\{S_{m_j}\}$ of $\{S_m\}$ such that

$$\lim_{j\to\infty} S^*_{m_j} = e^*_A, \quad (k,p)\text{-q.s.}.$$

It follows that $e^*_A \in \tilde{V}^A_{k,p}$. By eq. (2.11) we have

$$\begin{aligned} C_{k,p}(A) &\le \|e_A\|_{k,p} \le \sup_n \|e_n\|_{k,p} \\ &= \sup_n C_{k,p}(A_n). \end{aligned}$$

The inverse inequality is obvious.

2° Let $K = \bigcap_n K_n$. Then $\forall \epsilon > 0, \exists$ an open set $O_\epsilon \supset K$ such that $C_{k,p}(O_\epsilon) < C_{k,p}(K) + \epsilon$. Since $\{O_\epsilon^c \cap K_n\}$ is a non-increasing sequence of compact sets with intersection $O_\epsilon^c \cap K = \emptyset$, it follows that $\exists n_0$ such that $O_\epsilon^c \cap K_n = \emptyset$ whenever $n \ge n_0$, which means that $O_\epsilon \supset K_n$. Therefore

$$C_{k,p}(K_n) < C_{k,p}(K) + \epsilon.$$

Letting $\epsilon \downarrow 0$ we have

$$\lim_{n\to\infty} C_{k,p}(K_n) \le C_{k,p}(K).$$

The inverse inequality is also obvious. ∎

Remark. Theorem 2.14 shows that $C_{k,p}$ are *Choquet capacities* (cf. Dellacherie-Meyer[1]). According to Choquet's capacity theorem, for any Borel set B and any $\epsilon > 0$, there exists a compact set $K_\epsilon \subset B$ such that

$$C_{k,p}(K_\epsilon) \geq C_{k,p}(B) - \epsilon. \tag{2.17}$$

In other words, Borel sets are $C_{k,p}$-*capacitable*.

Apparently the definition of capacity depends on topological structure in X. However, in a general Gaussian probability space $(\Omega, \mathcal{F}, \mu; H)$, there are no topological structures in Ω. By fixing a base of H, we obtain a numerical model $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$, hence we can define (k,p)-capacities on $\mathbb{R}^\infty$. A natural question is: are the quasi-sure properties intrinsic? Or from the viewpoint of abstract Wiener spaces, if (H, X_1, μ_1) and (H, X_2, μ_2) are two abstract Wiener spaces, are the capacities defined on these two spaces equivalent? This question was put forward by Itô and afterward Albeverio, Fukushima et al.[1] had given an affirmative answer. The key to the answer is the tightness of capacities. In fact, restriction of any continuous injection to a compact set is a homeomorphism. The following theorem is a version of this result. For further discussion we refer to Gong and Ma[1].

Theorem 2.15 *Let (H, X, μ) be an abstract Wiener space. By choosing a base $\{e_j\}$ of H which is contained in X^*, the map $x \longmapsto \{\langle x, e_j\rangle\}_{j\in\mathbb{N}}$ defines a continuous injection $\iota : X \longrightarrow \mathbb{R}^\infty$ and establishes a numerical model $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$. Let $C_{k,p}$ and $\widetilde{C}_{k,p}$ be the (k,p)-capacities defined on X and $\mathbb{R}^\infty$ respectively. Then*

1° $(\iota X)^c$ is a slim set in $\mathbb{R}^\infty$;

2° $\forall k \in \mathbb{N}_0, p \in (1,\infty)$ and any subset A of X,

$$C_{k,p}(A) = \widetilde{C}_{k,p}(\iota A). \tag{2.18}$$

Proof. Obviously, $\gamma^\infty = \mu \circ \iota^{-1}$ and the topology induced by ι on X is weaker than the norm topology, hence

$$C_{k,p}(A) \leq \widetilde{C}_{k,p}(\iota A). \tag{2.19}$$

By the tightness of capacities, there exists a sequence $\{K_n\}$ of compact sets in X such that $C_{k,p}(K_n^c) < \frac{1}{n}$. Let $\widetilde{K}_n = \iota K_n$. Then $\widetilde{K}_n$ is a compact set in $\mathbb{R}^\infty$. Moreover,

$$\gamma^\infty(\widetilde{K}_n^c \setminus \iota(K_n^c)) = 0.$$

Consequently,

$$\varphi \geq 1 \ \gamma^\infty\text{-a.e. on } \widetilde{K}_n^c \Longleftrightarrow \varphi \circ \iota \geq 1 \ \mu\text{-a.e. on } K_n^c.$$

It follows that

$$\widetilde{C}_{k,p}((\iota X)^c) \leq \widetilde{C}_{k,p}(\widetilde{K}_n^c) = C_{k,p}(K_n^c) \longrightarrow 0,$$

thus 1° is proved.

Suppose that an open set $O \supset A$ such that

$$C_{k,p}(O) < C_{k,p}(A) + \epsilon.$$

$\iota : K_n \longrightarrow \widetilde{K}_n$ being a homeomorphism, $\exists \widetilde{O} \supset \iota A$ such that $\iota(O \cap K_n) = \widetilde{O} \cap \widetilde{K}_n$. It follows from $\iota^{-1}\widetilde{O} \subset (O \cap K_n) \cup K_n^c$ that

$$\begin{aligned}\widetilde{C}_{k,p}(\iota A) &\leq \widetilde{C}_{k,p}(\widetilde{O}) = C_{k,p}(\iota^{-1}\widetilde{O}) \\ &\leq C_{k,p}(O \cap K_n) + C_{k,p}(K_n^c) \\ &\leq C_{k,p}(A) + \epsilon + 1/n.\end{aligned}$$

Letting $n \to \infty, \epsilon \downarrow 0$, we obtain the inverse inequality of (2.19). ∎

2.4 Positive generalized functionals and measures with finite energy

In finite dimensional case, as we know, any positive generalized function is a measure. For abstract Wiener space (H, X, μ), Sugita[3] proved that any positive generalized functional (in the sense of S. Watanabe) is a Borel measure on X; In case of nuclear spaces, Kondratiev-Samoylenko[1] and Yokoi[1] also proved that any positive distribution (in the sense of T. Hida) is a measure. Since the space of Hida's distributions is larger than that of Meyer-Watanabe's generalized functionals and the proof in case of nuclear spaces is much simpler, here we only state Sugita's theorem without proof and refer the reader to Sugita[3] for details. A simple proof for classical Wiener spaces can be found in Huang[5].

We assume that $k \in \mathbb{N}_0, 1 < p < \infty, p^{-1} + q^{-1} = 1$.

Definition 2.16 Let $G \in \mathbb{D}^{-\infty}$. If for any $\varphi \in \mathbb{D}^\infty, \varphi \geq 0$ a.s., it holds that $\langle G, \varphi\rangle \geq 0$, then G is called a *positive generalized functional* and denoted by $G \in \mathbb{D}_+^{-\infty}$.

It is easy to see that, if $G \in \mathbb{D}_+^{-\infty} \cap L^p$, then $G \geq 0$ a.s., and that Q^k, T_t preserve the positivity.

Theorem 2.17 (Sugita[3]) *For any $G \in \mathbb{D}_+^{-\infty}$, there exists a unique finite measure ν_G on $(X, \mathcal{B}(X))$ such that*

$$\langle G, \varphi\rangle = \int_X \varphi^*(x)\nu_G(dx), \quad \forall \varphi \in \mathbb{D}^\infty, \tag{2.20}$$

where φ^* is the quasi-continuous modification of φ. If $G \in \mathbb{D}^q_{-k}$, then eq. (2.20) holds for all $\varphi \in \mathbb{D}^p_k$, where φ^* is the (k,p)-quasi-continuous modification of φ.

Remark. Since quasi-continuous modification is uniquely defined q.s. and ν_G never charges slim sets (cf. Corollary 2.20), expression (2.20) is well defined.

In particular, if $G \in L^q_+$, then eq. (2.20) holds for all $\varphi \in L^p$, hence $\nu_G << \mu$ and $d\nu_G = Gd\mu$.

Note that if $G \in \mathbb{D}^q_{-k}$, then $Q^k G \in L^q$. Since $q-1 = q/p$, $Q^k G|^{q-1} \in L^p$. Let

$$\varphi = Q^k(|Q^k G|^{q-2} Q^k G). \tag{2.21}$$

Then $\varphi \in \mathbb{D}^p_k$ and

$$G = Q^{-k}(|Q^{-k}\varphi|^{p-2}Q^{-k}\varphi). \tag{2.22}$$

Thus eqs. (2.21) and (2.22) have established a 1-1 correspondence between $\mathbb{D}^p_k$ and $\mathbb{D}^q_{-k}$.

Theorem 2.18 *Let e_A be (k,p)-equilibrium potential of subset A of X. Define*

$$G_A \equiv Q^{-k}(|Q^{-k}e_A|^{p-2}Q^{-k}e_A). \tag{2.23}$$

Then $G_A \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$ and the support of corresponding measure ν_A ((k,p)-equilibrium measure of A) is contained in $\overline{A}$. Moreover,

$$\begin{aligned}\int_X e^*_A(x)\nu_A(dx) &= \|e_A\|^p_{k,p} = \|G_A\|^q_{-k,q} \\ &= C^p_{k,p}(A),\end{aligned} \tag{2.24}$$

which is referred to as the (k,p)-energy of e_A or ν_A.

Proof. By eq. (2.21), $e_A = Q^k(|Q^k G_A|^{q-2}Q^k G_A)$. If $\psi \in \mathbb{D}^p_k$ and $\psi \geq 0$ a.s., then by Theorem 2.6, its (k,p)- quasi-continuous modification $\psi^* \geq 0$ (k,p)-q.s.. For $\lambda \geq 0$ we have $e_A + \lambda\psi^* \in \mathcal{V}^A_{k,p}$. Let

$$f(\lambda) \equiv \|e_A + \lambda\psi\|^p_{k,p} = \|Q^{-k}(e_A + \lambda\psi)\|^p_p.$$

Since $f(\lambda)$ attains its infimum at $\lambda = 0$, its right-hand derivative $f'_+(0) \geq 0$. But

$$\begin{aligned}f'_+(0) &= p\int |Q^{-k}e_A|^{p-2}(Q^{-k}e_A)(Q^{-k}\psi)d\mu \\ &= p\int (Q^k G_A)(Q^{-k}\psi)d\mu \\ &= p\langle G_A, \psi\rangle,\end{aligned}$$

it follows that $G_A \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$. Note that $\langle G_A, \psi\rangle = \int \psi^* d\nu_A \geq 0$ whenever $\psi \geq 0$ (k,p)-q.e. on A no matter what value of ψ outside A, we have (cf. Sugita[3] for details) $\mathrm{supp}(\nu_A) \subset \overline{A}$. However,

$$\begin{aligned}\int_X e^*_A d\nu_A &= \langle G_A, e_A\rangle \\ &= \langle G_A, Q^k(|Q^k G_A|^{q-2}Q^k G_A)\rangle \\ &= \langle Q^k G_A, |Q^k G_A|^{q-2}Q^k G_A\rangle \\ &= \|Q^k G_A\|^q_q = \|G_A\|^q_{-k,q}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle G_A, e_A\rangle &= \langle Q^{-k}(|Q^{-k}e_A|^{p-2}Q^{-k}e_A), e_A\rangle \\ &= \langle |Q^{-k}e_A|^{p-2}Q^{-k}e_A, Q^{-k}e_A\rangle \\ &= \|Q^{-k}e_A\|^p_p = \|e_A\|^p_{k,p} \\ &= C^p_{k,p}(A).\end{aligned}$$

The proof is complete. ∎

Theorem 2.19 *Let $G \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$, $\tilde{\nu}_G$ be the outer measure for ν_G. Then for any subset A of X it holds that*

$$\tilde{\nu}_G(A) \leq \|G\|_{-k,q} C_{k,p}(A). \tag{2.25}$$

Proof. Let $G_n = T_{1/n}G$. Then G_n weakly converges to G in $\mathbb{D}^q_{-k}$ hence (cf. Remark below) ν_{G_n} weakly converges to ν_G. Let O be an open set, e_O be its (k,p)-equilibrium potential. Then

$$\langle G_n, e_O\rangle \geq \int_O G_n d\mu = \nu_{G_n}(O). \tag{2.26}$$

Letting $n \to \infty$, we have $\underline{\lim}_{n\to\infty}\nu_{G_n}(O) \geq \nu_G(O)$. The left-hand side of above inequality converges to $\langle G, e_O\rangle$ and

$$\begin{aligned}\langle G, e_O\rangle &\leq \|G\|_{-k,q}\|e_O\|_{k,p} \\ &= \|G\|_{-k,q}C_{k,p}(O),\end{aligned}$$

hence (2.25) holds for any open set O. Taking infimum over all open sets O which contain A, we obtain inequality (2.25). ∎

Remark. From the tightness of capacities we have that, $\forall\epsilon > 0$, there exists a compact set K such that $C_{k,p}(K^c) < \epsilon$. Let e_{K^c} be the (k,p)-equilibrium potential. Then $\forall n$,

$$\begin{aligned}\nu_{G_n}(K^c) &= \int_{K^c} T_{1/n}G d\mu \leq \int_{K^c} e_{K^c}T_{1/n}G d\mu \\ &= \langle T_{1/n}e_{K^c}, G\rangle \leq \|T_{1/n}e_{K^c}\|_{k,p}\|G\|_{-k,q} \\ &\leq \epsilon\|G\|_{-k,q},\end{aligned}$$

hence $\{\nu_{G_n}\}$ is tight. Since in finite dimensional spaces any bounded continuous function can be approximated by smooth functions, it follows that all finite dimensional distributions of ν_{G_n} converge weakly to those of ν_G. From the tightness of $\{\nu_{G_n}\}$ we see that ν_{G_n} converges weakly to ν_G.

We then have some important consequences of Theorem 2.19.

Corollary 2.20 *If $G \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$, then ν_G never charges sets of (k,p)-capacity zero. In particular, if $G \in \mathbb{D}^{-\infty}_+$, then ν_G never charges slim sets.*

Corollary 2.21 *Let B be a Borel set. B is a set of (k,p)-capacity zero if and only if $\nu_G(B) = 0$, $\forall G \in \mathbb{D}^q_{-k} \cap \mathbb{D}^{-\infty}_+$. In particular, B is a slim set if and only if $\nu_G(B) = 0$, $\forall G \in \mathbb{D}^{-\infty}_+$.*

Proof. It suffices to prove the "only if" part. Suppose that $C_{k,p}(B) > 0$. Then there exists a compact set $K \subset B$ such that $C_{k,p}(K) > 0$. Let ν_K be the (k,p)-equilibrium measure of K. Then $\mathrm{supp}(\nu_K) \subset K$ and $\nu_K(K) > 0$ (otherwise by eq. (2.24) we have $C_{k,p}(K) = 0$). ∎

The measure which is expressed by some positive generalized functional is referred to as *measure with finite energy*. Hence, the slim sets are characterized as "common null sets" for all measures with finite energy.

As one of applications we have the following theorem of decomposition of integrals.

Theorem 2.22 *Let $F \in \mathbb{D}^\infty(\mathbb{R}^m)$ be non-degenerate, ρ_F be its density. For $y \in \mathcal{R}(F)$ (i.e. range of F), define*

$$\nu_y = \mu(\cdot|F = y). \tag{2.27}$$

Then $\exists k \in \mathbb{N}_0, p \in (1,\infty)$ so that the map $y \longmapsto \nu_y$ is continuous in $\mathbb{D}^p_{-k}$. Moreover, $\forall \varphi \in C^\infty(\mathbb{R}^m), G \in \mathbb{D}^\infty$,

$$\int_X G(\varphi \circ F)d\mu = \int_{\mathbb{R}^m} \varphi(y)\rho_F(y)dy \int_X G^* d\nu_y. \tag{2.28}$$

Proof. It follows from Theorem II.4.9 and Lemma II.4.8 that, if $k > m$, then the map $y \longmapsto \delta_y \circ F$ is continuous in $\mathbb{D}^p_{-k}$. Since $\delta_y \circ F \in \mathbb{D}^{-\infty}_+$, by eq. (II.4.23) we know that $\rho_F(y)^{-1}\delta_y \circ F$ is just the conditional probability ν_y and

$$\begin{aligned} E[G|F = y] &= \rho_F(y)^{-1}\langle \delta_y \circ F, G\rangle \\ &= \int_X G^* d\nu_y. \end{aligned}$$

Hence eq. (2.28) holds. ∎

Corollary 2.23 (principle of descent) *Under the conditions of Theorem 2.22, all propositions which hold quasi-surely also hold almost surely with respect to any conditional probability given $F = y, y \in \mathcal{R}(F)$.*

Proof. It follows from the fact that slim sets are "common null sets". ∎

2.5 Some quasi-sure sample properties of stochastic processes

In classical Wiener spaces, many properties of Brownian paths such as Hölder continuity, nowhere differentiability, law of the iterated logarithm, unattainability of single point sets (for dimension $d > 3$), absence of double points ($d > 5$) etc. are proved to be not only almost sure but also quasi-sure. For instance, see Fukushima[1] and Takeda[1]. Some classical limit theorems such as convergence of quadratic variations, large deviation principle, Doob's inequality and martingale

limit theorems etc. can be refined by replacing measures with capacities. For instance, see Yoshida[1], Ren[3] and Denis[1]. In this section we only give a brief introduction to some results on quasi-sure continuity of paths of stochastic processes and quasi-sure approximation of paths of diffusion processes. for details we refer to Ren[1,5].

Let us come back to classical Wiener space. Suppose that $\{X(t), t \in [0,1]^d\}$ is a random field. A random field $\{X^*(t), t \in [0,1]^d\}$ is called its *(k,p)-continuous modification* (or respectively, *∞-continuous modification*) if

1° $\forall t \in [0,1]^d$, $X^*(t)$ is the (k,p)-quasi-continuous modification (respectively, quasi-continuous modification) of $X(t)$;

2° for (k,p)-q.e.ω(respectively, for q.e.ω), $X^*(\cdot,\omega)$ is continuous.

We have the following extended Kolmogorov's criterion:

Theorem 2.24 (Ren[1]) *Let $k \in \mathbb{N}_0, 1 < p < \infty$. If there exist $\alpha > 0, c > 0$ and a positive integer β such that*

1° $\forall t \in [0,1]^d$, $X(t) \in \mathbb{D}^p_k$;

2° $\forall s,t \in [0,1]^d, (X(t) - X(s))^\beta \in \mathbb{D}^p_k$ *and*

$$\|(X(t) - X(s))^\beta\|_{k,p} \le c|t - s|^{d+\alpha}, \tag{2.29}$$

where $|t - s| \equiv \sum_{j=1}^d |t_j - s_j|$, then there exists a (k,p) continuous modification of X.

Proof. We may assume that $\forall t$, $X(t)$ itself is (k,p)-quasi- continuous (otherwise we take its (k,p)-quasi-continuous modification). Choose ν and δ such that $0 < \nu < \alpha/\beta$, $(1 - \delta)(\alpha - d - \beta\nu) > (1 + \delta)d$. Denote

$$T_n \equiv \{(i_1 2^{-n}, \cdots, i_d 2^{-n}) : 0 \le i_1, \cdots, i_d \le 2^n\},$$

$$T_n(\delta) \equiv \{(s,t) \in T_n \times T_n : |t - s| < 2^{-n(1-\delta)}\},$$

$T \equiv \cup_n T_n$, $l \equiv (1 - \delta)(\alpha + d - \beta\nu) - (1 + \delta)d > 0$. By Proposition 2.10 we obtain

$$\begin{aligned} &C_{k,p}\Big(\bigcup_{(s,t)\in T_n(\delta)} \{|X(t) - X(s)| > |t - s|^\nu\}\Big) \\ &\le c \sum_{(s,t)\in T_n(\delta)} |t - s|^{\alpha+d-\beta\nu} \le c2^{-nl}. \end{aligned}$$

Since $\sum_n 2^{-nl} < \infty$, by (2.7), for (k,p)-q.e.ω, $\exists n_0 = n_0(\omega)$ for $n \ge n_0$, $(s,t) \in T_n(\delta)$ we have

$$|X(t) - X(s)| \le |t - s|^\nu.$$

For any $s,t \in T$ with $|t - s| < 2^{-n_0(1-\delta)}$, we can choose $n \ge n_0$ such that $n/(n+1) > 1 - \delta$ and

$$2^{-(n+1)(1-\delta)} \le |t - s| < 2^{-n(1-\delta)}.$$

Proceeding one by one for every coordinate, we obtain

$$|X(t) - X(s)| \le c|t-s|^{\gamma}.$$

It means that $X(\cdot,\omega)$ is uniformly continuous on T except an ω set of (k,p)-capacity zero, hence it can be extended to a continuous function on $[0,1]^d$ which is denoted by $X^*(\cdot,\omega)$. For any $t \in [0,1]^d$ and $\{t_n\} \subset T$ such that $t_n \to t$, by (2.29) we know that $\{X(t_n)\}$ converges to $X(t)$ in $\mathbb{D}_k^p$. Hence there exists a subsequence $\{t_{n_j}\}$ such that $X^*(t_{n_j}) = X(t_{n_j}) \to X(t)$ (k,p)-q.s., therefore, $X^*(t) = X(t)$ (k,p)-q.s.. Evidently X^* is (k,p)-continuous modification of X. ∎

Corollary 2.25 *If $\forall k \in \mathbb{N}_0, p \in (1,\infty)$, $\exists \alpha = \alpha(k,p), c = c(k,p)$ and $\beta = \beta(k,p)$ satisfying conditions in Theorem 2.24, then X has an ∞-modification.*

Consider the Fisk-Stratonovich equation (1.27):

$$\begin{cases} dX_t = A_0(X_t)dt + A_i(X_t)\circ dW_t^i, \\ X_0 = x \qquad (0 \le t \le 1), \end{cases} \tag{2.30}$$

where $A_0, A_1, \cdots, A_d$ are C^∞ vector fields with bounded derivatives on $\mathbb{R}^m$. $\forall n \in \mathbb{N}$, let $t_n \equiv 2^{-n}[2^n t]$, $t_n^+ \equiv 2^{-n}([2^n t]+1)$, $0 \le t \le 1$, $\dot W_n^i(t) \equiv 2^n(W^i(t_n^+) - W^i(t_n))$, $i = 1,\cdots,d$. Consider the following sequence of ordinary differential equations approximating eq. (2.30):

$$\begin{cases} dX_n(t) = (A_0(X_n(t)) + A_i(X_n(t))\dot W_n^i(t))dt, \\ X_n(0) = x \qquad (0 \le t \le 1). \end{cases} \tag{2.31}$$

It is known that the solution $X_n(t)$ of equation (2.31) converges a.s. to the solution $X(t)$ of equation (2.30) (eg. cf. Bismut[2]). Ren[1] extended this result to quasi-sure convergence:

Theorem 2.26 *Under the above conditions, there exists a slim set S such that $\forall \omega \notin S$ it holds that*

$$\lim_{n\to\infty} \sup_{0\le t\le 1} |X_n(t,\omega) - X(t,\omega)| = 0 \tag{2.32}$$

Proof. We only give a sketch of the proof. For details refer to Ren[1,5].

Define

$$Z(s,t) = \begin{cases} X(t) & (s=0); \\ X_n(t) + (s - \frac{1}{n})(\frac{1}{n-1} - \frac{1}{n})^{-1}(X_{n-1}(t) - X_n(t)) \\ \qquad (\frac{1}{n-1} < s < \frac{1}{n}, n \in \mathbb{N}). \end{cases} \tag{2.33}$$

In view of $Z(0,t) = X(t)$, $Z(1/n,t) = X_n(t)$, we know that if random field (2.33): $\{Z(s,t),(s,t)\in[0,1]^2\}$ has ∞-modification, then the theorem follows.

By an estimate for stochastic integrals, $\forall p \ge 1$, there exists a constant c independent of n so that $\forall s,t \in [0,1]$,

$$\mathbb{E}[|X_n(t) - X_n(s)|^{2p}] \le c|t-s|^p; \tag{2.34}$$

$$\mathbb{E}[|X_n(t) - X(t)|^{2p}] \le c2^{-np}. \tag{2.35}$$

To prove that $Z(s,t)$ satisfies the conditions in Theorem 2.24, it suffices to prove that, $\forall k \in \mathbb{N}_0, p \in (1,\infty)$,

$$\sup_{0\le t\le 1}\sup_{n\in\mathbb{N}} \|X_n(t)\|_{k,p} < \infty. \tag{2.36}$$

This is done by some careful and tedious estimates for solutions of stochastic differential equations for derivatives of $X_n(t)$, using stochastic calculus of variation similar to that in the proof of Theorem 1.9. ∎

§3. Anticipating stochastic calculus

One of the notable features of the Itô's functional is that, as a stochastic process, it is adapted to the filtration generated by Brownian motion. It seems rather restrictive in applications. For example, in a terminal or boundary value problem for stochastic differential equations, we cannot expect that the solution would be an adapted process (unless augmenting the original filtration). We have noted that, Skorohod integral (or divergence operator δ) being an extension of Itô integral, it still makes sense for non-adapted processes. So the Malliavin calculus provided a powerful tool for anticipating stochastic calculus.

In this paragraph we consider Brownian motions on finite time interval. We suppose that $H = L^2([0,1];\mathbb{R}^d)$, $\Omega = C_0([0,1];\mathbb{R}^d)$, μ is the Wiener measure Ω, other notations are the same as in §1

3.1 Approximation of Skorohod integrals by Riemannian sums

Firstly we use Riemannian sums to approximate the Skorohod integrals. Let

$$\pi_n : 0 = t_0^n < t_1^n < \cdots < t_{k_n}^n = 1, \quad n \in \mathbb{N}$$

be a sequence of partitions for $[0,1]$. To simplify the notations, we shall omit the superscript n and denote

$$\Delta_j = (t_{j-1},t_j],\ |\Delta_j| \equiv t_j - t_{j-1},$$
$$W(\Delta_j) \equiv W(t_j) - W(t_{j-1})\ (1 \le j \le k_n),$$
$$|\pi_n| \equiv \max_{1\le j\le k_n} |\Delta_j|,$$
$$\mathcal{F}_{\Delta_j^c} \equiv \sigma\{W_t - W_s : (s,t] \cap \Delta_j = \emptyset\}, \quad 1 \le j \le k_n. \tag{3.1}$$

For $X \in L^2(\Omega; H) \cong L^2([0,1] \times \Omega; \mathbb{R}^d)$, define

$$\pi_n(X) \equiv \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\Big(\int_{\Delta_j} X_s ds\Big) 1_{\Delta_j}; \tag{3.2}$$

$$\tilde\pi_n(X) \equiv \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\Big(\int_{\Delta_j} \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]ds\Big) 1_{\Delta_j}. \tag{3.3}$$

Then we have the following simple proposition:

Proposition 3.1 *If $X \in L^2(\Omega; H)$, then both $\pi_n(X)$ and $\widetilde{\pi}_n(X)$ converge to X in $L^2(\Omega; H)$ as $|\pi_n| \to 0$; if $X \in \mathbb{D}_1^2(H)$, then the convergences are also in $\mathbb{D}_1^2(H)$.*

Proof. It is easy to see that the maps $X \longmapsto \pi_n(X)$ and $X \longmapsto \widetilde{\pi}_n(X)$ are linear operators in $L^2(\Omega; H)$. Let $\mathcal{B}_n$ be the finite σ-algebra in $[0,1]$ generated by the partition π_n, λ be the Lebesgue measure on $[0,1]$. Then $\lambda \times \mu$ is a probability measure on $[0,1] \times \Omega$. Moreover,

$$\pi_n(X) = \mathbb{E}_{\lambda\times\mu}[X|\mathcal{B}_n \times \mathcal{F}], \quad n \in \mathbb{N} \tag{3.4}$$

is an L^2 bounded martingale. By martingale convergence theorem, it converges to X in $L^2(\Omega; H)$. Similarly, let $\mathcal{G}_n$ be the σ-algebra in $[0,1] \times \Omega$ generated by $\{\Delta_j \times A_j : A_j \in \mathcal{F}_{\Delta_j^c}; 1 \le j \le k_n\}$. Then

$$\widetilde{\pi}_n(X) = \mathbb{E}_{\lambda\times\mu}[X|\mathcal{G}_n], \quad n \in \mathbb{N} \tag{3.5}$$

is again an L^2 bounded martingale converging to X in $L^2(\Omega; H)$.

If $X \in \mathbb{D}_1^2(H)$, for $r \in [0,1]$, we have

$$D_r\pi_n(X)_t = \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} D_rX_s ds\right)1_{\Delta_j}(t), \tag{3.6}$$

$$D_r\widetilde{\pi}_n(X)_t = \sum_j \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} \mathbb{E}[D_rX_s|\mathcal{F}_{\Delta_j^c}]ds\right)1_{\Delta_j^c}(r)1_{\Delta_j}(t) \tag{3.7}$$

(cf. Lemma 1.3 for the proof of the last equation). By a similar discussion we know that both $D\pi_n(X)$ and $D\widetilde{\pi}_n(X)$ converge to DX in $L^2(\Omega; H \otimes H)$, therefore, both $\pi_n(X)$ and $\widetilde{\pi}_n(X)$ converge to X in $\mathbb{D}_1^2(H)$. ∎

To simplify notations, in the sequel we assume that $d = 1$.

Proposition 3.2 *If $X \in L^2(\Omega; H)$, then $\widetilde{\pi}_n(X) \in \mathcal{D}(\delta)$ and*

$$\delta\widetilde{\pi}_n(X) = \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]ds\right) \cdot W(\Delta_j). \tag{3.8}$$

If $\delta\widetilde{\pi}_n(X)$ converges in $L^2(\Omega)$ when $|\pi_n| \to 0$, then $X \in \mathcal{D}(\delta)$ and the limit is δX.

Proof. Firstly we assume that $X \in \mathbb{D}_1^2(H)$. By eqs. (II.3.48) and (3.7) we have

$$\begin{aligned}\delta(\mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}]1_{\Delta_j}) &= \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}] \cdot W(\Delta_j) \\ &\quad - \int_0^1 \mathbb{E}[D_rX_s|\mathcal{F}_{\Delta_j^c}]1_{\Delta_j^c}(r)1_{\Delta_j}(r)dr \\ &= \mathbb{E}[X_s|\mathcal{F}_{\Delta_j^c}] \cdot W(\Delta_j).\end{aligned}$$

Since $\mathbb{D}_1^2(H)$ is dense in $L^2(\Omega; H)$, the above equality also holds for $X \in L^2(\Omega; H)$. Thus we obtain eq. (3.8). The last assertion follows from Proposition 3.1 and the closedness of operator δ. ∎

Proposition 3.3 If $X \in \mathbb{D}_1^2(H)$, then $\pi_n(X) \in \mathbb{D}_1^2(H)$ and

$$\begin{aligned}\delta\pi_n(X) = &\sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} X_s ds\right) \cdot W(\Delta_j) \\ &- \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\int_{\Delta_j}\int_{\Delta_j} D_rX_s drds.\end{aligned} \tag{3.9}$$

As $|\pi_n| \to 0$, both $\delta\pi_n(X)$ and $\delta\widetilde{\pi}_n(X)$ converge to δX in $L^2(\Omega)$.

Proof. By eq. (II.3.48), for $1 \le j \le k_n$ we have

$$\begin{aligned}\delta(X_s1_{\Delta_j}) &= X_s \cdot W(\Delta_j) - \int_0^1 D_rX_s1_{\Delta_j}(r)dr \\ &= X_s \cdot W(\Delta_j) - \int_{\Delta_j} D_rX_s dr,\end{aligned}$$

hence eq. (3.9) holds. The last assertion follows from Proposition 3.1 and the fact that δ is continuous from $\mathbb{D}_1^2(H)$ to $L^2(\Omega)$. ∎

Now we consider the Stratonovich integral.

Definition 3.4 Let $X = \{X_t, 0 \le t \le 1\}$ be a measurable process. If $\int_0^1 |X_t|dt < \infty$ a.s. and

$$S_n \equiv \sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\left(\int_{\Delta_j} X_s ds\right) \cdot W(\Delta_j) \tag{3.10}$$

converges in probability when $|\pi_n| \to 0$, then X is said to be *Stratonovich integrable*, this limit is called *Stratonovich integral* and denoted by $\int_0^1 X_t \circ dW_t$.

From eq. (3.9) we know that convergence of S_n is a consequence of convergence of the second sum in eq. (3.9) which involves regularity of D_rX_s along each side of the diagonal segment $r = s$. Here we give a sufficient condition:

If $X \in \mathbb{D}_1^2(H)$ and DX has a modification such that maps $t \longmapsto D_{s\wedge t}X_{s\vee t}$ and $t \longmapsto D_{s\vee t}X_{s\wedge t}$ from $[0,1]$ into $L^2(\Omega)$ are continuous uniformly with respect to $s \in [0,1]$ and that

$$\operatorname*{ess\,sup}_{0\le t,s\le 1} \mathbb{E}[|D_tX_s|^2] < \infty, \tag{3.11}$$

then we denote $X \in \widetilde{\mathbb{D}}_1^2(H)$. In particular, if D_tX_s has continuous modification, then $X \in \widetilde{\mathbb{D}}_1^2(H)$.

If $X \in \widetilde{\mathbb{D}}_1^2(H)$, then the following limits exist:

$$D_t^+X_t \equiv \lim_{s\downarrow 0} D_tX_{t\ldots}$$

$$D_t^- X_t \equiv \lim_{\epsilon \downarrow 0} D_t X_{t-\epsilon},$$

where convergence are in $L^2(\Omega)$ and uniform with respect to t. Denote $\nabla \equiv D^+ - D^-$. We have the following theorem.

Theorem 3.5 *If $X \in \widetilde{D}_1^2(H)$, then X is Stratonovich integrable and it holds that*

$$\int_0^1 X_t \circ dW_t = \int_0^1 X_t dW_t + \frac{1}{2}\int_0^1 (\nabla X)_t dt. \tag{3.12}$$

Proof. In view of Proposition 3.3, it suffices to prove that

$$\sum_{j=1}^{k_n} \frac{1}{|\Delta_j|}\int_{\Delta_j}\int_{\Delta_j} D_t X_s dt ds \longrightarrow \frac{1}{2}\int_0^1 (\nabla X)_t dt \tag{3.13}$$

as $|\pi_n| \to 0$. Note that

$$\begin{aligned}
&E\left|\sum_j \frac{1}{|\Delta_j|}\int_{\Delta_j} dt \int_t^{t_j} D_t X_s ds - \frac{1}{2}\int_0^1 D_t^+ X_t dt\right| \\
&\le E\left|\sum_j \frac{1}{|\Delta_j|}\int_{\Delta_j} dt \int_t^{t_j} (D_t X_s - D_t^+ X_t) ds\right| \\
&\qquad + E\left|\sum_j \int_{\Delta_j} \frac{t_j - t}{|\Delta_j|} D_t^+ X_t dt - \frac{1}{2}\int_0^1 D_t^+ X_t dt\right| \\
&\le \sup_{\substack{0 \le t < s \le 1 \\ s - t \le |\pi_n|}} E[|D_t X_s - D_t^+ X_t|] \\
&\qquad + E\left|\int_0^1 D_t^+ X_t \left(\sum_j \frac{t_j - t}{|\Delta_j|} 1_{\Delta_j}(t) - \frac{1}{2}\right) dt\right|.
\end{aligned}$$

As $|\pi_n| \to 0$, $\sum_j |\Delta_j|^{-1}(t_j - t) 1_{\Delta_j}(t)$ converges weakly in $L^2([0,1])$ to $1/2$. By definition of $\widetilde{D}_1^2(H)$ we know that both terms on left-hand side converge to 0. A similar procedure shows that

$$E\left|\sum_j \frac{1}{|\Delta_j|}\int_{\Delta_j} dt \int_{t_{j-1}}^t D_t X_s ds - \frac{1}{2}\int_0^1 D_t^- X_t dt\right| \longrightarrow 0,$$

hence (3.13) is true. ∎

Remark. If X is a progressive process, then $D_t^- X_t = 0, (\nabla X)_t = D_t^+ X_t$. If X is a continuous semi-martingale, then by eq. (3.12) the cross quadratic variation of X and W is

$$[X, W]_1 = \int_0^1 D_t^+ X_t dt. \tag{3.14}$$

3.2 Itô formula for anticipating processes

Let $V \in D_1^2(H)$. For $t \in [0,1]$, define

$$X_t = \delta(V 1_{[0,t]}) = \int_0^t V_s dW_s, \tag{3.15}$$

which is called *Skorohod indefinite integral*. Since V may not be adapted, X may not be a martingale and may have no continuous modification. A sufficient condition for existence of continuous modifications is the following

Proposition 3.6 *If $p > 2$, $V \in L^{2p}([0,1]; D_1^{2p})$, then $X_t = \int_0^t V_r dW_r$ has a continuous modification.*

Proof. $\delta : D_1^{2p}(H) \longrightarrow L^{2p}$ being continuous, $\exists C_p > 0$ such that

$$\|\delta V\|_{2p}^{2p} \le C_p(\|V\|_{2p}^{2p} + \|DV\|_{2p}^{2p}).$$

Replacing V by $V 1_{(s,t]}$ in above expression, by Hölder inequality we have

$$\begin{aligned}
E\left[\left|\int_s^t V_r dW_r\right|^{2p}\right] &\le C_p(t-s)^{p-1}\left\{E\left[\int_0^1 V_r^{2p} dr\right]\right. \\
&\qquad \left. + E\left[\int_0^1 \left(\int_0^1 (D_u V_r)^2 du\right)^p dr\right]\right\} \\
&= C_p(t-s)^{p-1}\int_0^1 \|V_r\|_{1,2p}^{2p} dr.
\end{aligned}$$

According to Kolmogorov's criterion, X has a continuous modification. ∎

Let $\{\pi_n\}$ be a sequence of partitions for $[0,1]$ and define

$$Q^{\pi_n}(V) = \sum_{j=1}^{k_n}\left(\int_{\Delta_j} V_s dW_s\right)^2 \tag{3.16}$$

to be the quadratic variation of X. We have the following convergence theorem:

Theorem 3.7 *If $V \in D_1^2(H)$, then $Q^{\pi_n}(V)$ converges in $L^1(\Omega)$ to $\int_0^1 V_s^2 ds$ as $|\pi_n| \longrightarrow 0$.*

Proof. If $U, V \in D_1^2(H)$, then $\exists C > 0$ such that

$$\begin{aligned}
&E[|Q^{\pi_n}(U) - Q^{\pi_n}(V)|] \\
&\le \left(E\left[\sum_j\left(\int_{\Delta_j}(U_s - V_s)dW_s\right)^2\right]\right)^{1/2} \\
&\quad \times \left(E\left[\sum_j\left(\int_{\Delta_j}(U_s + V_s)dW_s\right)^2\right]\right)^{1/2} \\
&\le C\|U - V\|_{1,2}\|U + V\|_{1,2}.
\end{aligned}$$

By density argument we may assume that V has the form

$$V = \sum_{i=1}^{m} \varphi_i \mathbf{1}_{(s_{i-1}, s_i]},$$

where $0 = s_0 < s_1 < \cdots < s_m = 1, \varphi_i \in \mathbb{D}_1^2$ and $\{s_i\}$ are division points of π_n. Therefore

$$\delta V = \sum_{i=1}^{m} \left\{ \varphi_i (W(s_i) - W(s_{i-1})) - \int_{s_{i-1}}^{s_i} D_s \varphi_i ds \right\},$$

$$Q^{\pi_n}(V) = \sum_{i=1}^{m} \sum_{j: s_{i-1} < t_j \le s_i} \left(\varphi_i W(\Delta_j) - \int_{\Delta_j} D_s \varphi_i ds \right)^2$$

$$= \sum_{i=1}^{m} \sum_{j} \left[\varphi_i^2 W(\Delta_j)^2 - 2\varphi_i W(\Delta_j) \int_{\Delta_j} D_s \varphi_i ds + \left(\int_{\Delta_j} D_s \varphi_i ds \right)^2 \right].$$

It follows from properties of quadratic variation of Brownian motion that, as $|\pi_n| \to 0$, the above expression converges in $L^1(\Omega)$ to $\sum_{i=1}^{m} \varphi_i^2 (s_i - s_{i-1}) = \int_0^1 V_s^2 ds$ ∎

Consequently, if $X_t = \int_0^t V_s dW_s$ has continuous modification of bounded variation, then $V = 0$.

Both Malliavin derivatives and Skorohod integrals have the following localizations.

Proposition 3.8 *Let $A \in \mathcal{F}, F \in \mathbb{D}_1^1$. Then*

$$F = 0 \text{ a.s. on } A \Longrightarrow D_t F = 0 \text{ a.e. on } [0,1] \times A.$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}), \varphi \ge 0, \varphi(0) = 1$ and $\mathrm{supp}(\varphi) \subset [-1,1]$. Denote $\varphi_\epsilon(t) = \varphi(t/\epsilon), \epsilon > 0$. Then $\mathrm{supp}(\varphi_\epsilon) \subset [-\epsilon, \epsilon]$. Define $\psi_\epsilon(t) = \int_{-\infty}^{t} \varphi_\epsilon(s) ds$. Then we have $D\psi_\epsilon(F) = \varphi_\epsilon(F) DF$. $\forall h \in H$,

$$|\mathbb{E}[\varphi_\epsilon(F) D_h F]| = |\mathbb{E}[D_h(\psi_\epsilon(F))]|$$

$$= \mathbb{E}[\psi_\epsilon(F) \delta h]| \le \epsilon \|\varphi\|_\infty \mathbb{E}[|\delta h|].$$

Letting $\epsilon \downarrow 0$, we have

$$\mathbb{E}\left[\mathbf{1}_{\{F=0\}} \int_0^1 (D_t F) \dot{h}(t) dt \right] = 0. \qquad ∎$$

Proposition 3.9 *Let $A \in \mathcal{F}, V \in \mathbb{D}_1^2(H)$. Then*

$$V_t = 0 \text{ a.e. on } [0,1] \times A \Longrightarrow \delta V = 0 \text{ a.s. on } A.$$

Proof. In view of Proposition 3.3, by choosing a.s. convergence subsequences in eq. (3.9) and using Proposition 3.8, the assertion is readily proved. ∎

By localization, we can extend domains of operators D and δ. Suppose that $k \in \mathbb{N}_0, p \ge 1, F$ is an E-valued functional. If there exist a sequence of sets $\{A_n\} \subset \mathcal{F}$ and a sequence of functionals $\{F_n\} \subset \mathbb{D}_k^p(E)$ having following properties:

1° $A_n \uparrow \Omega$ a.s.;

2° $\forall n \in \mathbb{N}, F = F_n$ a.s. on A_n,

then F is called a *local $\mathbb{D}_k^p(E)$ functional* and denoted by $F \in loc\mathbb{D}_k^p(E)$. Obviously, if $F \in loc\mathbb{D}_1^p$, we can define $DF = DF_n$ on each set A_n without ambiguity; and if $V \in loc\mathbb{D}_1^2(H)$, we can define $\delta V = \delta V_n$ on A_n. By localization, we can extend many propositions to the local $\mathbb{D}_k^p(E)$ spaces.

The following is the main theorem of this paragraph:

Theorem 3.10 (Itô formula) *Let $V \in locL^4([0,1]; \mathbb{D}_2^4)$, $U \in locL^4([0,1]; \mathbb{D}_1^4), f \in C^2(\mathbb{R})$. If*

$$X_t = \int_0^t V_s dW_s + \int_0^t U_s ds \tag{3.17}$$

has continuous modification, then

$$f(X_t) = f(0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s)(\nabla X)_s V_s ds, \tag{3.18}$$

where $\nabla = D^+ + D^-$.

Remark. It follows from Lemma 1.4 that

$$D_s X_t = V_s 1_{[0,t]}(s) + \int_0^t D_s V_r dW_r + \int_0^t D_s U_r dr. \tag{3.19}$$

hence

$$(\nabla X)_s = V_s + 2\int_0^s D_s V_r dW_r + 2\int_0^s D_s U_r dr. \tag{3.20}$$

In particular, if U and V are adapted processes, then the last two terms vanish and eq. (3.18) is reduced to usual Itô formula.

Proof. By localization, we may assume that f, f' and f'' are bounded and that $V \in L^4([0,1]; \mathbb{D}_2^4), U \in L^4([0,1]; \mathbb{D}_1^4)$. Let $\{\pi_n\}$ be a sequence of partitions for $[0,t]$. By Taylor expansion we have

$$f(X_t) = f(0) + \sum_{j=1}^{k_n} f'(X(t_{j-1}))(X(t_j) - X(t_{j-1})) + \frac{1}{2} \sum_{j=1}^{k_n} f''(\bar{X}_j)(X(t_j) - X(t_{j-1}))^2. \tag{3.21}$$

where $\tilde{X}_j$ is some random variable between $X(t_j)$ and $X(t_{j-1})$. Since for $j = 1, \cdots, k_n$,

$$X(t_j) - X(t_{j-1}) = \int_{\Delta_j} V_s dW_s + \int_{\Delta_j} U_s ds,$$

it follows from eq. (II.3.48) that

$$f'(X(t_{j-1})) \int_{\Delta_j} V_s dW_s = \int_{\Delta_j} f'(X(t_{j-1}))V_s dW_s + \int_{\Delta_j} V_s D_s f'(X(t_{j-1}))ds.$$

By eq. (3.19) it holds that

$$\begin{aligned} D_s f'(X_{t_{j-1}}) &= f''(X_{t_{j-1}}) D_s X_{t_{j-1}} \\ &= f''(X_{t_{j-1}}) \Big[V_s 1_{[0,t_{j-1}]}(s) + \int_0^{t_{j-1}} D_s V_r dW_r + \int_0^{t_{j-1}} D_s U_r dr \Big]. \end{aligned}$$

The first term on the right hand side vanishes when $s \in \Delta_j$. Therefore, it suffices to prove that, as $|\pi_n| \to 0$, the following sums converge in probability:

$$\sum_j f'(X(t_{j-1})) \int_{\Delta_j} U_s ds \longrightarrow \int_0^t f'(X_s) U_s ds, \tag{3.22}$$

$$\sum_j \int_{\Delta_j} f'(X(t_{j-1}))V_s dW_s \longrightarrow \int_0^t f'(X_s) V_s dW_s, \tag{3.23}$$

$$\sum_j \int_{\Delta_j} f''(X(t_{j-1}))V_s \Big(\int_0^{t_{j-1}} D_s U_r dr \Big) ds \longrightarrow \int_0^t f''(X_s) V_s \Big(\int_0^s D_s U_r dr \Big) ds, \tag{3.24}$$

$$\sum_j \int_{\Delta_j} f''(X(t_{j-1}))V_s \Big(\int_0^{t_{j-1}} D_s V_r dW_r \Big) ds \longrightarrow \int_0^t f''(X_s) V_s \Big(\int_0^s D_s V_r dW_r \Big) ds, \tag{3.25}$$

$$\sum_j f''(\tilde{X}_j)(X(t_j) - X(t_{j-1}))^2 \longrightarrow \int_0^t f''(X_s) V_s^2 ds. \tag{3.26}$$

The Stieltjes sum in (3.22) is obviously a.s. convergent. For (3.26) we note that

$$(X(t_j) - X(t_{j-1}))^2 = \Big(\int_{\Delta_j} V_s dW_s \Big)^2 + \Big(\int_{\Delta_j} U_s ds \Big)^2 + 2 \int_{\Delta_j} V_s dW_s \int_{\Delta_j} U_s ds.$$

Since f'' is bounded and continuous and $\int_0^t U_s ds$ has bounded variation, the sums over j of the last two terms converge to 0. On the other hand, it follows from Theorem 3.7 that

$$\sum_j f''(\tilde{X}_j) \Big(\int_{\Delta_j} V_s dW_s \Big)^2 \longrightarrow \int_0^t f''(X_s) V_s^2 ds,$$

hence (3.26) is proved. The proofs of (3.24) and (3.25) are similar. For instance, for (3.25) we have the following estimate:

$$\begin{aligned} &\Big| \sum_j \int_{\Delta_j} \Big[f''(X_{t_{j-1}}) \int_0^{t_{j-1}} D_s V_r dW_r - f''(X_s) \int_0^s D_s V_r dW_r \Big] V_s ds \Big| \\ &\le \Big| \sum_j \int_{\Delta_j} f''(X_{t_{j-1}}) \Big(\int_{t_{j-1}}^s D_s V_r dW_r \Big) V_s ds \Big| \\ &\quad + \Big| \sum_j \int_{\Delta_j} [f''(X_{t_{j-1}}) - f''(X_s)] \Big(\int_0^s D_s V_r dW_r \Big) V_s ds \Big| \\ &\le \|f''\|_\infty \sum_j \int_{\Delta_j} \Big| \int_{t_{j-1}}^s D_s V_r dW_r \Big| |V_s| ds \\ &\quad + \sup_j \sup_{s \in \Delta_j} |f''(X_{t_{j-1}}) - f''(X_s)| \int_0^t \Big| V_s \int_0^s D_s V_r dW_r \Big| ds. \end{aligned}$$

Since $V \in L^4([0,1]; \mathbb{D}_2^4) \subset L^2([0,1]; \mathbb{D}_2^2)$ and f'' is continuous, the second term on the right-hand side converges a.s. to 0. By Theorem 1.8 and Cauchy-Schwarz inequality, the expectation of the first term is less than

$$\|f''\|_\infty \Big(\mathbb{E}\Big[\int_0^1 V_s^2 ds \Big] \Big)^{1/2} \Big(\mathbb{E}\Big[\sum_j \int_{\Delta_j} \int_{t_{j-1}}^s |D_s V_r|^2 dr ds \Big] + \mathbb{E}\Big[\sum_j \int_{\Delta_j} \int_{t_{j-1}}^s \int_0^1 |D_u D_s V_r|^2 du dr ds \Big] \Big)^{1/2},$$

hence converges to 0. It remains to prove (3.23). We shall prove that it converges in $L^2(\Omega)$. Since the Skorohod integral operator $\delta : \mathbb{D}_1^2(H) \longrightarrow L^2(\Omega)$ is continuous, it suffices to prove that

$$V_s^{(n)} \equiv V_s \sum_{j=1}^{k_n} f'(X(t_{j-1})) 1_{\Delta_j}(s), \quad 0 \le s \le 1$$

converges in $\mathbb{D}_1^2(H)$ to $V_s f'(X_s)$. Obviously, $V^{(n)}$ converges in $L^2([0,1] \times \Omega)$ to $V f'(X)$. It remains to prove that $DV^{(n)}$ converges in $L^2([0,1]^2 \times \Omega)$ to

$D[Vf'(X)]$. However,

$$D_r V_s^{(n)} = (D_r V_s)\sum_j f'(X(t_{j-1}))1_{\Delta_j}(s) + V_s \sum_j f''(X(t_{j-1}))(D_r X(t_{j-1}))1_{\Delta_j}(s).$$

Since $V \in \mathbb{D}_1^2(H)$, the first sum obviously converges to $D_r V_s f'(X_s)$. To prove that the second sum converges to $V_s f''(X_s) D_r X_s$, we use eq. (3.19) to decompose $D_r X(t_{j-1})$ into three terms, apparently we have

$$V_s \sum_j f''(X(t_{j-1})) V_r 1_{[0,t_{j-1}]}(r) 1_{\Delta_j}(s) \longrightarrow V_s f''(X_s) V_r 1_{[0,s]}(r)$$

and

$$V_s \sum_j f''(X(t_{j-1}))\left(\int_0^s D_r U_u du\right) 1_{\Delta_j}(s) \longrightarrow V_s f''(X_s)\int_0^s D_r U_u du.$$

However,

$$\begin{aligned} &E\left[\sum_j \int_{\Delta_j}\int_0^1 V_s^2 f''(X(t_{j-1}))^2 \left(\int_{t_{j-1}}^s D_r U_u du\right)^2 dr ds\right] \\ &\le \|f''\|_\infty^2 \left(E\int_0^1 V_s^4 ds\right)^{1/2}\left(E\int_0^1 \left[\int_0^1 (D_r U_u)^2 dr\right]^2 du\right)^{1/2} |\pi_n| \\ &\longrightarrow 0. \end{aligned}$$

It follows that

$$V_s \sum_j f''(X(t_{j-1}))\left(\int_0^{t_{j-1}} D_r U_u du\right) 1_{\Delta_j}(s) \to V_s f''(X_s)\int_0^s D_r U_u du$$

in $L^2([0,1]^2 \times \Omega)$. Similarly we have

$$V_s \sum_j f''(X(t_{j-1}))\left(\int_0^{t_{j-1}} D_r V_u dW_u\right) 1_{\Delta_j}(s) \longrightarrow V_s f''(X_s)\int_0^s D_r V_u dW_u.$$

The proof is complete. ∎

For Stratonovich integrals, we also have a similar formula. If $V \in L^4([0,1];\mathbb{D}_2^4)$ and DV has a modification such that the maps $t \longmapsto D_{s\wedge t} V_{s\vee t}$ and $t \longmapsto$

$D_{s\vee t} V_{s\wedge t}$ are continuous from $[0,1]$ into $L^4(\Omega)$ uniformly with respect to $s \in [0,1]$, and if

$$\operatorname{esssup}_{0\le t,s\le 1} E[|D_t V_s|^4] < \infty,$$

then we denote $V \in \widehat{L}^4([0,1];\mathbb{D}_2^4)$.

Theorem 3.11 *Let $f \in C^2(\mathbb{R})$, $V \in loc\widehat{L}^4([0,1];\mathbb{D}_2^4)$, $DV \in locL^4([0,1]^2;\mathbb{D}_1^4)$, $U \in locL^4([0,1];\mathbb{D}_1^4)$. If*

$$X_t = \int_0^t V_s \circ dW_s + \int_0^t U_s ds \tag{3.27}$$

has continuous modification, then

$$f(X_t) = f(0) + \int_0^t f'(X_s) U_s ds + \int_0^t f'(X_s) V_s \circ dW_s. \tag{3.28}$$

Proof. By eq. (3.12) we have

$$X_t = \int_0^t V_s dW_s + \int_0^t U_s ds + \frac{1}{2}\int_0^t (\nabla V)_s ds.$$

According to Itô formula (3.18),

$$\begin{aligned} f(X_t) = f(0) &+ \int_0^t f'(X_s) U_s ds + \frac{1}{2}\int_0^t f'(X_s)(\nabla V)_s ds \\ &+ \int_0^t f'(X_s) V_s dW_s + \frac{1}{2}\int_0^t f''(X_s)(\nabla X)_s V_s ds. \end{aligned} \tag{3.29}$$

Again by eq. (3.12) we obtain

$$\int_0^t f'(X_s) V_s \circ dW_s = \int_0^t f'(X_s) V_s dW_s + \frac{1}{2}\int_0^t \nabla(f'(X_s) V_s) ds.$$

Replacing

$$\nabla(f'(X_s)V_s) = f'(X_s)(\nabla V)_s + f''(X_s) V_s (\nabla X)_s$$

into eq. (3.29) we obtain eq. (3.28). ∎

3.3 Anticipating stochastic differential equations

Consider the following stochastic differential equation:

$$X_t = \eta + \int_0^1 b(t,s,X_s) ds + \int_0^1 \sigma(t,s,X_s) dW_s, \quad 0 \le t \le 1,$$

where η is a random variable, $b(t,\cdot,x)$ and $\sigma(t,\cdot,x)$ are stochastic processes depending on parameters t, x (not necessarily adapted), the stochastic integral is in the sense of Skorohod. From Theorem 1.8 we know that the L^2 estimation

of solution involves that of its derivatives, and the latter involves that of second order derivatives. Therefore, the usual Picard's iterative procedure is never closed. Herein lies the main difficulty of this problem. For this kind of equations there have been many special methods. In some special cases the existence and uniqueness, even explicit expressions of solutions have been obtained. However, as a whole, the theory is far from being complete. We refer the reader to the survey paper of Pardoux[1] for present situation of its development. In this section we only introduce the method of Girsanov transformation developed by Buckdahn[1,2].

In classical Wiener spaces, Girsanov[1] extended Cameron-Martin's result of quasi-invariance of Wiener measure under translations (Theorem II.2.5) to the case of "random translations", but the stochastic processes involved are adapted. Ramer[1] and Kusuoka[1] extended the Girsanov theorem to non-adapted case in abstract Wiener spaces. The anticipating Girsanov transformation has been investigated by many authors recently. For example, see Buckdahn[1], Enchev - Stroock[1], Ustunel - Zakai[3,5], Y. Zhang[1]. Here we briefly introduce Kusuoka's theorem.

Definition 3.12 Let (H, X, μ) be an abstract Wiener space, $J : H \longrightarrow X$ be embedding map, V an H-valued functional on X. If for μ-a.e. $x \in X$, the map $h \longmapsto V(x + Jh)$ is continuously differentiable in H, that is, there exists a Hilbert-Schmidt operator $DV(x) : H \longrightarrow H$ so that

1° $\|V(x + Jh) - V(x) - DV(x)h\| = o(\|h\|)$;

2° $DV(x + J\cdot) : H \longrightarrow \mathcal{L}_{(2)}(H)$ is continuous,

then V is said to be H *continuously differentiable* and denoted by $V \in C^1_H(H)$.

It is proved that $C^1_H(H) \subset loc\mathbb{D}^2_1(H)$ and DV coincides with Malliavin's derivative (noting that $\mathcal{L}_{(2)}(H) \cong H \otimes H$). For example, see Kusuoka[1]. We give a lemma in finite dimensional case:

Lemma 3.13 *Let $(\mathbb{R}^n, \mathcal{B}^n, \gamma^n; \mathbb{R}^n)$ be finite dimensional Gaussian probability space, $\psi \in S_M(\mathbb{R}^n)$, $D\psi = \{\partial_i\psi_j\}_{1\le i,j\le n}$ be Jacobian matrix. Define*

$$\eta(x) = |\det(I + D\psi(x))| \exp\{-(\psi(x), x) - \tfrac{1}{2}|\psi(x)|^2\}. \tag{3.30}$$

If the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $Tx = x + \psi(x)$ is injective, then $\gamma^n \circ T^{-1} \ll \gamma^n$; if T is bijective, then there exists a probability measure ν on $\mathcal{B}^n$ such that $\nu \sim \gamma^n$, $\nu \circ T^{-1} = \gamma^n$ and that $d\nu/d\gamma^n = \eta$.

Proof. For any non-negative bounded measurable function f on $\mathbb{R}^n$ we have

$$\begin{aligned}&\int_{\mathbb{R}^n} f(Tx)\eta(x)\gamma^n(dx)\\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(Tx)\exp\{-\tfrac{1}{2}|Tx|^2\}|\det(I + D\psi(x))|dx\\ &= \int_{T(\mathbb{R}^n)} f(y)\gamma^n(dy) \le \int_{\mathbb{R}^n} f(y)\gamma^n(dy).\end{aligned} \tag{3.31}$$

Hence $\gamma^n \circ T^{-1} \ll \gamma^n$. If T is bijective, then the last inequality in (3.31) becomes equality. Letting $\nu(dx) = \eta(x)\gamma^n(dx)$ we obtain $\nu \circ T^{-1} = \gamma^n$. ∎

To extend this result to infinite dimensional spaces, we need the *Carleman-Fredholm determinant*. Let $S \in \mathcal{L}_{(2)}(H)$ with eigenvalues $\{\lambda_j\}$. The determinant $\det(I + S) = \prod_j(1 + \lambda_j)$ makes sense only for $S \in \mathcal{L}_{(1)}(H)$. We define

$$\det{}_2(I + S) = \prod_j (1 + \lambda_j)e^{-\lambda_j}. \tag{3.32}$$

It is called the Carleman-Fredholm determinant which still makes sense for $S \in \mathcal{L}_{(2)}(H)$. Moreover, the map $S \longmapsto \det_2(I + S)$ is uniformly continuous on any bounded set in $\mathcal{L}_{(2)}(H)$ (eg. cf. B. Simon[2]). Obviously, if $S \in \mathcal{L}_{(1)}(H)$, then

$$\det{}_2(I + S) = \det(I + S)e^{-S}. \tag{3.33}$$

By formula (II.2.4), $\delta\psi(x) = (\psi(x), x) - \mathrm{Tr}(D\psi(x))$, we can rewrite eq. (3.30) as

$$\eta(x) = |\det{}_2(I + D\psi(x))| \exp\{-\delta\psi(x) - \tfrac{1}{2}|\psi(x)|^2\}. \tag{3.34}$$

Using finite dimensional approximation, we can prove the following theorem.

Theorem 3.14 (Kusuoka[1]) *Let (H, X, μ) be an abstract Wiener space, $V \in C^1_H(H)$, $Tx = x + JV(x)$. If $T : X \longrightarrow X$ is bijective and for μ-a.e. $x \in X$ the map $I + DV(x) :\ H \to H$ is invertible (i.e. $\det_2(I + DV(x)) \neq 0$), then there exists a probability measure ν on $\mathcal{B}(X)$ which is equivalent to μ such that $\nu \circ T^{-1} = \mu$ and*

$$\frac{d\nu}{d\mu} = |\det{}_2(I + DV)| \exp\{-\delta V - \tfrac{1}{2}\|V\|^2\}. \tag{3.35}$$

In particular, in the classical Wiener space $\Omega = C_0([0, 1])$, the transformation $T : \Omega \longrightarrow \Omega$ takes the following form:

$$T\omega = \omega + \int_0^{\cdot} V_s(\omega)ds. \tag{3.36}$$

If V is adapted, then for $t > s$, $D_tV_s = 0$ a.s.. It follows that $|\det_2(I + DV)| = 1$ a.s., hence eq. (3.35) is reduced to usual Girsanov formula:

$$\frac{d\nu}{d\mu} = \exp\Big\{ - \int_0^1 V_s dW_s - \frac{1}{2}\int_0^1 V_s^2 ds\Big\}. \tag{3.37}$$

Let us investigate the following linear anticipating stochastic differential equation:

$$X_t = X_0 + \int_0^t b_s X_s ds + \int_0^t \sigma_s X_s dW_s, \quad 0 \le t \le 1, \tag{3.38}$$

where X_0 is a bounded random variable (need not be $\mathcal{F}_0$ measurable), $b = \{b_s(\omega), 0 \le s \le 1\}$ and $\sigma = \{\sigma_s(\omega), 0 \le s \le 1\}$ are bounded stochastic processes (need not be adapted). Buckdahn[2] defined a family of transformations:

$$T_t\omega = \omega + \int_0^{t\wedge\cdot} \sigma_s(T_s\omega)ds, \quad 0 \le t \le 1, \tag{3.39}$$

and proved the following theorem:

Theorem 3.15 *If $\sigma \in L^\infty([0,1]\times\Omega)$, $D\sigma \in L^\infty([0,1]^2\times\Omega)$, then eq. (3.39) defines a unique family of invertible transformations $\{T_t, 0 \le t \le 1\}$, and $\forall t \in [0,1]$, $\mu \circ T_t^{-1} \ll \mu$, its Radon-Nikodym derivative $L_t \equiv d\mu\circ T_t^{-1}/d\mu$ satisfies the following equation:*

$$L_t = 1 + \int_0^t \sigma_s L_s dW_s, \quad 0 \le t \le 1. \tag{3.40}$$

If $b \in L^\infty([0,1]\times\Omega)$, $X_0 \in L^\infty(\Omega)$, then

$$X_t = X_0(T_t^{-1}) \exp\left\{\int_0^t b_s(T_sT_t^{-1})ds\right\}L_t \tag{3.41}$$

is the unique solution of equation (3.38) in the space $L^1([0,t]\times\Omega)$.

Proof. We only give a sketch of the proof. For details see Buckdahn[2].

Firstly we prove the theorem for σ of the following form:

$$\sigma_t = f_t(W_{t_1}, \cdots, W_{t_n}), \quad n \ge 1, t_1, \cdots, t_n \in [0,1], \tag{3.42}$$

where $f \in L^\infty([0,1]\times\mathbb{R}^n)$ such that $\forall t \in [0,1]$, $f_t \in C_b^\infty(\mathbb{R}^n)$. Note that σ_t satisfies the following condition:

$$|\sigma_t(\omega) - \sigma_t(\omega')| \le C \sup_{0\le s\le 1} |\omega(s) - \omega'(s)|, \quad \forall \omega, \omega' \in \Omega, t \in [0,1].$$

By Picard's iteration we prove that equation (3.39) has a unique Ω-valued solution $\{T_t\omega, 0 \le t \le 1\}$, its inverse $\{A_t\omega, 0 \le t \le 1\}$ satisfies the following equation:

$$A_t\omega = \omega - \int_0^{t\wedge\cdot} \sigma_s(T_sA_t\omega)ds. \tag{3.43}$$

Denote $\sigma(T) = \{\sigma_t(T_t\omega), 0 \le t \le 1\}$. By a straightforward computation of its Malliavin derivatives and Carleman-Fredholm determinant, we know that $\sigma(T) \in C_H^1(H)$ and $I + D\sigma(T)$ is invertible. By Theorem 3.14 we obtain

$$\begin{aligned}\frac{d\mu\circ A_t^{-1}}{d\mu} = \exp\bigg\{ &-\int_0^t \sigma_s(T_s)dW_s - \frac{1}{2}\int_0^t \sigma_s(T_s)^2 ds \\ &- \int_0^t\int_0^s (D_r\sigma_s)(T_s)D_s[\sigma_r(T_r)]drds\bigg\}.\end{aligned} \tag{3.44}$$

Denote the right-hand side of eq. (3.44) by η_t. Then

$$L_t = \frac{d\mu\circ T_t^{-1}}{d\mu} = \eta_t(A_t)^{-1} \tag{3.45}$$

satisfies equation (3.40). Since the class of processes having form (3.42) is dense in $\mathbb{D}_1^2(H)$, for σ satisfying conditions of theorem, there exists a sequence of processes $\{\sigma^n\}$ of form (3.42) which converges in $\mathbb{D}_1^2(H)$ to σ such that $\{\sigma^n\}$ and $\{D\sigma^n\}$ are uniformly bounded. As just proved, for every n, there exists a transformation T_t^n together with its inverse A_t^n, R-N derivative L_t^n and η_t^n satisfying eqs. (3.44), (3.45) and (3.40). It is not hard to prove that, as $n \to \infty$, $\{\sigma^n(T^n)\}$ converges in $\mathbb{D}_1^2(H)$ and $\{L_t^n : 0 \le t \le 1, n \ge 1\}$ are uniformly integrable, hence the first half of the theorem follows.

It is obvious that X defined by eq. (3.41) belongs to $L^1([0,t]\times\Omega)$. To verify that it satisfies equation (3.38), for any $G \in S_M$, we compute $\mathbb{E}[\int_0^t \sigma_s X_s D_s G ds]$. By a transformation of measure $\mu \longmapsto \mu\circ T_s^{-1} = L_s\cdot\mu$ we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \sigma_s X_s D_s G ds\right] \\ &= \mathbb{E}\left[\int_0^t \sigma_s X_0(A_s)L_s \exp\left\{\int_0^s b_r(T_rA_s)dr\right\}D_sGds\right] \\ &= \mathbb{E}\left[\int_0^t \sigma_s(T_s)X_0 \exp\left\{\int_0^s b_r(T_r)dr\right\}(D_sG)(T_s)ds\right].\end{aligned}$$

Noting that $\frac{d}{ds}G(T_s) = \sigma_s(T_s)(D_sG)(T_s)$ which follows from (3.39), using integration by parts and again a transformation of measure we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^t \sigma_s X_s D_s G ds\right] \\ &= \mathbb{E}\left[\int_0^t X_0 \exp\left\{\int_0^s b_r(T_r)dr\right\}\frac{d}{ds}G(T_s)ds\right] \\ &= \mathbb{E}\bigg[X_0 \exp\left\{\int_0^t b_s(T_s)ds\right\}G(T_t) - X_0G \\ &\quad - \int_0^t X_0 b_s(T_s)\exp\left\{\int_0^s b_r(T_r)dr\right\}G(T_s)ds\bigg] \\ &= \mathbb{E}\left[X_0(A_t)\exp\left\{\int_0^t b_s(T_sA_t)ds\right\}L_tG\right] - \mathbb{E}[X_0G] \\ &\quad - \mathbb{E}\left[\int_0^t X_0(A_s)b_s \exp\left\{\int_0^s b_r(T_rA_s)dr\right\}L_sGds\right] \\ &= \mathbb{E}\left[\left(X_t - X_0 - \int_0^t b_sX_sds\right)G\right].\end{aligned}$$

Therefore, by eq. (II.3.47) we have

$$\int_0^t \sigma_s X_s dW_s = X_t - X_0 - \int_0^t b_s X_s ds, \quad \text{a.s.}.$$

The uniqueness can be proved also by approximation and integration by parts.∎

Remark. If σ is adapted, then eq. (3.43) is reduced to

$$A_t\omega = \omega - \int_0^{t\wedge\cdot} \sigma_s(\omega)ds,$$

while eq. (3.44) has the form

$$\eta_t = \exp\left\{ \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right\}.$$

Moreover,

$$L_t = \exp\left\{ \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right\}$$

if b is also adapted, then eq. (3.41) is reduced to

$$X_t = X_0(A_t)\exp\left\{ \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t b_s ds\right\}.$$

When X_0 is $\mathcal{F}_0$ measurable, it is nothing but the solution for classical Itô stochastic differential equation.

Chapter IV

General Theory of White Noise Analysis

White noise analysis was initiated by T. Hida in 1975. This is an infinite dimensional stochastic analysis, the basic idea of which is to view Wiener functionals as functionals of white noise. More precisely, let Ω denote the space of all continuous functions f on $\mathbb{R}$, null at 0, equipped with the topology of uniform convergence on bounded sets. Then Ω is a Fréchet space. Let $\mathcal{B}(\Omega)$ denote the Borel σ-field on Ω and $\mathbb{P}$ the standard Wiener measure on $(\Omega, \mathcal{B}(\Omega))$. Put

$$W_t(\omega) = \omega(t), \quad t \in \mathbb{R}, \ \omega \in \Omega.$$

Then $\{W_t, t \in \mathbb{R}\}$ is a Brownian motion, and $W_0 = 0$, a.s.. Thus a Wiener functional $f(\omega)$ on Ω can be regarded as a functional of the sample paths of Brownian motion W. Furthermore, Put

$$\Omega_1 = \left\{\omega \in \Omega : \lim_{|t|\to\infty} (1+t^2)^{-1/2}|\omega(t)| = 0\right\}.$$

Then from the sample path properties of Brownian motion, we know that $\mathbb{P}(\Omega_1) = 1$. Let $\mathcal{F} = \Omega_1 \cap \mathcal{B}(\Omega)$. Then $\{W_t, t \in \mathbb{R}\}$ is still a Brownian motion when restricted to $(\Omega_1, \mathcal{F}, \mathbb{P})$. Since $\Omega_1 \subset \mathcal{S}^*(\mathbb{R})$ ($\mathcal{S}^*(\mathbb{R})$ is the tempered distrubition space of Schwartz), we may regard Brownian motion W and the generalized derivative $\dot{W}$ of its sample paths as $\mathcal{S}^*(\mathbb{R})$-valued random elements. $\dot{W}$ is the so-called "*white noise*", its probability distribution μ on the sample path space $(\mathcal{S}^*(\mathbb{R}), \mathcal{B}(\mathcal{S}^*(\mathbb{R}))$ is called the *white noise measure*. A measurable function on $(\mathcal{S}^*(\mathbb{R}), \mathcal{B}(\mathcal{S}^*(\mathbb{R})), \mu)$ is called a *white noise functional*. In this circumstance, we may regard a Wiener functional $f(\omega)$ as a white noise functional $F(x)$, where $F = f \circ J^{-1}$, $J(\omega) = \dot{\omega}$ is a measurable map from Ω_1 to $\mathcal{S}^*(\mathbb{R})$. The advantage of this viewpoint is that we can make full use of the linear topological structure of $\mathcal{S}^*(\mathbb{R})$ as the dual of a nuclear space, and by means of second quantization, we can construct the Sobolev space $(\mathcal{S})_p$ of white noise functionals from the Sobolev space $\mathcal{S}_p(\mathbb{R})$ over $\mathbb{R}$. Then let $(\mathcal{S})$ be the projective limit of $\{(\mathcal{S})_p, p \in \mathbb{N}_0\}$ and $(\mathcal{S})^*$ the inductive limit of $\{(\mathcal{S})_{-p}, p \in \mathbb{N}_0\}$. $(\mathcal{S})$ and $(\mathcal{S})^*$ are called Hida's testing functional and distribution spaces respectively. Thus from the Gel'fand triplet $\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}, dx) \hookrightarrow \mathcal{S}^*(\mathbb{R})$ we obtain a new

Gel'fand triplet $(S) \hookrightarrow L^2(S^*(\mathbb{R}),\mu) \hookrightarrow (S)^*$. The latter is the classical framework for white noise analysis. We shall see that Hida's testing functional space (S) is smaller than Meyer-Watanabe's testing functional space $\mathbb{D}^\infty$, and (S) can be continuously and densely imbedded in $\mathbb{D}^\infty$. Hence Hida's distribution space $(S)^*$ is larger than Meyer-Watanabe's distribution space $\mathbb{D}^{-\infty}$. Many formal manipulations in quantum physics can be legalized mathematically in this framework. Therefore, the white noise analysis has obtained successful application in quantum physics. The method of second quantization can be easily generalized to more general Gel'fand triplet (see Kondratiev-Leukert-Potthoff-Streit-Westerkamp[1]). The following method of constructing white noise calculus, first proposed by Kondratiev-Streit[1], is a natural generalization of "second quantization". The basic idea of this method is to multiply the term of n-th chaos by factor $(n!)^\beta$ when defining the Sobolev norms of Wiener functionals in order to obtain a smaller testing functional space (and correspondingly, a larger distribution space). This provides a flexible framework for applications of the white noise analysis .

§1. General framework for white noise analysis

We have studied functionals on the so-called irreducible Gauss space $(\Omega, \mathcal{F}, \mu; H)$ in Chapter 2. No assumption was made on the fundamental probability space $(\Omega,\mathcal{F},\mu)$ there, and Ω was not required to have a topological structure. We have used the so-called numerical model $(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$ only for the convenience of proof and even though, the topological structure of $\mathbb{R}^\infty$ was not used. In that chapter, we have introduced Meyer-Watanabe's distributions. However, in the chaos decomposition of such a distribution, the sequence $\{f_n\}$ consists still of elements of $\{H^{\widehat{\otimes}n}\}$. This is a restriction for certain application purpose (e.g. for quantum physics). In this section we shall introduce a general framework for white noise analysis, which extends the region of Meyer-Watanabe's distributions. This extension depends crucialy on the linear topological structure of the fundamental probability space $(\Omega,\mathcal{F},\mu)$. More precisely, Ω is required to be the dual space of a countably Hilbertian nuclear space.

1.1 Wick tensor products and the Wiener-Itô-Segal isomorphism

Let $E \hookrightarrow H \hookrightarrow E^*$ be a Gel'fand triplet, that is, H is a real separable Hilbert space, E a countably Hilbertian nuclear space which is densely and continuously imbedded in H, and E^* is the dual of E (identify the dual of H with itself). The inner product and norm in H or $H^{\otimes n}$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$ respectively. Henceforth, we view $H^{\widehat{\otimes}n}$ as a subspace of $H^{\otimes n}$, and the original inner product in $H^{\widehat{\otimes}n}$ will be converted to that in $H^{\otimes n}$, i.e., $(\cdot,\cdot)_{H^{\widehat{\otimes}n}} = n!(\cdot,\cdot)$. By Minlos' theorem (in Chapter 1), there exists a unique Gaussian measure μ on $(E^*,\mathcal{B}(E^*))$

such that

$$\int_{E^*} e^{i\langle x,f\rangle}\mu(dx) = \exp\{-\tfrac{1}{2}|f|^2\}\ , \quad f \in E. \tag{1.1}$$

Here and henceforth $\langle\cdot,\cdot\rangle$ denotes the canonical bilinear form on $E^* \times E$ (or $E \times E^*$).

Let $f \in E$ and $W_f(x) = \langle x, f\rangle$, $x \in E^*$. Then W_f is a Gaussian random variable on (E^*,μ), and by (1.1),

$$\mathbb{E}[W_f] = 0\ , \quad \mathbb{E}[W_f^2] = |f|^2\ .$$

Thus the linear map $f \mapsto W_f$ from E to $L^2(E^*,\mu)$ can be extended to a linear isometry from H to $L^2(E^*,\mu)$. Consequently, associated with any Gel'fand triplet $E \hookrightarrow H \hookrightarrow E^*$, there is a Gauss probability space $(E^*, \mathcal{B}(E^*), \mu; H)$. We call such a Gauss probability space a *canonical Gauss probability space*. The classical white noise space $(S^*(\mathbb{R}^d), \mathcal{B}(S^*(\mathbb{R}^d)), \mu)$ together with $L^2(\mathbb{R}^d, dx)$ constitutes a canonical Gauss probability space.

In Chapter 2, for a particular case where $H = L^2(T,\mathcal{B},\lambda)$ we have established the isomorphism between H and $L^2(\Omega,\mathcal{F},\mu)$ by means of multiple Wiener-Itô integral. In the following, we shall construct such an isomorphism for a canonical Gauss probability space. For this purpose, let's introduce the notion of Wick tensor product in E^*. By means of this notion, we can introduce an analogue of multiple Wiener-Itô integral.

Definition 1.1 Let $\tau \in E^*\widehat{\otimes}E^*$ be defined as

$$\langle \tau, f\otimes g\rangle = (f,g)\ , \quad f,g \in E\ . \tag{1.2}$$

For $x \in E^*$, put $:x^{\otimes 0}: \equiv 1$, $:x^{\otimes 1}: \equiv x$, and define inductively

$$:x^{\otimes n}: = x\widehat{\otimes}:x^{\otimes n-1}: -(n-1)\tau\widehat{\otimes}:x^{\otimes n-2}:, \quad n \geq 2.$$

Then

$$:x^{\otimes n}: = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!2^k} x^{\otimes(n-2k)}\widehat{\otimes}\tau^{\widehat{\otimes}k}\ , \ n \geq 1\ . \tag{1.3}$$

Obviously, $:x^{\otimes n}: \in E^{*\widehat{\otimes}n}$. We call $:x^{\otimes n}:$ the *n-fold Wick tensor product* of x.

Remark. By expressions (A.11) and (A.3) of Hermite polynomials, it is easy to prove

$$:x^{\otimes n}: = \int_{E^*} (x \pm iy)^{\otimes n}\mu(dy)\ , \tag{1.4}$$

the integral in r.h.s is in Bochner sense. Moreover, we have

$$x^{\otimes n} = \int_{E^*} :(x+y)^{\otimes n}: \mu(dy)\ , \tag{1.5}$$

$$x^{\otimes n} = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!2^k} :x^{\otimes(n-2k)}:\widehat{\otimes}\tau^{\widehat{\otimes}k}\ , \tag{1.6}$$

Lemma 1.2 *We have*

$$\langle f^{\otimes n}, : x^{\otimes n} : \rangle = |f|^n H_n(|f|^{-1}\langle f, x\rangle), \quad f \in E, \tag{1.7}$$

$$\int_{E^*} \langle f_n, : x^{\otimes n} : \rangle \langle g_m, : x^{\otimes m} : \rangle \mu(dx) = \delta_{mn} n! (f_n, g_m), \tag{1.8}$$

where $f_n \in E^{\hat{\otimes} n}$, $g_m \in E^{\hat{\otimes} m}$. H_n *is* n*-th Hermite polynomial.*

Proof. By (1.3) and (A.3) we get immediately (1.7). Let $E^{\circ n}$ be the linear space spaned by $\{f^{\otimes n}, f \in E\}$. By a corollary to the polarization formula (Chapter 1, (2.12)) we know that $E^{\circ n}$ is dense in $E^{\hat{\otimes} n}$. By (1.7) and (A.2),

$$\exp\{\langle f, x\rangle - \tfrac{1}{2}|f|^2\} = \textstyle\sum_{n=0}^{\infty} \frac{1}{n!}\langle f^{\otimes n}, : x^{\otimes n} : \rangle.$$

Consequently,

$$\begin{aligned}
&\int \sum_{n=0}^{\infty} \frac{u^n}{n!} \langle f^{\otimes n}, : x^{\otimes n} : \rangle \sum_{m=0}^{\infty} \frac{v^m}{m!} \langle g^{\otimes m}, : x^{\otimes m} : \rangle \mu(dx) \\
&= \int \exp\left[\langle uf + vg, x\rangle\right] \mu(dx) \exp\{-\tfrac{1}{2}(u^2|f|^2 + v^2|g|^2)\} \\
&= \exp\{uv(f, g)\} \\
&= \sum_{n=0}^{\infty} \frac{u^n v^n}{n!} (f, g)^n.
\end{aligned}$$

Hence for $f_n = f^{\otimes n}$ and $g_m = g^{\otimes m}$, (1.8) holds. This implies that (1.8) also holds for $f_n \in E^{\circ n}$ and $g_m \in E^{\circ m}$. Thus the map $f_n \mapsto (n!)^{-1/2}\langle f_n, : x^{\otimes n} : \rangle$ from $E^{\circ n}$ to $L^2(E^*, \mu)$ can be uniquely extended to be a linear isometry from $H^{\hat{\otimes} n}$ to $L^2(E^*, \mu)$. In particular, (1.8) holds true for $f_n \in E^{\hat{\otimes} n}$ and $g_m \in E^{\hat{\otimes} m}$. ∎

Remark. We shall denote by $I_n(f_n)$ the linear extension to $H^{\hat{\otimes} n}$ of the map $f_n \to \langle f_n, : x^{\otimes n} : \rangle$. It is an analogue of the multiple Wiener-Itô integral. From the proof of Lemma 1.2 we know that

$$I_n(f^{\otimes n}) = |f|^n H_n(|f|^{-1} W_f), \quad f \in H, \tag{1.9}$$

$$\int_{E^*} I_n(f_n) I_m(g_m) d\mu = \delta_{nm} n! (f_n, g_m), \quad f_n \in H^{\hat{\otimes} n}, g_m \in H^{\hat{\otimes} m}. \tag{1.10}$$

We shall also use $\langle f_n, : x^{\otimes n} : \rangle$ to denote $I_n(f_n)(x)$ formally. This should not be confused with the canonical bilinear form $\langle \cdot, \cdot \rangle$ on $E^{\otimes n} \times E^{*\otimes n}$.

The following theorem establishes an isomprphism between $L^2(E^*, \mu)$ and the symmetric Fock space $\Gamma(H)$ over H, which is called the *Wiener-Itô-Segal isomorphism*.

Theorem 1.3 *For* $\varphi \in L^2(E^*, \mu)$, *there exists a unique sequence* $\{f_n\}_{n \in \mathbb{N}_0} \in \Gamma(H)$ *such that*

$$\varphi = \sum_{n=0}^{\infty} I_n(f_n), \tag{1.11}$$

where the series converges in L^2*-sense. Moreover, we have*

$$\|\varphi\|^2 = \sum_{n=0}^{\infty} n! |f_n|^2. \tag{1.12}$$

Conversely, for $\{f_n\}_{n \in \mathbb{N}_0} \in \Gamma(H)$, *(1.11) defines an element of* $L^2(E^*, \mu)$. *We shall denote by* $\varphi \sim \{f_n\}$ *the correspondence between* $L^2(E^*, \mu)$ *and* $\Gamma(H)$, *determined by (1.11).*

Proof. Let

$$L^2(E^*, \mu) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{1.13}$$

be the Wiener-Itô decomposition of $L^2(E^*, \mu)$ (see Theorem 1.5 of Chapter 2). Put $\mathcal{G}_n = \{I_n(f_n), f_n \in H^{\hat{\otimes} n}\}$. By (1.9) and (1.10), we have

$$\bigoplus_{n=0}^{k} \mathcal{H}_n = \mathcal{P}_k = \bigoplus_{n=0}^{k} \mathcal{G}_n, \quad \forall k \geq 0,$$

where $\mathcal{P}_k$ is the closed subspace of $L^2(E^*, \mu)$ generated by $\{W_f^n, 0 \leq n \leq k, f \in H\}$. Hence, $\mathcal{H}_n = \mathcal{G}_n, \forall n \geq 0$. The conclusion follows. ∎

Remark. Let $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal base of H. For $\alpha \in \Lambda$ (Λ is the set of all sequences of non-negative numbers with a finite number of non zero terms), let $\widehat{e}_\alpha$ be defined as (2.13) in Chapter 1. Since $|\widehat{e}_\alpha|^2 = \alpha!/|\alpha|!$, $\{(\alpha!)^{-1/2} I_n(\widehat{e}_\alpha), \alpha \in \Lambda_n\}$ constitutes an orthonormal base of $\mathcal{H}_n$, where $\Lambda_n = \{\alpha \in \Lambda : |\alpha| = n\}$. Proceed similarly as the proof of (1.46) in Chapter 2, we have $\forall \alpha \in \Lambda$,

$$\prod_j H_{\alpha_j}(W_{e_j}) = I_{|\alpha|}(\widehat{e}_\alpha). \tag{1.14}$$

1.2 Testing functional space and distribution space

Let $E \hookrightarrow H \hookrightarrow E^*$ be a Gel'fand triplet. By the definition of countably Hilbertian nuclear space, there exists a sequence of increasing and compatible Hilbert norm $\{|\cdot|_p\}_{p \in \mathbb{N}}$ such that E is the projective limit of $\{H_p\}_{p \in \mathbb{N}}$ (H_p being the completion of E w.r.t $|\cdot|_p$), and for any $p \in \mathbb{N}$, there exists $p' > p$ such that the embedding $I_{pp'}$ from $H_{p'}$ to H_p is a Hilbert-Schmidt operator. Moreover, we may impose $|\cdot|_p \geq |\cdot|$ ($|\cdot|$ being the norm on H). Henceforth we call the sequence of Hilbert norms satisfying the above conditions a *standard sequence of norms* on E and, as a covention, put $H_0 = H$, $|\cdot|_0 = |\cdot|$.

Let $(L^2) = L^2(E^*, \mu)$. In the following we shall use a standard sequence of norms on E to construct a family of dense subspaces $\{(E)^\beta, \beta \geq 0\}$ of (L^2) by means of second quantization such that each $(E)^\beta$ is a countably Hilbertian nuclear space. We shall call $(E)^\beta$ a testing functional space and its dual the distribution space.

Let $\{|\cdot|_p\}$ be a standard sequence of norms on E, H_p the completion of E w.r.t. $|\cdot|_p$. For $p, q \in \mathbb{N}$, $\beta \geq 0$, put

$$\|\varphi\|^2_{p,q,\beta} \equiv \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |f_n|^2_p, \quad \varphi \sim \{f_n\}, \ f_n \in H_p^{\hat{\otimes} n}, \tag{1.15}$$

$$(H_{p,q,\beta}) \equiv \left\{\varphi \in (L^2) : \ \|\varphi\|_{p,q,\beta} < \infty\right\}. \tag{1.16}$$

Let $(E)^\beta$ be the projective limit of $\{(H_{p,q,\beta}), p, q \in \mathbb{N}\}$, i.e.

$$(E)^\beta = \bigcap_{p,q \geq 1} (H_{p,q,\beta}), \tag{1.17}$$

and we furnish $(E)^\beta$ with the projective limit topology.

Theorem 1.4 *$(E)^\beta$ is a countably Hilbertian nuclear space. It can be continuously and densely imbedded in (L^2). Moreover, $(E)^\beta$ and its topology do not depend on the choice of the standard sequence of norms on E.*

Proof. Fix $p, q \in \mathbb{N}$. Let $p' > p$ be such that the embedding $I_{pp'}$ from $H_{p'}$ to H_p is a Hilbert-Schmidt operator. Let $q' > q$ satisfy

$$\sum_{n=0}^{\infty} 2^{n(q-q')} \|I_{pp'}\|^{2n}_{\mathrm{HS}} < \infty. \tag{1.18}$$

Denote by I the embedding from $(H_{p',q',\beta})$ to $(H_{p,q,\beta})$. We shall prove that I is a Hilbert-Schmidt operator. To this end, let $\{e_i\}_{i \in \mathbb{N}}$ and $\{e'_i\}_{i \in \mathbb{N}}$ be orthonormal bases of H_p and $H_{p'}$ respectively. For $\alpha \in \Lambda$, put

$$\varphi_\alpha = (|\alpha|!)^{-\beta/2} 2^{-|\alpha|q/2} (\alpha!)^{-1/2} I_{|\alpha|}(\widehat{e_\alpha}). \tag{1.19}$$

We define φ'_α similarly. Then $\{\varphi_\alpha, \alpha \in \Lambda\}$ is an orthonormal base of $(H_{p,q,\beta})$, and $\{\varphi'_\alpha, \alpha \in \Lambda\}$ is an orthonormal base of $(H_{p',q',\beta})$. Hence we have

$$\begin{aligned}
\|I\|^2_{\mathrm{HS}} &= \sum_{\alpha \in \Lambda} \|\varphi'_\alpha\|^2_{p,q,\beta} \\
&= \sum_{\alpha, \sigma \in \Lambda} (\varphi'_\alpha, \varphi_\sigma)^2_{p,q,\beta} \\
&= \sum_{n=0}^{\infty} \sum_{\alpha, \sigma \in \Lambda_n} 2^{n(q-q')} (\alpha!\sigma!)^{-1} (\widehat{e'_\alpha}, \widehat{e_\sigma})^2_p
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty} 2^{n(q-q')} \|I_{pp'}^{\otimes n}\|^2_{\mathrm{HS}} \\
&= \sum_{n=0}^{\infty} 2^{n(q-q')} \|I_{pp'}\|^{2n}_{\mathrm{HS}} < \infty.
\end{aligned}$$

Thus $(E)^\beta$ is a countably Hilbertian nuclear space. Let

$$\mathcal{P}(E^*) = \left\{ \sum_{n=0}^{m} I_n(f_n), \ f_n \in E^{\hat{\otimes} n}, \ m \in \mathbb{N} \right\}.$$

Obviously, $\mathcal{P}(E^*) \subset (E)^\beta$, and $\mathcal{P}(E^*)$ is dense in (L^2). Hence $(E)^\beta$ can be continuously and densely imbedded in (L^2).

Finally, we prove that $(E)^\beta$ and its topology do not depend on the choice of the standard sequence of norms on E. Let $\{|\cdot|'_k\}$ be another sequence of standard norms on E and H'_p the completion of E w.r.t. $|\cdot|'_p$, $(E)'^\beta = \bigcap_{k,l \geq 1} (H'_{k,l,\beta})$. We furnish $(E)'^\beta$ with the projective limit topology. In order to prove $(E)^\beta = (E)'^\beta$ and the two topologies coincide, it suffices to prove that for any $k, l \in \mathbb{N}$, there exist $p, q \in \mathbb{N}$ such that $\forall \varphi \in \mathcal{P}(E^*)$, $\|\varphi\|'^2_{k,l,\beta} \leq \|\varphi\|^2_{p,q,\beta}$. Take a $p \in \mathbb{N}$ and a constant $c > 0$ such that $\forall f \in E$, $|f|'_k \leq c|f|_p$ (since $|\cdot|'_k$ is continuous w.r.t. the topology on E, such p and c do exist). Choose a $q \in \mathbb{N}$ such that $2^{(q-l)/2} \geq c$. Then for $\varphi \in \mathcal{P}(E^*)$ with $\varphi = \sum_{n=0}^{N} I_n(f_n)$,

$$\begin{aligned}
\|\varphi\|'^2_{k,l,\beta} &= \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nl} |f_n|'^2_k \\
&\leq \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nl} c^{2n} |f_n|^2_p \\
&\leq \|\varphi\|^2_{p,q,\beta}.
\end{aligned}$$

This completes the proof. ∎

Remark 1. From the proof of Theorem 1.4 we see that we may use a more general doubly indexed sequence $\{C_{q,n}\}$ to construct the second quantization spaces of E. In fact, let $\{C_{q,n}, q \geq 1, n \geq 1\}$ be a doubly indexed sequence of positive numbers satisfying: (i) for fixed n, the sequence is increasing w.r.t. q; (ii) for any $q \in \mathbb{N}$, $K > 0$, there exists $q' > q$ such that $\sum_n C_{q,n} C^{-1}_{q',n} K^n < \infty$. Use $C_{q,n}$ instead of $(n!)^\beta 2^{nq}$ in (1.15) to construct the space $\mathcal{G}_{p,q}$, and let $\mathcal{G}$ be the projective limit of $\{\mathcal{G}_{p,q}\}$. Then $\mathcal{G}$ is still a countably Hilbertian nuclear space, dense in (L^2). Further, we define an equivalence relation between the doubly indexed sequences satisfying (i) and (ii) above as: $\{C_{q,n}\} \sim \{C'_{k,n}\}$ if and only if $\forall K > 0$, $q \in \mathbb{N}$, $k \in \mathbb{N}$, there exist $p \in \mathbb{N}$, $l \in \mathbb{N}$, such that

$$C_{q,n} C'^{-1}_{l,n} \leq K^n, \quad C'_{k,n} C^{-1}_{p,n} \leq K^n.$$

Then the second quantization spaces constructed from equivalent sequences coincide with each other. In particular, 2^{nq} in (1.16) may be replaced by c^{nq}, provided $c > 1$.

Remark 2. If there exists a constant $0 < c < 1$ such that $|\cdot|_p \le c|\cdot|_{p+1}$, $\forall p \in \mathbb{N}_0$, then

$$|f_n|_p \le c^n |f_n|_{p+1},\ \forall n \ge 1,\ f_n \in E^{\hat{\otimes} n}.$$

$(E)^\beta$ is the projective limit of $\{H_{p,0,\beta},\ p \in \mathbb{N}\}$.

Remark 3. If Λ is a directed set, E is the projective limit of $\{H_\lambda,\ \lambda \in \Lambda\}$, then we can still use (1.16) to define the Hilbert space $(H_{\lambda,q,\beta})$. In this case, $(E)^\beta$ is the projective limit of $\{H_{\lambda,q,\beta},\ \lambda \in \Lambda, q \in \mathbb{N}\}$.

For $\beta \ge 0$, denote by $(E)^{-\beta}$ the dual of $(E)^\beta$. For $\beta = 0$, denote $(E)^0$ by (E) and $(E)^{-0}$ by $(E)^*$. $(E)^\beta$ is called the *testing functional space* and $(E)^{-\beta}$ is called the *distribution space*. Since for the particular case $E = \mathcal{S}(\mathbb{R}^d)$ (the space of C^∞ rapidly decreasing functions on $\mathbb{R}^d$), (E) and $(E)^*$ are the classical Hida's testing functional and distribution spaces respectively, we also call (E) and $(E)^*$ *Hida's testing functional* and *distribution spaces* respectively.

We give now a concrete construction of $(E)^{-\beta}$. Let H_{-p} be the dual of H_p (identify the dual of H with itself). For $p, q \in \mathbb{N}$, $\beta \ge 0$, define $(H_{-p,-q,-\beta})$ as follows:

$$\|F\|^2_{-p,-q,-\beta} \equiv \sum_{n=0}^\infty (n!)^{1-\beta} 2^{-nq} |g_n|^2_{-p},\ F \sim \{g_n\}, g_n \in H_{-p}^{\hat{\otimes} n}, \tag{1.20}$$

$$(H_{-p,-q,-\beta}) \equiv \Big\{ F : \|F\|_{-p,-q,-\beta} < \infty \Big\}. \tag{1.21}$$

Then $(H_{-p,-q,-\beta})$ is the dual of $(H_{p,q,\beta})$, and the canonical bilinear form on $(H_{p,q,\beta}) \times (H_{-p,-q,-\beta})$ is given by

$$\langle\langle \varphi, F \rangle\rangle = \sum_{n=0}^\infty n! \langle f_n, g_n \rangle\ , \tag{1.22}$$

where $\varphi \in (H_{p,q,\beta})$, $F \in (H_{-p,-q,-\beta})$, $\varphi \sim \{f_n\}$, $F \sim \{g_n\}$, and $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $H_p^{\hat{\otimes} n} \times H_{-p}^{\hat{\otimes} n}$.

The next theorem follows from the general theory on the dual of a nuclear space.

Theorem 1.5 *Let $(E)^{-\beta}$ be the inductive limit of $\{H_{-p,-q,-\beta}\}$:*

$$(E)^{-\beta} = \bigcup_{p,q \ge 1} (H_{-p,-q,-\beta})\ .$$

Then $(E)^{-\beta}$ is the dual of $(E)^\beta$, the canonical bilinear form on $(E)^{-\beta} \times (E)^\beta$ is defined by (1.22), where $\varphi \in (E)^\beta$, $F \in (E)^{-\beta}$, $\varphi \sim \{f_n\}$, $F \sim \{g_n\}$, and $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E \times E^*$.

Remark. Henceforth, $\forall p \in \mathbb{Z}$, $q, r \in \mathbb{R}$, we define the following Hilbert space:

$$(H_{p,q,r}) = \Big\{\varphi \sim \{g_n\} : g_n \in H_p^{\hat{\otimes} n}, \forall n \in \mathbb{N}_0,$$
$$\|\varphi\|^2_{p,q,r} \equiv \sum_{n=0}^\infty (n!)^{1+r} 2^{nq} |g_n|_p^2 < \infty\Big\}. \tag{1.23}$$

Obviously, $(H_{p,q,r})$ is a dense subspace of $(E)^{-|r|}$, and the embedding from $(H_{p,q,r})$ to $(E)^{-|r|}$ is continuous. Moreover, $(H_{p,q,r})$ and $(H_{-p,-q,-r})$ are dual to each other.

Let $E \hookrightarrow H \hookrightarrow E^*$ be a Gel'fand triplet. We can construct Meyer-Watanabe's testing functional space $\mathbb{D}^\infty$ and distribution space $\mathbb{D}^{-\infty}$ on $(E^*, \mathcal{B}(E^*), \mu)$ as in Chapter 2. The following theorem shows that (E) is smaller than $\mathbb{D}^\infty$.

Theorem 1.6 $(E) \subset \mathbb{D}^\infty$, *$(E)$ is dense in $\mathbb{D}^\infty$, and the embedding map is continuous.*

Proof. Denote by $\mathcal{P}$ the polynomial functionals on E^*, i.e., any element φ of $\mathcal{P}$ has the form

$$\varphi(x) = f(W_{\xi_1}(x), \cdots, W_{\xi_n}(x)),\ \xi_1, \cdots, \xi_n \in E,\ n \in \mathbb{N},$$

where f is a polynomial on $\mathbb{R}^n$. Obviously, $\mathcal{P} \subset \mathbb{D}^\infty \cap (E)$, and $\mathcal{P}$ is dense in $\mathbb{D}^\infty$ and in (E). Let $\mathcal{L}$ be the OU operator. In order to prove the theorem, it sufficies to prove: $\forall k \ge 1, r \ge 2$, there exists $q > 0$ such that

$$\|(I - \mathcal{L})^k \varphi\|_{L^r} \le \|\varphi\|_{0,q,0},\ \forall \varphi \in \mathcal{P}. \tag{1.24}$$

In fact, let $\varphi \in \mathcal{P}$, $\varphi \sim \{f_n\}$ and $t = \frac{1}{2}\log(r-1)$, then by the hypercontractivity of the semigroup $T_t = e^{t\mathcal{L}}$ (Theorem 3.5 in §3 of Chapter 2),

$$\begin{aligned}
\|(I - \mathcal{L})^k \varphi\|_{L^r} &\le \|e^{-t\mathcal{L}}(I - \mathcal{L})^k \varphi\|_{L^2} \\
&= \Big\|\sum_n e^{tn}(1 + n)^k I_n(f_n)\Big\|_{L^2} \\
&\le \Big\|\sum_n e^{(t+k)n} I_n(f_n)\Big\|_{L^2} \\
&= \|\varphi\|_{0,q,0},
\end{aligned}$$

where q satisfies $2^{q/2} = e^{t+k}$. ∎

Corollary 1.7 $\mathbb{D}^{-\infty} \subset (E)^*$, *$\mathbb{D}^{-\infty}$ is dense in $(E)^*$, and the embedding is continuous.*

1.3 Classical framework for white noise analysis

In many practical problems, the Gel'fand triplet $E \hookrightarrow H \hookrightarrow E^*$ and the standard sequence of norms on E are generated by a self-adjoint operator as

follows: let H be a real separable Hilbert space with norm $|\cdot|_0$, A a positive self-adjoint operator on H satisfying: 1) $\|A^{-1}\| < 1$; 2) there exists $p_0 > 0$ such that $\|A^{-p_0}\|_{\mathrm{HS}} < \infty$. For $p \geq 1$, let $H_p = \mathcal{D}(A^p)$ with norm $|x|_p = |A^p x|_0$. If $p' > p \geq 0$, then by $\|A^{-1}\| < 1$ we have $H_{p'} \hookrightarrow H_p$ and $|\cdot|_p \leq |\cdot|_{p'}$. Let $I_{p,p'}$ denote the embedding from $H_{p'}$ into H_p. Then

$$\|I_{p,p+p_0}\|^2_{\mathrm{HS}} = \|A^{-p_0}\|^2_{\mathrm{HS}} < \infty.$$

Hence if we denote by E the projective limit of $\{H_p, |\cdot|_p\}$, then E is a countably Hilbertian nuclear space, and $\{|\cdot|_p, p \geq 0\}$ is a standard sequence of norms on E. Let E^* be the dual space of E (identify the dual of H with itself). Then $E \hookrightarrow H \hookrightarrow E^*$ is a Gel'fand triplet. We shall say that the above triplet is *generated by* (H, A), and $\{|\cdot|_p, p \geq 0\}$ is the standard sequence of norms determined by A. A typical example is: $\mathcal{S}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}^*(\mathbb{R})$. It is generated by $L^2(\mathbb{R})$ and the harmonic oscillator $A = -\frac{d^2}{dt^2} + t^2 + 1$ (see the example in Section 3.2 of Chapter 1).

We shall give a more direct construction for Hida's testing functional space (E) and distribution space $(E)^*$. First, let's recall the definition of second quantization $\Gamma(A)$ given in Section 2 of Chapter 1. There the operator $\Gamma(A)$ was defined on the symmetric Fock space over H. However, since there is a natural isometric isomorphism between the Fock space $\Gamma(H)$ and (L^2) (Theorem 1.3), we may regard $\Gamma(A)$ as a positive self-adjoint operator on (L^2). Let $\varphi \in \mathcal{D}(\Gamma(A))$, $\varphi \sim \{f_n\}$. Then $\Gamma(A)\varphi \sim \{A^{\otimes n} f_n\}$. More generally, for $p \geq 0$, $\varphi \in \mathcal{D}(\Gamma(A)^p)$, $\varphi \sim \{f_n\}$, put $\Gamma(A)^p\varphi \sim \{(A^p)^{\otimes n} f_n\}$. Since

$$|f_n|_p = |(A^p)^{\otimes n} f_n|_0, \quad f_n \in \mathcal{D}((A^p)^{\otimes n}), \ p \geq 0,$$

the $(H_{p,0,0})$ defined in (1.16) is precisely $\mathcal{D}(\Gamma(A)^p)$. Denote $\mathcal{D}(\Gamma(A)^p)$ by $(E)_p$, and the norm $\|\cdot\|_p$ by

$$\|\varphi\|_p = \|\Gamma(A)^p\varphi\|_0 .$$

Denote by $(E)_{-p}$ the dual of $(E)_p$, $\|\cdot\|_{-p}$ the dual norm on $(E)_{-p}$.

On the other hand, since $\|A^{-1}\| < 1$, we have

$$|f_n|_p \leq \|A^{-1}\|^n |f_n|_{p+1}, \quad n \in \mathbb{N}_0, \ f_n \in E^{\widehat{\otimes} n}, \ p \geq 0. \tag{1.25}$$

Hence the condition in the Remark 2 of Theorem 1.4 is satisfied, and (E) is the projective limit of $\{(E)_p, p \in \mathbb{N}_0\}$, $(E)^*$ is the inductive limit of $\{(E)_{-p}, p \in \mathbb{N}_0\}$.

Remark. One can prove that $\Gamma(A)^{-p_0}$ is a Hilbert-Schmidt operator on (L^2).

§2. Characterization of functional spaces

We adopt the notations in §1. For any topological linear space K on $\mathbb{R}$, we denote by $K_{\mathbb{C}}$ its *complexification*, i.e., $K_{\mathbb{C}} = K + iK$. If K is a Hilbert space, then the inner product on $K_{\mathbb{C}}$ of $f_1 + ig_1$ and $f_2 + ig_2$ with $f_1, f_2, g_1, g_2 \in K$ is

$$(f_1 + ig_1, f_2 + ig_2) \equiv (f_1, f_2) + (g_1, g_2) + i[(f_2, g_1) - (f_1, g_2)] . \tag{2.1}$$

We still denote by $|\cdot|_p$ the norm on the complexification $H_{p,\mathbb{C}}$ of H_p or on $H^{\widehat{\otimes} n}_{p,\mathbb{C}}$.

The canonical bilinear form on $K_{\mathbb{C}} \times K_{\mathbb{C}}$ is

$$\langle f_1 + ig_1, f_2 + ig_2\rangle \equiv (f_1, f_2) - (g_1, g_2) + i[(f_2, g_1) + (f_1, g_2)] . \tag{2.2}$$

It is related to the inner product on $K_{\mathbb{C}}$ as follows:

$$\langle F, G\rangle = (F, \overline{G}) , \quad F, G \in K_{\mathbb{C}}. \tag{2.3}$$

For $f_n \in (H^{\widehat{\otimes} n})_{\mathbb{C}}$ $(\cong (H_{\mathbb{C}})^{\widehat{\otimes} n})$, $f_n = g_n + ih_n$, $g_n, h_n \in H^{\widehat{\otimes} n}$, put

$$I_n(f_n) \equiv I_n(g_n) + iI_n(h_n).$$

Then $(L^2)_{\mathbb{C}}$ is isomorphic to $\Gamma(H_{\mathbb{C}})$. We still denote by $\varphi \sim \{f_n\}$ the relation defined by (1.11).

This section is devoted to the characterization of $(E)^{-\beta}_{\mathbb{C}}$ $(0 \leq \beta \leq 1)$ and $(E)^{\beta}_{\mathbb{C}}$ $(0 \leq \beta < \infty)$. The characterization of $(E)^{-\beta}_{\mathbb{C}}$ $(1 < \beta < \infty)$ is postponed to §4.

2.1 S-transform and characterization of $(E)^{-\beta}_{\mathbb{C}}$ $(0 \leq \beta < 1)$

The following result is a generalization of (1.9).

Lemma 2.1 *We have*

$$I_n(f^{\otimes n}) = \langle f, f\rangle^{n/2} H_n(\langle f, f\rangle^{-1/2} W_f) , \quad f \in H_{\mathbb{C}}. \tag{2.4}$$

Proof. It suffices to prove (2.4) for $f \in E_{\mathbb{C}}$. Let $f = g + ih$, $g, h \in E$. Then $\forall k \geq 1$,

$$\begin{aligned}
\langle f^{\otimes 2k}, \tau^{\widehat{\otimes} k}\rangle &= \langle f^{\otimes 2}, \tau\rangle^k = \langle (g + ih)^{\otimes 2}, \tau\rangle^k \\
&= \langle g^{\otimes 2} - h^{\otimes 2} + 2ig\widehat{\otimes} h, \tau\rangle^k \\
&= \Big((g, g) - (h, h) + 2i(h, g)\Big)^k \\
&= \langle f, f\rangle^k.
\end{aligned}$$

Now (2.4) follows readily from (1.3) and (A.3). ∎

Let $f \in H_{\mathbb{C}}$, $\mathcal{E}_f$ be the exponential functional associated to f:

$$\mathcal{E}_f = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes n}) . \tag{2.5}$$

By (2.4) and (A.2),

$$\mathcal{E}_f = \exp\{W_f - \tfrac{1}{2}\langle f, f\rangle\} \tag{2.6}$$

Lemma 2.2 *Let $f \in E_{\mathbb{C}}$. Then $\mathcal{E}_f \in (E)^{\beta}_{\mathbb{C}}$, $\forall \beta \in [0,1)$.*

Proof. For $p, q \geq 1$, we have

$$\|\mathcal{E}_f\|^2_{p,q,\beta} = \sum_{n=0}^{\infty} (n!)^{\beta-1} 2^{nq} |f|_p^{2n} < \infty .$$

Hence $\mathcal{E}_f \in (E)^{\beta}_{\mathbb{C}}$ by the definition. ∎

Definition 2.3 For $0 \leq \beta < 1$, $\Phi \in (E)^{-\beta}_{\mathbb{C}}$, put

$$S\Phi(f) = \langle\langle \Phi, \mathcal{E}_f \rangle\rangle , \quad f \in E_{\mathbb{C}} .$$

The restriction of $S\Phi$ to E is called the *S-transform* of Φ, and $S\Phi$ is called the *extended S-transform* of Φ.

If $\Phi \in (L^2)_{\mathbb{C}}$, then by Cameron-Martin theorem,

$$S\Phi(f) = \mathbb{E}[\Phi \mathcal{E}_f] = \int_{E^*} \Phi(x + f)\mu(dx) , \quad f \in E . \tag{2.7}$$

If $\Phi \sim \{f_n\}$, then by (1.22),

$$S\Phi(f) = \sum_{n=0}^{\infty} \langle f_n, f^{\otimes n} \rangle , \quad f \in E_{\mathbb{C}} . \tag{2.8}$$

By Proposition 2.14 in Chapter 1, the complex linear space generated by the exponential functionals $\{\mathcal{E}_f, f \in E\}$ is dense in $(E)^{\beta}_{\mathbb{C}}$. Thus any element of $(E)^{-\beta}_{\mathbb{C}}$ is uniquely determined by its S-transform. A natural question arises: for $0 \leq \beta < 1$, how to characterize $(E)^{-\beta}_{\mathbb{C}}$ by S-transform? In order to answer this question, we first recall some results from complex analysis on locally convex spaces (see Direen[1]).

Denote by $\mathcal{L}_s(E^n_{\mathbb{C}})$ the set of all symmetric n-linear forms from $E^n_{\mathbb{C}}$ to $\mathbb{C}$. For $L \in \mathcal{L}_s(E^n_{\mathbb{C}})$, put

$$\widehat{L}(f) = L(f, \cdots, f) , \quad f \in E_{\mathbb{C}} . \tag{2.9}$$

$\widehat{L}$ is called the *n-homogeneous polynomial* corresponding to L. Denote by $\mathcal{P}_n(E_{\mathbb{C}})$ the set of all n-homogeneous polynomials on $E_{\mathbb{C}}$. By the polarization formula (Chapter 1 (2.11)), the map $L \mapsto \widehat{L}$ is injective from $\mathcal{L}_s(E^n_{\mathbb{C}})$ to $\mathcal{P}_n(E_{\mathbb{C}})$.

Definition 2.4 Let U be a non-empty open subset of $E_{\mathbb{C}}$. A functional on U is said to be *G-holomorphic* if $\forall \eta \in U$, $\forall \xi \in E_{\mathbb{C}}$, the map $\lambda \mapsto F(\eta + \lambda\xi)$ is holomorphic on some neighbourhood of $0 \in \mathbb{C}$. A functional on U is said to be *holomorphic* on U if it is continuous and G-holomorphic on U. A functional is said to be *holomorphic at* ξ_0 if it is holomorphic on some neighbourhood of $\xi_0 \in E_{\mathbb{C}}$. A G-holomorphic (holomorphic) functional on $E_{\mathbb{C}}$ is said to be *G-entire analytic (entire analytic)*, or simply *G-entire (entire)*.

From complex analysis we know that any locally bounded G-holomorphic functionals on U is continuous, hence holomorphic on U.

Denote by $H_G(U)$ and $H(U)$ the set of all G-holomorphic and holomorphic functions on U, respectively. For $F \in H_G(U)$ and any $\eta \in U$, there exists a unique sequence of symmetric n-linear forms $\{F^{(n)}_\eta, n \in \mathbb{N}\}$, $F^{(n)}_\eta \in \mathcal{L}_s(E^n_{\mathbb{C}})$, such that

$$F(\eta + \xi) = F(\eta) + \sum_{n=1}^{\infty} \frac{1}{n!} \widehat{F}^{(n)}_\eta(\xi) , \tag{2.10}$$

where ξ belongs to some open neighbourhood of the origin of $E_{\mathbb{C}}$. If $F \in H_G(E_{\mathbb{C}})$, then (2.10) holds for any $\xi \in E_{\mathbb{C}}$.

The next result is well known in the theory of several complex variables.

Lemma 2.5 *Let $n \in \mathbb{N}$, $n \geq 2$, f a complex function on $\mathbb{R}^n$. If for any $1 \leq k \leq n$ and $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \mathbb{R}^{n-1}$, the map $x_k \mapsto f(x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n)$ has an entire analytic continuation on $\mathbb{C}$, then f has an entire analytic continuation on $\mathbb{C}^n$.*

Definition 2.6 *Let F be a complex function on E, $0 \leq \beta < 1$. If F satisfies*

(C.1) for any $f, g \in E$, the map $\lambda \longrightarrow F(g + \lambda f)$ has an entire analytic continuation on $\mathbb{C}$;

(C.2) there exist constants $C, K > 0$ and $p \in \mathbb{N}_0$, such that

$$|F(zf)| \leq C \exp\{K|z|^{2/(1-\beta)}|f|_p^{2/(1-\beta)}\} , \forall f \in E, z \in \mathbb{C} . \tag{2.11}$$

then F is called a U_β-functional.

Lemma 2.7 *Let $0 \leq \beta < 1$. Any U_β-functional F has a unique entire analytic continuation on $E_{\mathbb{C}}$. Moreover, if F satisfies (2.11), then for any $0 < \rho < 1$, $\xi \in E_{\mathbb{C}}$, we have*

$$|F(\xi)| \leq C' \exp\{K'|\xi|_p^{2/(1-\beta)}\} , \tag{2.12}$$

where $C' = C(1-\rho)^{-\frac{1+\beta}{2}}$, $K' = (2e^2)^{\frac{1}{1-\beta}} \frac{2K}{1-\beta} \rho^{-\frac{2-\beta}{1-\beta}}$.

Proof. We first prove that F has a unique G-entire analytic continuation on $E_{\mathbb{C}}$. Denote still by F the functional $F(g_0 + \lambda g_1)$ on $E_{\mathbb{C}}$ in Definition 2.6(C.1), where $g_0, g_1 \in E$, $\lambda \in \mathbb{C}$. For $g_2, g_3 \in E$, consider the map

$$(x_1, x_2, x_3) \longmapsto F(g_0 + x_1 g_1 + x_2 g_2 + x_3 g_3) , \ (x_1, x_2, x_3) \in \mathbb{R}^3 .$$

By condition (C.1) and Lemma 2.5, this functional has an entire analytic continuation on $\mathbb{C}^3$. In particular, this means that the extended functional F on $E_{\mathbb{C}}$ is G-entire analytic. The uniqueness of G-entire analytic continuation is obvious.

Now we prove (2.12), which implies that F is locally bounded, hence entire. By (2.10),

$$F(\lambda\xi) = F(0) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \widehat{F}^{(n)}_0(\xi) . \tag{2.13}$$

We have the following Cauchy formula (see, e.g. Nachbin[1]):

$$\frac{1}{n!}\widehat{F}_0^{(n)}(\xi) = \frac{1}{2\pi i}\int_{|z|=R} \frac{F(z\xi)}{z^{n+1}}\,dz\,. \tag{2.14}$$

Let $f \in E$, $|f|_p = 1$, $R = \left(\frac{n(1-\beta)}{2K}\right)^{\frac{1-\beta}{2}}$. Then by (2.14) and (2.11),

$$\begin{aligned}\left|\frac{1}{n!}\widehat{F}_0^{(n)}(f)\right| &\le CR^{-n}e^{K(R|f|_p)^{\frac{2}{1-\beta}}}\\ &= C\left(\frac{2eK}{n(1-\beta)}\right)^{\frac{n(1-\beta)}{2}}\,. \end{aligned}\tag{2.15}$$

Hence by the homogeneity of $\widehat{F}_0^{(n)}$, we have

$$\begin{aligned}\left|\frac{1}{n!}\widehat{F}_0^{(n)}(f)\right| &= |f|_p^n\left|\frac{1}{n!}\widehat{F}_0^{(n)}(|f|_p^{-1}f)\right|\\ &\le C\left(\frac{2eK}{n(1-\beta)}\right)^{\frac{n(1-\beta)}{2}}|f|_p^n, \quad \forall f \in E\,. \end{aligned}\tag{2.16}$$

By the polarization formula (Chapter 1 (2.11)), $\forall f_1,\cdots,f_n \in E$,

$$\left|\frac{1}{n!}F_0^{(n)}(f_1,\cdots,f_n)\right| \le C\frac{n^n}{n!}\left(\frac{2eK}{n(1-\beta)}\right)^{\frac{n(1-\beta)}{2}}\prod_{j=1}^n|f_j|_p\,. \tag{2.17}$$

Since $F_0^{(n)}$ is an n-symmetric linear form on $E_{\mathbb{C}}^n$, by (2.17) and the inequality $(x+y)^2 \le 2(x^2+y^2)$, we obtain: $\forall \xi_1,\cdots,\xi_n \in E_{\mathbb{C}}$,

$$\begin{aligned}\left|\frac{1}{n!}F_0^{(n)}(\xi_1,\cdots,\xi_n)\right| &\le C\frac{n^n}{n!}\left(\frac{2eK}{n(1-\beta)}\right)^{\frac{n(1-\beta)}{2}}2^{\frac{n}{2}}\prod_{j=1}^n|\xi_j|_p\\ &= C\left(\frac{1}{n!}\right)^{\frac{1-\beta}{2}}\left(\frac{n^n}{n!}\right)^{\frac{1+\beta}{2}}\left(\frac{2eK}{1-\beta}\right)^{\frac{n(1-\beta)}{2}}2^{\frac{n}{2}}\prod_{j=1}^n|\xi_j|_p\\ &\le C\left(\frac{1}{n!}\right)^{\frac{1-\beta}{2}}\left[2e^2\left(\frac{2K}{1-\beta}\right)^{1-\beta}\right]^{\frac{n}{2}}\prod_{j=1}^n|\xi_j|_p\,. \end{aligned}\tag{2.18}$$

Here we have used the inequality $n^n/n! \le e^n$. Finally, by using Hölder inequality (taking $s = 2/(1-\beta)$ and $t = 2/(1+\beta)$ as a pair of conjugate exponents, i.e. $s^{-1}+t^{-1}=1$), from (2.13) and (2.18) we deduce (2.12). ∎

Lemma 2.8 *Let $0 \le \beta < 1$, F a U_β-functional satisfying (2.11). Let $p' > p$ be such that the embedding $I_{pp'}$ from $H_{p'}$ to H_p is Hilbert-Schmidt, and $q \in \mathbb{R}$ with $2^q > e^2(\frac{2K}{1-\beta})^{1-\beta}\|I_{pp'}\|_{\mathrm{HS}}^2$. Then there exists a unique $\Phi \in (H_{-p',-q,-\beta})_{\mathbb{C}}$ such that F is the S-transform of Φ. Moreover, we have*

$$\|\Phi\|_{-p',-q,-\beta} \le C\left[1-2^{-q}e^2\left(\frac{2K}{1-\beta}\right)^{1-\beta}\|I_{pp'}\|_{\mathrm{HS}}^2\right]^{-1/2}\,. \tag{2.19}$$

Proof. We use the notations in the proof of Lemma 2.7. By (2.18), $\forall n \ge 1$, $F_0^{(n)}$ is continuous on $E_{\mathbb{C}}$ with respect to every variable $\xi_j \in E_{\mathbb{C}}$. By the nuclear theorem (Chapter 1, Theorem 3.17), there exists $g_n \in (E_{\mathbb{C}}^*)^{\widehat{\otimes}n}$ such that

$$\langle g_n, \xi_1\widehat{\otimes}\cdots\widehat{\otimes}\xi_n\rangle = \frac{1}{n!}F_0^{(n)}(\xi_1,\cdots,\xi_n),\ \xi_1,\cdots,\xi_n \in E_{\mathbb{C}}\,. \tag{2.20}$$

Let $\{e_k, k \in \mathbb{N}\}$ be an orthonormal base of $H_{p'}$ and $e_k \in E$, $\forall k \in \mathbb{N}$. Since $\{e_{k_1}\otimes\cdots\otimes e_{k_n} : k_i \in \mathbb{N}, 1 \le i \le n\}$ is an orthonormal base of $H_{p',\mathbb{C}}^{\otimes n}$, we have (by (2.20),(2.17) and the inequality $n^n/n! \le e^n$)

$$\begin{aligned}|g_n|_{-p'}^2 &= \sum_{k_1,\cdots,k_n}\left|\langle g_n, e_{k_1}\otimes\cdots\otimes e_{k_n}\rangle\right|^2\\ &= \sum_{k_1,\cdots,k_n}(n!)^{-2}\left|F_0^{(n)}(e_{k_1},\cdots,e_{k_n})\right|^2\\ &\le C^2(n!)^{\beta-1}\left[e^2\left(\frac{2K}{1-\beta}\right)^{1-\beta}\right]^n\left(\sum_{k=1}^\infty|I_{pp'}e_k|_p^2\right)^n\\ &= C^2(n!)^{\beta-1}\left[e^2\left(\frac{2K}{1-\beta}\right)^{1-\beta}\right]^n\|I_{pp'}\|_{\mathrm{HS}}^{2n}\,. \end{aligned}\tag{2.21}$$

Put $g_0 = F(0)$, $\Phi \sim \{g_n\}$. Then (2.19) follows from (2.21), hence $\Phi \in (H_{-p',-q,-\beta})_{\mathbb{C}}$. In particular, $\Phi \in (E)_{\mathbb{C}}^{-\beta}$, and for $f \in E$, by (2.20) and (2.13), we obtain

$$\begin{aligned}S\Phi(f) &= \sum_{n=0}^\infty\langle g_n, f^{\otimes n}\rangle\\ &= F(0)+\sum_{n=1}^\infty\frac{1}{n!}\widehat{F}_0^{(n)}(f) = F(f)\,. \end{aligned}$$

This means that F is the S transform of Φ. ∎

The following theorem gives a characterization of spaces $(E)_{\mathbb{C}}^{-\beta}$, $0 \le \beta < 1$, in terms of the S-transform.

Theorem 2.9 *Let $0 \le \beta < 1$. The map $F : E \longrightarrow \mathbb{C}$ is the S-transform of some element of $(E)_{\mathbb{C}}^{-\beta}$ if and only if F is a U_β-functional.*

Proof. The sufficiency follows from Lemma 2.8. It remains to prove the necessity. For $\Phi \in (E)_{\mathbb{C}}^{-\beta}$, there exist $p, q \in \mathbb{N}$ such that $\Phi \in (H_{-p,-q,-\beta})_{\mathbb{C}}$. Put $F(f) = S\Phi(f)$, $f \in E$. Then the extended S-transform of Φ (still denoted by F) is just the entire analytic continuation of F onto $E_{\mathbb{C}}$. Let $\Phi \sim \{g_n\}$. Then by

(2.8),

$$\begin{aligned}
|F(zf)| &\le \sum_{n=0}^{\infty} |\langle g_n, (zf)^{\otimes n}\rangle| \\
&\le \sum_{n=0}^{\infty} |z|^n |g_n|_{-p} |f|_p^n \\
&\le \Big(\sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |g_n|^2_{-p}\Big)^{1/2} \Big(\sum_{n=0}^{\infty} \Big(\frac{1}{n!}\Big)^{1-\beta} 2^{qn} |z|^{2n} |f|_p^{2n}\Big)^{1/2} \\
&= \|\Phi\|_{-p,-q,-\beta} \Big(\sum_{n=0}^{\infty} \Big(\frac{1}{n!}\Big)^{1-\beta} 2^{qn} |z|^{2n} |f|_p^{2n}\Big)^{1/2} . \qquad (2.22)
\end{aligned}$$

Take $0 < \rho < 1$. By Hölder inequality (taking $1/\beta$ and $1/(1-\beta)$ as a pair of conjugate exponents),

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Big(\frac{1}{n!}\Big)^{1-\beta} 2^{qn} |z|^{2n} |f|_p^{2n} \\
&\le \Big(\sum_{n=0}^{\infty} \rho^n\Big)^{\beta} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} 2^{\frac{qn}{1-\beta}} |z|^{\frac{2n}{1-\beta}} |f|_p^{\frac{2n}{1-\beta}} \rho^{-\frac{\beta n}{1-\beta}}\Big)^{1-\beta} \\
&= (1-\rho)^{-\beta} \exp\{(1-\beta) 2^{\frac{q}{1-\beta}} \rho^{-\frac{\beta}{1-\beta}} |z|^{\frac{2}{1-\beta}} |f|_p^{\frac{2}{1-\beta}}\} .
\end{aligned}$$

This means that F satisfies condition (C.2) of Definition 2.6, hence is a U_β-functional. ∎

Remark. We shall denote by $S^{-1}F$ the distribution corresponding to the U_β-functional F.

The following two theorems are important corollaries of Theorem 2.9.

Theorem 2.10 *Let $0 \le \beta < 1$ and $\{F_n, n \in \mathbb{N}\}$ a sequence of U_β-functionals. If the following conditions are satisfied:*

(1) $\forall f \in E$, $\{F_n(f), n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathbb{C}$;

(2) there exist $p \in \mathbb{N}_0$ and $C, K > 0$ such that

$$|F_n(zf)| \le C \exp K(|z||f|_p)^{\frac{2}{1-\beta}} , \ \forall f \in E, \forall z \in \mathbb{C}, \forall n \ge 1, \qquad (2.23)$$

then $\{S^{-1}F_n, n \in \mathbb{N}\}$ converges strongly in $(E)_{\mathbb{C}}^{-\beta}$.

Proof. By Lemma 2.8, there exist $p', q \in \mathbb{N}$, $\rho \in (0,1)$, such that

$$\|S^{-1}F_n\|_{-p',-q,-\beta} \le C(1-\rho)^{-1/2} . \qquad (2.24)$$

Condition (1) implies that: $\forall f \in E$, the sequence $\{\langle\langle S^{-1}F_n, \mathcal{E}_f\rangle\rangle, n \in \mathbb{N}\}$ is Cauchy in $\mathbb{C}$. But the linear space spaned by $\{\mathcal{E}_f, f \in E\}$ is dense in $(H_{p',q,\beta})$, hence by (2.24), $\{S^{-1}F_n, n \in \mathbb{N}\}$ converges weakly in $(H_{-p',-q,-\beta})_{\mathbb{C}}$ and in $(E)_{\mathbb{C}}^{-\beta}$. However, since the strong convergence and weak convergence for sequences in $(E)_{\mathbb{C}}^{-\beta}$ are equivalent (see Chapter 1, Theorem 3.12), we conclude the theorem. ∎

Remark. By inequality (2.22), the converse of this theorem is also true.

Theorem 2.11 *Let $0 \le \beta < 1$, $(\Omega, \mathcal{F}, \nu)$ a measure space, $\omega \longmapsto \Phi_\omega$ a map from Ω to $(E)_{\mathbb{C}}^{-\beta}$, $F_\omega = S\Phi_\omega$. If F_ω satisfies*

(1) $\forall f \in E$, the map $\omega \longmapsto F_\omega(f)$ is measurable on $(\Omega, \mathcal{F})$;

(2) there exist $K > 0$, $p \in \mathbb{N}$ and a non-negative ν-integrable function $C(\omega)$ on Ω such that for ν-a.e. ω,

$$|F_\omega(zf)| \le C(\omega) \exp K(|z||f|_p)^{\frac{2}{1-\beta}}, \ \forall f \in E, \forall z \in \mathbb{C}, \qquad (2.25)$$

then there exist $p', q \in \mathbb{N}$, such that $\omega \longmapsto \Phi_\omega$ is Bochner integrable on $(H_{-p',-q,-\beta})_{\mathbb{C}}$ and

$$S\Big(\int_\Omega \Phi_\omega d\nu(\omega)\Big)(f) = \int_\Omega S\Phi_\omega(f) d\nu(\omega) , \ \forall f \in E . \qquad (2.26)$$

Proof. By Lemma 2.8, there exist $p', q \in \mathbb{N}$, $\rho \in (0,1)$, such that

$$\|\Phi_\omega\|_{-p',-q,-\beta} \le C(\omega)(1-\rho)^{-1/2} , \ \nu\text{-a.e. } \omega . \qquad (2.27)$$

But condition (1) implies that, $\forall f \in E$, $\omega \longmapsto \langle\langle \Phi_\omega, \mathcal{E}_f\rangle\rangle$ is measurable. Hence $\forall \varphi \in (H_{p',q,\beta})$, $\omega \longmapsto \langle\langle \Phi_\omega, \varphi\rangle\rangle$ is measurable. Now by (2.27) and condition (2), as an $(H_{-p',-q,-\beta})_{\mathbb{C}}$-valued functional, $\omega \longmapsto \Phi_\omega$ is Bochner integrable, and (2.26) holds (since $SG(f) = \langle\langle G, \mathcal{E}_f\rangle\rangle$). ∎

2.2 Local S-transform and characterization of $(E)_{\mathbb{C}}^{-1}$

We now turn to characterize $(E)_{\mathbb{C}}^{-1}$. Let $f \in E_{\mathbb{C}}$. Then $\mathcal{E}_f \in (H_{p,q,1})_{\mathbb{C}}$ if and only if $2^q |f|_p^2 < 1$. In fact,

$$\begin{aligned}
\|\mathcal{E}_f\|^2_{p,q,1} &= \sum_{n=0}^{\infty} (n!)^2 2^{nq} |(n!)^{-1} f^{\otimes n}|_p^2 \\
&= \sum_{n=0}^{\infty} 2^{nq} |f|_p^{2n} .
\end{aligned}$$

Hence $\mathcal{E}_f \in (E)^1_{\mathbb{C}}$ if and only if $f = 0$ and we cannot define the S-transform on $(E)_{\mathbb{C}}^{-1}$ as before. However, since

$$(E)_{\mathbb{C}}^{-1} = \bigcup_{p,q \in \mathbb{N}} (H_{-p,-q,-1})_{\mathbb{C}} ,$$

we can define a "local S-transform" on $(E)_{\mathbb{C}}^{-1}$. Put

$$U_{p,q} = \{\xi \in E_{\mathbb{C}} : \ 2^q |\xi|_p^2 < 1\} , \quad p \in \mathbb{Z}, q \in \mathbb{R} .$$

Definition 2.12 Let $\Phi \in (E)_{\mathbb{C}}^{-1}$. Take $p, q \in \mathbb{N}$ such that $\Phi \in (H_{-p,-q,-1})_{\mathbb{C}}$. For $\Phi \sim \{g_n, n \in \mathbb{N}_0\}$, put

$$S\Phi(\xi) = \langle\langle \Phi, \mathcal{E}_\xi \rangle\rangle = \sum_{n=0}^{\infty} \langle g_n, \xi^{\otimes n}\rangle , \ \xi \in U_{p,q} .$$

Clearly, $S\Phi$ is a holomorphic functional on $U_{p,q}$, it is called the *local S-transform* of Φ.

Denote by $\mathrm{Hol}_0(E_{\mathbb{C}})$ the space of all functionals holomorphic at $0 \in E_{\mathbb{C}}$ (see Definition 2.4). Thus, $\forall \Phi \in (E_{\mathbb{C}})^{-1}$, $S\Phi \in \mathrm{Hol}_0(E_{\mathbb{C}})$. Let $F_1, F_2 \in \mathrm{Hol}_0(E_{\mathbb{C}})$. If there exists an open neighbourhood U of $0 \in E_{\mathbb{C}}$ such that F_1 and F_2 coincide on U, then F_1 and F_2 are called equivalent. We shall denote this equivalence relation by $F_1 \sim F_2$.

$(E)_{\mathbb{C}}^{-1}$ is characterized by the local S-transform as follows.

Theorem 2.13 (1) *If $\Phi \in (E)_{\mathbb{C}}^{-1}$, then $S\Phi \in \mathrm{Hol}_0(E_{\mathbb{C}})$.*

(2) *If $F \in \mathrm{Hol}_0(E_{\mathbb{C}})$, then there exists a unique $\Phi \in (E)_{\mathbb{C}}^{-1}$ such that $S\Phi \sim F$. More precisely, assume that F is holomorphic on $U_{p,q}$ for some $p, q \in \mathbb{N}$ and $|F(\xi)| \leq C$ for some $C > 0$ and all $\xi \in U_{p,q}$. Let $p' > p$ be such that the embedding $I_{pp'}$ from $H_{p'}$ to H_p is Hilbert-Schmidt, and $\rho = 2^{-(q'-2q-2)}e^2\|I_{pp'}\|_{\mathrm{HS}}^2 < 1$ for some $q' \in \mathbb{R}$. Then Φ corresponding to F belongs to $(H_{-p',-q',-1})_{\mathbb{C}}$, and we have the estimate*

$$\|\Phi\|_{-p',-q',-1} \leq C(1-\rho)^{-1/2} . \tag{2.28}$$

Proof. (1) is easily verified. We shall prove (2). For $|\xi|_p = 1$, $\xi \in E_{\mathbb{C}}$ and $|z| < 2^{-q}$, $z \in \mathbb{C}$, we have

$$F(z\xi) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \widehat{F}_0^{(n)}(\xi) . \tag{2.29}$$

By Cauchy formula (2.14) (taking $R = 2^{-(q+1)}$),

$$\left|\frac{1}{n!}\widehat{F}_0^{(n)}(\xi)\right| \leq C2^{n(q+1)} .$$

By the homogeneity of $\widehat{F}_0^{(n)}$ and the polarization formula (Chapter 1, (2.11)),

$$\left|\frac{1}{n!}F_0^{(n)}(\xi_1, \cdots, \xi_n)\right| \leq C2^{n(q+1)}e^n \prod_{j=1}^{n} |f_j|_p . \tag{2.30}$$

Here we have used the inequality $(n!)^{-1}n^n \leq e^n$. Thus by the nuclear theorem, $\forall n \geq 1$, there exists $g_n \in (E_{\mathbb{C}}^*)^{\widehat{\otimes} n}$ such that (2.20) holds. Moreover, proceeding similarly as the proof of Lemma 2.8, we have

$$\begin{aligned} |g_n|^2_{-p'} &\leq C^2(e2^{(q+1)}\|I_{pp'}\|_{\mathrm{HS}})^{2n} \\ &= C^2 2^{q'n}\rho^n . \end{aligned} \tag{2.31}$$

Let $\Phi \sim \{g_n, n \in \mathbb{N}_0\}$ with $g_0 = F(0)$. Then $\Phi \in (H_{-p',-q',-1})_{\mathbb{C}}$, and (2.28) holds. Clearly, $S\Phi$ and F coincide on $U_{p',q'}$. ∎

Similar to the proofs of Theorems 2.10 and 2.11, we can prove the following two important corollaries of Theorem 2.13.

Theorem 2.14 Let $F_n \in \mathrm{Hol}_0(E_{\mathbb{C}})$, $n \in \mathbb{N}$. If the following two conditions are satisfied:

(1) there exist $p \in \mathbb{N}$, $q \in \mathbb{R}$ and $C > 0$, such that for any F_n holomorphic on $U_{p,q}$, $|F_n(\xi)| \leq C$, $\forall \xi \in U_{p,q}$;

(2) for any $\xi \in U_{p,q}$, $\{F_n(\xi), n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathbb{C}$,

then $\{S^{-1}F_n\}$ converges strongly in $(E)_{\mathbb{C}}^{-1}$.

Theorem 2.15 *Let $(\Omega, \mathcal{F}, \nu)$ be a measure space, $\omega \mapsto \Phi_\omega$ a measurable map from Ω to $(E)_{\mathbb{C}}^{-1}$. If there exists $p \in \mathbb{N}$, $q \in \mathbb{R}$ such that the local S-transform F_ω of Φ_ω is well-defined on $U_{p,q}$ and*

(1) *$\forall \xi \in U_{p,q}$, $\omega \mapsto F_\omega(\xi)$ is measurable;*

(2) *there exists a non negative ν-integrable function $C(\omega)$ such that for ν-a.e. ω,*

$$|F_\omega(\xi)| \leq C(\omega), \ \forall \xi \in U_{p,q},$$

then there exist $p', q' \in \mathbb{N}$ such that $\omega \mapsto \Phi_\omega$ is Bochner integrable in $(H_{-p',q',-1})_{\mathbb{C}}$, and

$$S\Big(\int_\Omega \Phi_\omega d\nu(\omega)\Big)(\xi) = \int_\Omega S\Phi_\omega(\xi)d\nu(\omega) \ , \ \forall \xi \in U_{p,q} . \tag{2.32}$$

2.3 Two characterizations for testing functional spaces

We have characterized the spaces $(E)_{\mathbb{C}}^{-\beta}$ $(0 \leq \beta < 1)$ and $(E)_{\mathbb{C}}^{-1}$ by means of S-transform and local S-transform, respectively. We now turn to the characterization of the testing functional spaces $(E)_{\mathbb{C}}^{\beta}$ $(0 \leq \beta < \infty)$. First, since $(E)_{\mathbb{C}}^{\beta}$ is a subspace of $(E)_{\mathbb{C}}^{-\beta}$, we can characterize $(E)_{\mathbb{C}}^{\beta}$ by means of S-transform.

Theorem 2.16 *A functional F on E is the S-transform of some element in $(E)_{\mathbb{C}}^{\beta}$ if and only if it satisfies the condition (C.1) of Definition 2.6 and*

(C.3) *$\forall p \in \mathbb{N}$, $\forall \epsilon > 0$, there exists $C_{p,\epsilon} > 0$ such that*

$$|F(zf)| \leq C_{p,\epsilon}\exp\{\epsilon|z|^{\frac{2}{1+\beta}}|f|_{-p}^{\frac{2}{1+\beta}}\} \ , \forall f \in E, \ z \in \mathbb{C}. \tag{2.33}$$

Proof. Necessity. Let $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $F = S\varphi$. Obviously, F satisfies (C.1). For $p \in \mathbb{N}_0$, similar to the proof of (2.22), we can prove that, $\forall f \in E$, $z \in \mathbb{C}$,

$$|F(zf)| \leq \|\varphi\|_{p,q,\beta}\Big(\sum_{n=0}^{\infty}\Big(\frac{1}{n!}\Big)^{1+\beta}2^{-qn}|z|^{2n}|f|_{-p}^{2n}\Big)^{1/2} . \tag{2.34}$$

For a given $\epsilon > 0$, take a q large enough such that $\rho = 2^{-q}(\frac{2}{1+\beta})^{-(1+\beta)} < 1$. By (2.34),

$$|F(zf)| \leq \|\varphi\|_{p,q,\beta}\Big(\sum_{n=0}^{\infty}\rho^n\Big)^{1/2}\exp\{\epsilon|z|^{\frac{2}{1+\beta}}|f|_{-p}^{\frac{2}{1+\beta}}\} . \tag{2.35}$$

Here we have used the inequality $(n!)^{-(1+\beta)}x^n \leq \exp\{(1+\beta)x^{\frac{1}{1+\beta}}\}$. Thus (C.3) holds.

Sufficiency. Suppose that F satisfies (C.1) and (C.3). In (2.14), put

$$R = \left(\frac{n(1+\beta)}{2\epsilon}\right)^{\frac{1+\beta}{2}} .$$

Just as the proof of Lemma 2.7, $\forall f_1, \cdots, f_n \in E$, we have

$$\begin{aligned} \left|\frac{1}{n!}F_0^{(n)}(f_1, \cdots, f_n)\right| &\le C_{p,\epsilon}\frac{n^n}{n!}\left(\frac{2e\epsilon}{n(1+\beta)}\right)^{\frac{n(1+\beta)}{2}} \prod_{j=1}^{n} | f_j |_{-p} \\ &\le C_{p,\epsilon}\left(\frac{1}{n!}\right)^{\frac{1-\beta}{2}}\left[e^2\left(\frac{2\epsilon}{1+\beta}\right)^{1-\beta}\right]^{\frac{n}{2}} \prod_{j=1}^{n} | f_j |_{-p} . \end{aligned} \tag{2.36}$$

Now similar to the proof of Lemma 2.8, there exists $\varphi \in (E)^{\beta}_{\mathbb{C}}$, $\varphi \sim \{f_n\}$, such that $S\varphi = F$, and for any $p, q \in \mathbb{N}$, when ϵ is sufficiently small,

$$|f_n|_p^2 \le C^2_{p,\epsilon}(n!)^{-(1+\beta)}\left[e^2\left(\frac{2\epsilon}{1+\beta}\right)^{1+\beta}\right]^n \|I_{pp'}\|^{2n}_{\mathrm{HS}} ,$$

$$\begin{aligned} \|\varphi\|^2_{p,q,\beta} &= \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{nq}|f_n|^2_p \\ &\le C^2_{p,\epsilon}\sum_{n=0}^{\infty}\left(2^q e^2\left(\frac{2\epsilon}{1+\beta}\right)^{1+\beta}\|I_{pp'}\|^2_{\mathrm{HS}}\right)^n < \infty , \end{aligned}$$

which means $\varphi \in (E)^{\beta}_{\mathbb{C}}$. ∎

We shall give an alternative characterization of $(E)^{\beta}_{\mathbb{C}}$ which is more transparent than that through S-transform and is more convenient to use.

Definition 2.17 *Let $0 < \alpha \le 2$ and φ a complex function on $E^*_{\mathbb{C}}$. If $\forall p \in \mathbb{N}_0$, φ is an entire function on $H_{-p,\mathbb{C}}$ (see Definition 1.9), and $\forall \epsilon > 0$, there exists $C_{p,\epsilon} > 0$ such that*

$$|\varphi(z)| \le C_{p,\epsilon}\exp\{\epsilon|z|^{\alpha}_{-p}\} , \forall z \in H_{-p,\mathbb{C}}, \tag{2.37}$$

then φ is called an entire function of growth α and minimal type.

Denote by $A^{\alpha}(E^*_{\mathbb{C}})$ the set of all entire functions on $E^*_{\mathbb{C}}$ of growth α and minimal type and $A^{\alpha}(E^*)$ the restriction of $A^{\alpha}(E^*_{\mathbb{C}})$ to E^*, i.e.,

$$A^{\alpha}(E^*) = \{\varphi|_{E^*} : \varphi \in A^{\alpha}(E^*_{\mathbb{C}})\}. \tag{2.38}$$

Here $\varphi|_{E^*}$ denotes the restriction of φ to E^*.

$(E)^{\beta}_{\mathbb{C}}$ being a subspace of $(L^2)_{\mathbb{C}}$, every element of $(E)^{\beta}_{\mathbb{C}}$ is defined almost everywhere on E^*. Thus $(E)^{\beta}_{\mathbb{C}}$ is a space of μ-equivalence classes of functions. We shall identify the μ-equivalence class with its representative. In this sense we have

Theorem 2.18 *Let $0 \le \beta < \infty$. Then $(E)^{\beta}_{\mathbb{C}} = A^{\frac{2}{1+\beta}}(E^*)$. Moreover, let $p_0 \in \mathbb{N}$ be such that the embedding from H_{p_0} to H is of trace class. Then $\forall p \ge p_0, \epsilon > 0$, $\exists\, q \in \mathbb{N}$, $C_{q,\epsilon} > 0$, such that*

$$|\widetilde{\varphi}(x)| \le \|\varphi\|_{p,q,\beta}C_{p,\epsilon}\exp\left\{\epsilon|x|^{\frac{2}{1+\beta}}_{-p}\right\}, \tag{2.39}$$

where $\widetilde{\varphi}$ is the continuous version of φ.

Proof. Let $\varphi \in (E)^{\beta}_{\mathbb{C}}$, $\varphi \sim \{f_n\}$, i.e.,

$$\varphi(x) = \sum_{n=0}^{\infty}\langle f_n, :x^{\otimes n}:\rangle , \quad x \in E^* , \tag{2.40}$$

where the series converges in L^2-sense. For $y, z \in H_{-p,\mathbb{C}}$, we have

$$\begin{aligned} |(z+iy)^{\otimes n}|_{-p} &\le (|z|_{-p}+|y|_{-p})^n \\ &= [(|z|_{-p}+|y|_{-p})^{\frac{2}{1+\beta}}]^{\frac{1+\beta}{2}n} \\ &\le [2^{\frac{(1-\beta)^+}{1+\beta}}(|z|^{\frac{2}{1+\beta}}_{-p}+|y|^{\frac{2}{1+\beta}}_{-p})]^{\frac{1+\beta}{2}n} . \end{aligned} \tag{2.41}$$

Here we have used the C_r-inequality: $(a+b)^r \le (2^{r-1} \vee 1)(a^r+b^r), r > 0$, $a, b \ge 0$. Now by the Minlos-Sazanov theorem, for any $p \ge p_0$, the Gaussian measure μ is supported by H_{-p}. Thus for $p \ge p_0$ and q sufficiently large,

$$C_1^2 = \int_{E^*}\exp\left\{(1+\beta)2^{\frac{(1-\beta)^+-q}{1+\beta}}|y|^{\frac{2}{1+\beta}}_{-p}\right\}\mu(dy) < \infty . \tag{2.42}$$

By (1.4) (it also holds for $x \in E^*_{\mathbb{C}}$) we have

$$\begin{aligned} \sum_{n=0}^{\infty}|\langle f_n, :z^{\otimes n}:\rangle| &\le \sum_{n=0}^{\infty}|f_n|_p\int_{E^*}|(z+iy)^{\otimes n}|_{-p}\,\mu(dy) \\ &\le \sum_{n=0}^{\infty}(n!)^{\frac{1+\beta}{2}}2^{nq/2}|f_n|_p \\ &\quad\times\int_{E^*}(n!)^{-\frac{1+\beta}{2}}\left[2^{\frac{(1-\beta)^+-q}{1+\beta}}(|z|^{\frac{2}{1+\beta}}_{-p}+|y|^{\frac{2}{1+\beta}}_{-p})\right]^{\frac{1+\beta}{2}n}\mu(dy) \\ &\le \|\varphi\|_{p,q,\beta}\left[\sum_{n=0}^{\infty}\left(\int_{E^*}\cdots\mu(dy)\right)^2\right]^{1/2} \\ &\le \|\varphi\|_{p,q,\beta}\left(\int_{E^*}\sum_{n=0}^{\infty}(n!)^{-(1+\beta)}\right. \\ &\quad\left.\times\left[2^{\frac{(1-\beta)^+-q}{1+\beta}}(|z|^{\frac{2}{1+\beta}}_{-p}+|y|^{\frac{2}{1+\beta}}_{-p})\right]^{(1+\beta)n}\mu(dy)\right)^{1/2} \\ &\le \|\varphi\|_{p,q,\beta}\left[\int_{E^*}\exp\left\{(1+\beta)[2^{\frac{(1-\beta)^+-q}{1+\beta}}(|z|^{\frac{2}{1+\beta}}_{-p}+|y|^{\frac{2}{1+\beta}}_{-p})]\right\}\mu(dy)\right]^{1/2} \\ &= \|\varphi\|_{p,q,\beta}C_1\exp\left\{\frac{1+\beta}{2}2^{\frac{(1-\beta)^+-q}{1+\beta}}|z|^{\frac{2}{1+\beta}}_{-p}\right\} . \end{aligned} \tag{2.43}$$

Hence by (2.43), for $p \geq p_0$, the following function $\tilde{\varphi}$ defined on $H_{-p,\mathbb{C}}$ is an entire function:

$$\tilde{\varphi}(z) = \sum_{n=0}^{\infty} \langle f_n, :z^{\otimes n}: \rangle , \tag{2.44}$$

and $\forall \epsilon > 0$, $\forall p \geq p_0$, when q is sufficiently large,

$$|\, \tilde{\varphi}(z) \,| \leq \|\varphi\|_{p,q,\beta} C_{q,\epsilon} \exp\{\epsilon |z|_{-p}^{\frac{2}{1+\beta}}\} . \tag{2.45}$$

On the other hand, for $0 \leq p \leq p_0$, we have $H_{-p,\mathbb{C}} \subset H_{-p_0,\mathbb{C}}$ and $|z|_{-p_0} \leq |z|_{-p}$. Thus $\tilde{\varphi}$ is an entire function on $H_{-p,\mathbb{C}}$ and (2.45) holds. This means $\tilde{\varphi} \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*_{\mathbb{C}})$. Since $\tilde{\varphi}(x) = \varphi(x)$, μ-a.e. $x \in E^*$, we know that $\varphi = \tilde{\varphi}|_{E^*} \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$.

Conversely, let $\varphi \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*_{\mathbb{C}})$. Without loss of generality we take $p_0 \in \mathbb{N}$ in Definition 2.17 such that $\mu(H_{-p_0}) = 1$. Then $\forall p \geq p_0$, $\forall \epsilon > 0$, by (2.37) we have (using C_r-inequality)

$$\begin{aligned} |\, \varphi(x+z) \,| &\leq C_{p,\epsilon} \exp\Big\{\epsilon |x+z|_{-p}^{\frac{2}{1+\beta}}\Big\} \\ &\leq C_{p,\epsilon} \exp\Big\{\epsilon 2^{\frac{(1-\beta)^+}{1+\beta}} |x|_{-p}^{\frac{2}{1+\beta}}\Big\} \exp\Big\{\epsilon 2^{\frac{(1-\beta)^+}{1+\beta}} |z|_{-p}^{\frac{2}{1+\beta}}\Big\} . \end{aligned} \tag{2.46}$$

Put

$$\psi(z) = \int_{E^*} \varphi(x+z)\mu(dx) , \quad z \in H_{-p,\mathbb{C}} . \tag{2.47}$$

By the Fernique theorem (I.4.20), we can take $0 < \epsilon_1 \leq \epsilon$ sufficiently small in (2.46) such that the above integral exists and ψ is an entire function on $H_{-p,\mathbb{C}}$. Now by (2.46),

$$|\, \psi(z) \,| \leq C'_{p,\epsilon} \exp\Big\{\epsilon 2^{\frac{(1-\beta)^+}{1+\beta}} |z|_{-p}^{\frac{2}{1+\beta}}\Big\} . \tag{2.48}$$

But $\varphi|_{E^*}$ is obviously in $(L^2)_{\mathbb{C}}$, thus by (2.7),

$$S\, \varphi|_{E^*} (f) = \psi(f) , \quad f \in E . \tag{2.49}$$

It follows from (2.47) and (2.48) that $S\varphi|_{E^*}$ satisfies the conditions of Theorem 2.16. Hence $\varphi|_{E^*} \in (E)^{\beta}_{\mathbb{C}}$. ∎

Remark. By the above theorem, every element of $(E)^{\beta}_{\mathbb{C}}$ has a unique continuous version. Henceforth, concerning elements of $(E)^{\beta}_{\mathbb{C}}$, we shall always take their continuous versions.

Corollary 2.19 *Let $0 \leq \beta < \infty$, $\varphi_n \in (E)^{\beta}_{\mathbb{C}}$, $n \in \mathbb{N}$. If $\varphi_n \to \varphi$ in $(E)^{\beta}_{\mathbb{C}}$, then $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, $\forall x \in E^*$.*

Proof. Let $x \in H_{-p}$, $p \geq p_0$, where p_0 satisfies $\mu(H_{p_0}) = 1$. Then by (2.43),

$$|\varphi_n(x) - \varphi(x)| \leq \|\varphi_n - \varphi\|_{p,q,\beta} C_1 \exp\{\alpha(q)|x|_{-p}^{\frac{2}{1+\beta}}\} ,$$

thus $\varphi_n(x) \to \varphi(x)$. ∎

Corollary 2.20 *Let $0 \leq \beta < \infty$. Then $(E)^{\beta}_{\mathbb{C}}$ is closed under multiplication.*

2.4 Some examples of distributions

We present here some examples of distributions, many of which will be used in the sequel.

Example 2.21 (*Generalized exponential functionals*). For $z \in E^*_{\mathbb{C}}$, put $\mathcal{E}_z \sim \{\frac{1}{n!} z^{\otimes n}\}$. Then $\mathcal{E}_z \in (E)^*_{\mathbb{C}}$. If suppose $z \in H_{-p,\mathbb{C}}$, then $\forall q \in \mathbb{N}$,

$$\|\mathcal{E}_z\|^2_{-p,-q,0} = \sum_{n=0}^{\infty} (n!)^{-1} 2^{-nq} |z|_{-p}^{2n} = \exp\left\{2^{-q}|z|^2_{-p}\right\}. \tag{2.50}$$

$\mathcal{E}_z$ is called a (generalized) exponential functional, it is a Hida distribution with S-transform

$$S\mathcal{E}_z(f) = e^{\langle z, f\rangle}, \quad \forall f \in E.$$

Example 2.22 (*Gaussian measure with complex parameter*). First for $\lambda > 0$, put

$$\mu^{(\lambda)}(B) = \mu(\lambda^{-1}B) , \quad B \in \mathcal{B}(E^*) . \tag{2.51}$$

$\forall f \in E$, put

$$\begin{aligned} G(f) &= \int_{E^*} \mathcal{E}_f(x)\mu^{(\lambda)}(dx) \\ &= \int_{E^*} \exp\{\langle x, f\rangle - \tfrac{1}{2}|f|^2\}\mu^{(\lambda)}(dx) \\ &= \int_{E^*} \exp\{\langle x, \lambda f\rangle - \tfrac{1}{2}|f|^2\}\mu(dx) \\ &= \int_{E^*} \mathcal{E}_{\lambda f}(x)\mu(dx)\exp\{\tfrac{\lambda^2-1}{2}|f|^2\} \\ &= \exp\{\tfrac{\lambda^2-1}{2}|f|^2\} . \end{aligned}$$

Then by Theorem 2.9, G is the S-transform of some element in $(E)^*$. Denote $S^{-1}G$ by $F(\lambda)$. Then $F(\lambda)$ can be regarded as the *generalized Radon-Nikodym derivative* of $\mu^{(\lambda)}$ with respect to μ, and it is a Hida distribution. In this sense, we can view the Gaussian measure with parameter λ^2 as a Hida distribution.

Now for $\lambda \in \mathbb{C}$, put $G(f) = \exp\{\frac{\lambda^2-1}{2}|f|^2\}$, $f \in E$. Then G is still the S-transform of some element in $(E)^*_{\mathbb{C}}$. We denote $S^{-1}G$ by $F(\lambda)$. We shall give its chaos decomposition. Since

$$\begin{aligned} \exp\{\tfrac{\lambda^2-1}{2}|f|^2\} &= \sum_{k=0}^{\infty} \frac{(\lambda^2-1)^k}{k!2^k} |f|^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda^2-1)^k}{k!2^k} \langle \tau^{\hat{\otimes} k}, f^{\otimes 2k}\rangle, \end{aligned}$$

Comparing this with (2.8), we have

$$F(\lambda) \sim \{g_n, n \geq 0\},$$

$$g_{2k+1} = 0\,, \quad g_{2k} = \frac{(\lambda^2 - 1)^k}{k!2^k}\tau^{\widehat{\otimes} k}\,,\ k \geq 0\,. \tag{2.52}$$

Take $p \in \mathbb{N}$ such that the embedding I_{0p} from H_p to H is an H-S operator, and let $\{e_i, i \in \mathbb{N}\}$ be an orthonormal base of H_p. Then by (1.2),

$$\begin{aligned}|\tau|^2_{-p} &= \sum_{i,j=1}^{\infty} \langle \tau, e_i \otimes e_j\rangle^2 = \sum_{i,j=1}^{\infty} (e_i, e_j)^2 \\ &\leq \sum_{i,j=1}^{\infty} |e_i|^2 |e_j|^2 = \|I_{0p}\|^4_{\mathrm{H.S}}\,.\end{aligned} \tag{2.53}$$

Thus when $q \in \mathbb{R}$ and $2^q > |\lambda^2 - 1|\|I_{0p}\|^2_{\mathrm{H.S}}$, we have $F(\lambda) \in (H_{-p,-q,0})_{\mathbb{C}}$.

Example 2.23 (*δ-functional*). Motivated by the Schwartz' theory of distributions, $\forall y \in E^*$, we want to define a distribution δ_y with the property that $\forall \varphi \in (E)_{\mathbb{C}}$, $\langle\langle \varphi, \delta_y\rangle\rangle = \varphi(y)$. If such a δ_y exists, then for any $f \in E$,

$$S\delta_y(f) = \langle\langle \mathcal{E}_f, \delta_y\rangle\rangle = \mathcal{E}_f(y) = \exp\{\langle y, f\rangle - \tfrac{1}{2}|f|^2\}\,. \tag{2.54}$$

The r.h.s. of (2.54) is indeed the S-transform of some Hida distribution. We denote this distribution by δ_y (called δ-functional). Since $\{\mathcal{E}_f, f \in E\}$ is total in (E), by (2.54) and Corollary 2.19,

$$\langle\langle \varphi, \delta_y\rangle\rangle = \varphi(y)\,,\ \forall \varphi \in (E)_{\mathbb{C}},\ \forall y \in E^*_{\mathbb{C}}\,. \tag{2.55}$$

More generally, $\forall y \in E^*_{\mathbb{C}}$, the r.h.s. of (2.54) is the S-transform of some generalized Hida functional, we still denote this distribution by δ_y.

Now we are going to deduce the chaos decomposition of δ_y from its S-transform. By (2.54),

$$\begin{aligned}S\delta_y(f) &= \exp\{\langle y, f\rangle - \tfrac{1}{2}|f|^2\} \\ &= \sum_{l=0}^{\infty} \frac{\langle y^{\otimes l}, f^{\otimes l}\rangle}{l!} \sum_{k=0}^{\infty} \frac{(-1)^k |f|^{2k}}{k!2^k} \\ &= \sum_{l,k=0}^{\infty} \frac{(-1)^k}{l!k!2^k} \langle y^{\otimes l}\widehat{\otimes}\tau^{\widehat{\otimes} k}, f^{\otimes(2k+l)}\rangle \\ &= \sum_{n=0}^{\infty} \langle \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(n-2k)!k!2^k} y^{\otimes(n-2k)}\widehat{\otimes}\tau^{\widehat{\otimes} k}, f^{\otimes n}\rangle,\end{aligned}$$

which implies $\delta_y \sim \{g_n\}$. Here

$$g_n = \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(n-2k)!k!2^k} y^{\otimes(n-2k)}\widehat{\otimes}\tau^{\widehat{\otimes} k}. \tag{2.56}$$

Remark. By (2.54), $\delta_0 = F(0)$.

Example 2.24 (*Composition of a Schwartz distribution with W_f*). For $f \in H$, $f \neq 0$, $T \in S^*(\mathbb{R})$, we shall define the composition of T with W_f such that it is a generalization of the usual composition of functions. First suppose that T is a bounded Borel function. Since $T(W_f) \in (L^2)$, its S-transform can be computed by (2.7) as follows:

$$\begin{aligned}ST(W_f)(\xi) &= \int_{E^*} T(W_f)(x + \xi)\mu(dx) \\ &= \int_{E^*} T\Big(W_f + (f, \xi)\Big)\mu(dx) \\ &= \int_{\mathbb{R}} T(t + (f, \xi))p_{|f|^2}(t)dt \\ &= T * p_{|f|^2}((f, \xi))\,,\end{aligned} \tag{2.57}$$

where " $*$ " denotes the convolution, $p_{|f|^2}(t)$ denotes the density of the normal distribution with variance $|f|^2$, i.e.,

$$p_{|f|^2}(t) = \frac{1}{\sqrt{2\pi}|f|}\exp\Big\{-\frac{t^2}{2|f|^2}\Big\}\,.$$

Now let $T \in S^*(\mathbb{R})$, since $p_{|f|^2} \in S(\mathbb{R})$, $T * p_{|f|^2}$ makes sense as a tempered continuous function. Thus there exist $c_1, c_2 > 0$ such that

$$\Big|T * p_{|f|^2}(x)\Big| \leq c_1 e^{c_2|x|^2}\,.$$

Hence $\xi \mapsto T * p_{|f|^2}((f, \xi))$ is a U_0-functional on E, and by Theorem 2.9, it is the S-transform of some Hida distribution. We call this distribution the composition of T with W_f, denoted by $T(W_f)$.

If $a \in \mathbb{R}$, $T = \delta_a$, $\delta_a(W_f)$ is usually called *Donsker δ-functional* (see Chapter 2, section 4.4, Example 1).

Remark. If $T_n \in S^*(\mathbb{R})$, $n \geq 1$, and $\{T_n\}$ converges to T in $S^*(\mathbb{R})$, then from the proof of Theorem 2.10 we know that the sequence $\{T_n(W_f)\}$ converges to $T(W_f)$ in $(E)^*$. In particular, for any $T \in S^*(\mathbb{R})$, we can select $\{T_n\} \subset S(\mathbb{R})$ such that $\{T_n\}$ converges to T in $S^*(\mathbb{R})$. Thus $T(W_f)$ is the limit of the sequence $\{T_n(W_f)\}, n \geq 1\}$ in $(E)^*$. This shows that the composition of T with W_f defined above is reasonable.

Example 2.25 (*Local time of Brownian motion*). Consider the classical framework for white noise analysis: $E = S(\mathbb{R})$, $H = L^2(\mathbb{R})$, $E^* = S^*(\mathbb{R})$. For $s \geq 0$, we denote $W_{1_{[0,s]}}$ by W_s. Clearly $\{W_s, s \geq 0\}$ is a Gaussian process and $\mathbb{E}(W_t - W_s)^2 = |t - s|$. The continuous version of $\{W_s, s \geq 0\}$ is just a Brownian motion on $(S^*(\mathbb{R}), \mathcal{B}(S^*(\mathbb{R})), \mu)$. For $a \in \mathbb{R}$, the local time L^a_t of $\{W_s, s \geq 0\}$ at a can be formally defined as

$$L^a_t = \int_0^t \delta(W_s - a)ds = \int_0^t \delta_a(W_s)ds. \tag{2.58}$$

We shall prove that the above integral exists in Bochner sense. In fact, for any $\xi \in S(\mathbb{R})$, by (2.57),

$$S\delta_a(W_s)(\xi) = \frac{1}{\sqrt{2\pi s}} \exp\Big\{ -\frac{1}{2s}\Big(\int_0^s \xi(u)du - a\Big)^2 \Big\}.$$

Hence

$$\begin{aligned} |S\delta_a(W_s)(z\xi)| &\leq \frac{e^{-\frac{a^2}{2s}}}{\sqrt{2\pi s}} \exp\{\tfrac{1}{2s}(2|a|\,|z|\sqrt{s}|\xi| + |z|^2 s|\xi|^2)\} \\ &\leq \frac{e^{-\frac{a^2}{2s}}}{\sqrt{2\pi s}} \exp\{\tfrac{1}{2s}(a^2 + 2|z|^2 s|\xi|^2)\} \\ &= \frac{1}{\sqrt{2\pi s}} \exp\{|z|^2|\xi|^2\}, \qquad z \in \mathbb{C}. \end{aligned}$$

Here $|\xi|^2 = \int_{\mathbb{R}} |\xi(s)|^2 ds$. Now the conclusion follows from Theorem 2.11.

Example 2.26 (*Poisson measure*). Still consider the classical framework for white noise analysis: $E = S(\mathbb{R})$, $H = L^2(\mathbb{R})$, $E^* = S^*(\mathbb{R})$. Let $\{P_t^{(i)}, t \geq 0\}$, $i = 1, 2$, be two independent Poisson processes on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Put

$$P_t(\omega) = \begin{cases} P_t^{(1)}, & t \geq 0, \\ P_{-t}^{(2)}, & t < 0. \end{cases}$$

It is easy to prove that $\forall \xi \in S(\mathbb{R})$,

$$\mathbb{E}\Big[e^{i\int_{\mathbb{R}} \xi(t)dP_t}\Big] = \exp\Big\{ \int_{\mathbb{R}} (e^{i\xi(t)} - 1)dt \Big\}. \tag{2.59}$$

Denote by π the probability distribution of $\dot{P}$ on $S^*(\mathbb{R})$. π is called a Poisson measure. Now by (2.59),

$$\int_{S^*(\mathbb{R})} e^{i\langle x, \xi\rangle}\pi(dx) = \exp\Big\{ \int_{\mathbb{R}} (e^{i\xi(t)} - 1)dt \Big\}. \tag{2.60}$$

Put

$$F(f) = \exp\Big\{ \int_{\mathbb{R}} (e^{f(t)} - 1)dt - \frac{1}{2}\int_{\mathbb{R}} f(t)^2 dt \Big\}, \quad f \in S(\mathbb{R})_{\mathbb{C}}. \tag{2.61}$$

Obviously F is an entire analytic function on $S(\mathbb{R})_{\mathbb{C}}$. Thus by Theorem 2.13, the Poisson measure π (or, its generalized Radon-Nikodym derivative $d\pi/d\mu$ with respect to μ) can be regarded as an element of $(E)^{-1}$ with local S-transform F. By (2.61), when $\beta \in [0,1)$, F is not a U_β-functional, hence $d\pi/d\mu \notin (E)^{-\beta}$.

Example 2.27 (***Multiple intersection local time of Brownian motion***). Let $d \geq 2$. Consider the classical framework of white noise analysis $(E) \hookrightarrow (L^2) \hookrightarrow (E)^*$,

where $E = S(\mathbb{R})^d$, $H = L^2(\mathbb{R})^d$, $E^* = S^*(\mathbb{R})^d$. Let $\{B_t\} = \{B_t^1, \cdots, B_t^d\}$ be a d-dimensional Brownian motion realized on (E^*, μ), i.e., for any $f \in S(\mathbb{R})^d$,

$$\int_{-\infty}^{\infty} B_t(x) f(t)dt = \langle x, f\rangle, \quad \mu \text{ a.e. } x.$$

Let δ_0 be the Dirac measure at 0. Then the local time $L_t^0(B)$ of $\{B_t\}$ at 0 can be expressed as the Bochner integral

$$L_t^0(B) = \int_0^t \delta_0(B_s)ds.$$

It is natural to guess that the self-intersection local time of $\{B_t\}$ can be expressed as

$$\int_{0 \leq u < v \leq t} \delta_0(B_v - B_u)dudv$$

It can be proved that the above integral makes sense for $d = 2$. But for $d \geq 3$, the above integral does not exist in $(E)^*$. We now show how to "renormalize" this integral so that it makes sense.

First, let $\Lambda^{(n)}$ be the subset of d-tuple indices defined as

$$\Lambda^{(n)} = \{\alpha = (\alpha_1, \cdots, \alpha_d) : \alpha_j \in \mathbb{N}_0, 1 \leq j \leq d, \textstyle\sum_{j=1}^d \alpha_j = n\}.$$

Clearly, any $\varphi \in L^2(E^*, \mu)$ has the following chaos decomposition:

$$\varphi = \textstyle\sum_{n=0}^{\infty} \sum_{\alpha \in \Lambda^{(n)}} I_\alpha(f_\alpha),$$
$$\|\varphi\|^2 = \textstyle\sum_{n=0}^{\infty} \sum_{\alpha \in \Lambda^{(n)}} \alpha! |f_\alpha|^2,$$

where $f_\alpha \in \otimes_{i=1}^d \widehat{L^2(\mathbb{R}^{\alpha_i})}$,

$$I_\alpha(f_\alpha) = \int f_\alpha(t_1, \cdots, t_n)dB_{t_1}^1 \cdots dB_{t_{\alpha_1}}^1 \cdots dB_{t_{n-\alpha_d+1}}^d \cdots dB_{t_n}^d$$

In general, any $\varphi \in (E)^*$ has a similar chaos decomposition with $f_\alpha \in \otimes_{i=1}^d \widehat{S^*(\mathbb{R}^{\alpha_i})}$. Denote by $|\cdot|_0$ and $|\cdot|_p$ the norms on $L^2(\mathbb{R}^k)$ and on $S_p(\mathbb{R}^k)$, respectively (see Chapter I, Section 3.2).

For $a = (a_1, \cdots, a_d) \in \mathbb{R}^d$, let δ_a be the δ-function at a. It can be easily verified that the S-transform of $\delta_a(B_v - B_u)$ is given by (see Example 2.24)

$$S\delta_a(B_v - B_u)(\xi) = \frac{1}{[2\pi(v-u)]^{d/2}} \exp\Big\{ -\frac{1}{2}\sum_{j=1}^d \Big(a_j - \int_u^v \xi_j(r)dr\Big)^2 \Big\}.$$

From this it is easy to obtain the chaos decomposition of $\delta_a(B_v - B_u)$: $\delta_a(B_v - B_u) \sim \{\psi_\alpha^{(a)}(v, u), \alpha \in \Lambda\}$, where Λ is the set of all d-tuple indices,

$$\begin{aligned} &\psi_\alpha^{(a)}(u, v)(x) \\ &= [2(v-u)]^{-d/2}\pi^{-d/4}e^{-\sum_{j=1}^d x_j^2} \prod_{j=1}^d \frac{1}{\sqrt{\alpha_j!}} e_{\alpha_j}(x_j)\Big(\frac{1_{[u,v]}}{\sqrt{v-u}}\Big)^{\otimes|\alpha|}. \end{aligned}$$

Here e_k is k-th order Hermite function. Moreover we can prove that $\forall p > 0$,

$$\delta_a(B_u - B_v) \in (E)_{-p}.$$

If $a \neq 0$, then $\forall t \geq 0, p > 0$,

$$G_d^a(t) \equiv \int_{0 \leq u < v \leq t} \delta_a(B_v - B_u) du dv \in (E)_{-p}.$$

By a delicate calculation, He, Yang et al.[1] have proved that for any $t > 0$,

$$G_d(t) = \int_{0 \leq u < v < t} \Big[\delta_0(B_v - B_u) - \sum_{n=0}^{d-2} \sum_{\alpha \in \Lambda^{(n)}} I_\alpha\Big(\psi_\alpha^{(0)}(u,v)\Big)\Big] du dv$$

is a Hida distribution (called the self-intersection local time of d-dimensional Brownian motion).

Let $\Delta_k^{(t)} = \{(t_1, \cdots, t_{k+1}) \in [0,t]^{k+1} : t_1 < \cdots < t_{k+1}\}$. The $(k+1)$-tuple self-intersection local time of $\{B_t\}$ can be expressed formally as

$$\delta_d^k(t) = \int_{\Delta_k^{(t)}} \prod_{i=1}^{k} \delta_0(B_{t_{i+1}} - B_{t_i}) dt_1 \cdots dt_{k+1}.$$

For $d = 2$, Imkeller and Yan[2] have proved that the following renormalization of $\delta_2^k(t)$,

$$\tilde{\delta}_2^k(t) = \int_{\Delta_k^{(t)}} \prod_{i=1}^{k} \Big[\delta_0(B_{t_{i+1}} - B_{t_i}) - \frac{1}{2\pi(t_{i+1} - t_i)}\Big] dt_1 \cdots dt_{k+1}$$

is a Hida distribution.

§3. Products and Wick products of functionals

3.1 Products of functionals

In §1 of Chapter 2, we have established a formula for the product of two Wiener functionals $I_m(f_m)$ and $I_n(g_n)$, where $f_m \in H^{\widehat{\otimes} m}$, $g_n \in H^{\widehat{\otimes} n}$, $H = L^2(T, \mathcal{B}, \lambda)$. In the product formula, there appear contractions of f_m and g_n: $f_m \bar{\otimes}_k g_n$, $0 \leq k \leq m \wedge n$. Now we define the contraction in tensor products of Hilbert spaces.

For $\xi_i \in H_{\mathbb{C}}$, $1 \leq i \leq m$, $\eta_i \in H_{\mathbb{C}}$, $1 \leq i \leq n$, put

$$f_m = \otimes_{i=1}^m \xi_i, \quad g_n = \otimes_{i=1}^n \eta_i. \tag{3.1}$$

Let $0 \leq k \leq m \wedge n$. We define the contraction $f_m \otimes_k g_n$ of f_m with g_n as

$$f_m \otimes_k g_n = \prod_{i=1}^{k} (\xi_{m-k+i}, \eta_{n-k+i}) \Big(\otimes_{i=1}^{m-k} \xi_i\Big) \otimes \Big(\otimes_{i=1}^{n-k} \eta_i\Big). \tag{3.2}$$

Then $f_m \otimes_k g_n \in H_{\mathbb{C}}^{\otimes m+n-2k}$.

Let $\{e_i, i \geq 1\}$ be a base of $H_{\mathbb{C}}$. Put

$$e_\alpha = \otimes_{j=1}^{n} e_{\alpha_j}, \quad \alpha = (\alpha_1, \cdots \alpha_n). \tag{3.3}$$

Then $\{e_\alpha, \alpha \in N^n\}$ is a base of $H_{\mathbb{C}}^{\otimes n}$. It can be easily verified that for f_m and g_n in (3.1),

$$f_m \otimes_k g_n = \sum_{\sigma \in N^{m-k}, \delta \in N^{n-k}} \sum_{\alpha \in N^k} (f_m, e_\sigma \otimes e_\alpha)(g_n, e_\delta \otimes e_\alpha) e_\sigma \otimes e_\delta. \tag{3.4}$$

Moreover, for general $f_m \in H_{\mathbb{C}}^{\otimes m}$ and $g_n \in H_{\mathbb{C}}^{\otimes n}$, the series in the r.h.s. of (3.4) converges in $H_{\mathbb{C}}^{\otimes n+m-2k}$, and

$$| f_m \otimes_k g_n | \leq | f_m || g_n |. \tag{3.5}$$

Thus we can define the *contraction* $f_m \otimes_k g_n$ by (3.4). When $k = n$, we denote $f_m \otimes_n g_n$ by $\langle f_m, g_n \rangle$. In this way we obtain a bilinear continuous mapping from $H_{\mathbb{C}}^{\otimes m} \times H_{\mathbb{C}}^{\otimes n}$ to $H_{\mathbb{C}}^{\otimes m+n-2k}$. In particular, the contraction defined by (3.4) does not depend on the choice of bases of $H_{\mathbb{C}}$. Furthermore, for $f_m \in H^{\otimes m}$, $g_n \in H^{\otimes n}$, we can take a base of H rather than that of $H_{\mathbb{C}}$ in defining the contraction of f_m and g_n. This gives the same result.

Now for $f_m \in H_{\mathbb{C}}^{\widehat{\otimes} m}$, $g_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$, denote the symmetrization of $f_m \otimes_k g_n$ by $f_m \hat{\otimes}_k g_n$. Then

$$| f_m \hat{\otimes}_k g_n | \leq | f_m || g_n |, \tag{3.6}$$

which follows from the fact that the symmetrization does not increase the norm.

Lemma 3.1 *Let $\beta \geq 0$, $m, n \in N_0$. Then*

$$\sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} [(m+n-2k)!]^{\frac{1+\beta}{2}} \leq (m!n!)^{\frac{1+\beta}{2}} 2^{(m+n)(1+\beta)}. \tag{3.7}$$

Proof. The r.h.s. of (3.7) can be rewritten as

$$\sum_{k=0}^{m \wedge n} (k!)^{-\beta} \Big[\binom{m}{k} \binom{n}{k}\Big]^{\frac{1-\beta}{2}} \binom{m+n-2k}{n-k}^{\frac{1+\beta}{2}} (m!n!)^{\frac{1+\beta}{2}}.$$

Thus when $0 \leq \beta < 1$,

$$\begin{aligned} \text{l.h.s. of (3.7)} &\leq \sum_{k=0}^{m \wedge n} \binom{m+n}{2k}^{\frac{1-\beta}{2}} \binom{m+n-2k}{n-k}^{\frac{1+\beta}{2}} (m!n!)^{\frac{1+\beta}{2}} \\ &\leq \Big[\sum_{k=0}^{m \wedge n} \binom{m+n}{2k}\Big]^{\frac{1-\beta}{2}} \Big[\sum_{k=0}^{m \wedge n} \binom{m+n-2k}{n-k}\Big]^{\frac{1+\beta}{2}} (m!n!)^{\frac{1+\beta}{2}} \\ &\leq 2^{(m+n)(1-\beta)/2} 2^{(m+n)(1+\beta)/2} (m!n!)^{(1+\beta)/2} \\ &= 2^{m+n} (m!n!)^{(1+\beta)/2}. \end{aligned} \tag{3.8}$$

If $\beta \geq 1$, then

$$
\begin{aligned}
&\text{l.h.s. of (3.7)} \\
&\leq \sum_{k=0}^{m\wedge n} \binom{m+n-2k}{n-k}^{\frac{1-\beta}{2}} (m!n!)^{\frac{1+\beta}{2}} \\
&\leq \left[\sum_{k=0}^{m\wedge n} \binom{m+n-2k}{n-k}\right]^{\frac{1+\beta}{2}} (m!n!)^{\frac{1+\beta}{2}} \\
&\leq 2^{(m+n)(1+\beta)/2}(m!n!)^{(1+\beta)/2} . \qquad (3.9)
\end{aligned}
$$

Thus (3.7) follows from (3.8) and (3.9). ∎

Now let $f_m \in H_{\mathbb{C}}^{\widehat{\otimes} m}$, $g_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$. Similar to the proof of Proposition 1.8 in Chapter 2, we have

$$
I_m(f_m)I_n(g_n) = \sum_{k=0}^{m\wedge n} k! \binom{m}{k}\binom{n}{k} I_{m+n-2k}(f_m \widehat{\otimes}_k g_n) , \tag{3.10}
$$

where $f_m \widehat{\otimes}_k g_n$ is the symmetrization of $f_m \otimes_k g_n$.

In the sequel, let $p \in \mathbb{N}_0$, $q \in \mathbb{R}_+$, $\beta \geq 0$, and $(H_{p,q,\beta})$ be defined as (1.16).

Theorem 3.2 *If $\epsilon > 0$, $q' = q + 2(1+\beta) - \epsilon$, $\varphi, \psi \in (H_{p,q',\beta})_{\mathbb{C}}$, then $\varphi\psi \in (H_{p,q,\beta})_{\mathbb{C}}$ and*

$$
\|\varphi\psi\|_{p,q,\beta} \leq (1-2^{-\epsilon})^{-1}\|\varphi\|_{p,q',\beta}\|\psi\|_{p,q',\beta} . \tag{3.11}
$$

Proof. Let $\varphi = \sum_m I_m(f_m)$, $\psi = \sum_n I_n(g_n)$. By (3.10),(1.15) and (3.7), we have

$$
\begin{aligned}
\|\varphi\psi\|_{p,q,\beta} &\leq \sum_{m,n=0}^{\infty} \sum_{k=0}^{m\wedge n} k! \binom{m}{k}\binom{n}{k} \|I_{m+n-2k}(f_m \widehat{\otimes}_k g_n)\|_{p,q,\beta} \\
&\leq \sum_{m,n=0}^{\infty} \sum_{k=0}^{m\wedge n} k! \binom{m}{k}\binom{n}{k} [(m+n-2k)!]^{\frac{1+\beta}{2}} 2^{\frac{(m+n-2k)q}{2}} |f_m|_p |g_n|_p \\
&\leq \sum_{m,n=0}^{\infty} (m!n!)^{(1-\beta)/2} 2^{(m+n)(1+\beta)+(m+n)q/2} |f_m|_p |g_n|_p \\
&= \Big(\sum_{m=0}^{\infty} 2^{-(\epsilon m)/2} (m!)^{\frac{1+\beta}{2}} 2^{m(1+\beta-(q+\epsilon)/2)} |f_m|_p\Big) \\
&\quad \times \Big(\sum_{n=0}^{\infty} 2^{-\epsilon n/2} (n!)^{\frac{1+\beta}{2}} 2^{n(1+\beta+(q+\epsilon)/2)} |g_n|_p\Big) \\
&\leq (1-2^{-\epsilon})^{-1}\|\varphi\|_{p,q',\beta}\|\psi\|_{p,q',\beta}.
\end{aligned}
$$

∎

Remark. Let $\varphi, \psi \in (H_{p,q,\beta})_{\mathbb{C}}$, $q > 2(1+\beta)$. If $\varphi \sim \{f_m\}$, $\psi \sim \{g_n\}$, then by (3.10) and the proof of Theorem 3.2, $\varphi\psi \sim \{h_l\}$,

$$
h_l = \sum_{m+n=l} \sum_{k=0}^{\infty} k! \binom{m+k}{k}\binom{n+k}{k} f_{m+k} \widehat{\otimes}_k g_{n+k} , \tag{3.12}
$$

where the series in the r.h.s. of (3.12) converges in $H_{p,\mathbb{C}}^{\widehat{\otimes} l}$.

Corollary 3.3 *For $0 \leq \beta < \infty$, the map $\{\varphi, \psi\} \longmapsto \varphi\psi$ is continuous from $(E)_{\mathbb{C}}^{\beta} \times (E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$.*

Remark. Let $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $F \in (E)_{\mathbb{C}}^{-\beta}$. Then by Corollary 3.3, there exists a unique $G \in (E)_{\mathbb{C}}^{-\beta}$ such that

$$
\langle\langle G, \psi\rangle\rangle = \langle\langle \varphi\psi, F\rangle\rangle , \ \forall \psi \in (E)_{\mathbb{C}}^{\beta} . \tag{3.13}
$$

We call G the product of φ and F and denote it by φF.

3.2 Wick products of distributions

In general, the product of distributions is not well-defined, but the "Wick product" makes sense. First consider two exponential functionals (see Example 2.21) $\mathcal{E}_y$ and $\mathcal{E}_z$, here $y, z \in E_{\mathbb{C}}^*$. We call $\mathcal{E}_{y+z}$ the Wick product of $\mathcal{E}_y$ and $\mathcal{E}_z$. In this case, we have

$$
S\mathcal{E}_{y+z}(\xi) = S\mathcal{E}_y(\xi) S\mathcal{E}_z(\xi) \ , \forall \xi \in E . \tag{3.14}
$$

More generally, for $0 \leq \beta < 1$, φ, $\psi \in (E)_{\mathbb{C}}^{-\beta}$, put

$$
F(\xi) = S\varphi(\xi) S\psi(\xi) \ , \ \xi \in E . \tag{3.15}
$$

Then by Theorem 2.9, F is a U_β-functional. We call $S^{-1}F$ the *Wick product* of φ and ψ, and denote it by $\varphi \diamond \psi$.

The chaos decomposition of Wick product can be expressed as follows.

Theorem 3.4 *Let $0 \leq \beta < 1$, $\varphi, \psi \in (E)_{\mathbb{C}}^{-\beta}$, $\varphi \sim \{f_n\}$, $\psi \sim \{g_n\}$. Then $\varphi \diamond \psi \sim \{h_n\}$, where*

$$
h_l = \sum_{m+n=l} f_m \widehat{\otimes} g_n . \tag{3.16}
$$

Proof. By the definition of Wick product, we have

$$
\begin{aligned}
\sum_{l=0}^{\infty} \langle h_l, \xi^{\otimes l}\rangle &= S(\varphi \diamond \psi)(\xi) \\
&= S\varphi(\xi) S\psi(\xi) \\
&= \sum_{m=0}^{\infty} \langle f_m, \xi^{\otimes m}\rangle \sum_{n=0}^{\infty} \langle g_n, \xi^{\otimes n}\rangle \\
&= \sum_{l=0}^{\infty} \Big(\sum_{m+n=l} f_m \widehat{\otimes} g_n, \xi^{\otimes l}\Big) \ , \ \forall \xi \in E ,
\end{aligned}
$$

thus (3.16) follows. ∎

Example. Let $F(\lambda)$, δ_y and $\mathcal{E}_y$ be defined as in §4. Then by (2.54), $\delta_y = \mathcal{E}_y \diamond F(0)$, $\mathcal{E}_y = \delta_y \diamond F(\sqrt{2})$.

Now suppose that $\beta \ge 1$, $\varphi, \psi \in (E)_{\mathbb{C}}^{-\beta}$, $\varphi \sim \{f_n\}$, $\psi \sim \{g_n\}$. The S-transform of φ or ψ perhaps do not exist, thus we cannot define their Wick product by (3.15). But (3.16) still makes sense and one naturally asks whether the sequence $\{h_n\}$ corresponds to an element of $(E)_{\mathbb{C}}^{-\beta}$? The next theorem gives an affirmative answer.

Theorem 3.5 *Let $p \in \mathbb{Z}$, $q, r \in \mathbb{R}$, $\varphi, \psi \in (H_{p,q,r})_{\mathbb{C}}$, $\varphi \sim \{f_n\}$, $\psi \sim \{g_n\}$, and $\{h_n\}$ be defined by (3.16). Then $\forall \epsilon > (1+r)^+$, $\{h_n\}$ corresponds to an element of $(H_{p,q-\epsilon,r})_{\mathbb{C}}$, denoted by $\varphi \diamond \psi$, called the Wick product of φ and ψ. Moreover, we have*

$$\|\varphi \diamond \psi\|_{p,q-\epsilon,r} \le (1 - 2^{-\epsilon+(1+r)^+})^{-1/2}\|\varphi\|_{p,q,r}\|\psi\|_{p,q,r}. \tag{3.17}$$

Proof. By (3.16),

$$\begin{aligned}
&(n!)^{1+r}2^{n(q-\epsilon)}|h_n|_p^2 \\
&\le (n!)^{1+r}2^{n(q-\epsilon)}\Big(\sum_{k+j=n}|f_k|_p|g_j|_p\Big)^2 \\
&= 2^{-n\epsilon}\Big(\sum_{k+j=n}\binom{n}{k}^{\frac{1+r}{2}}2^{\frac{nq}{2}}(k!)^{\frac{1+r}{2}}(j!)^{\frac{1+r}{2}}|f_k|_p|g_j|_p\Big)^2 \\
&\le 2^{n[-\epsilon+(1+r)^+]}\Big(\sum_{k=0}^n 2^{kq}(k!)^{1+r}|f_k|_p^2\Big)\Big(\sum_{j=0}^n 2^{jq}(j!)^{1+r}|g_j|_p^2\Big) \\
&\le 2^{n[-\epsilon+(1+r)^+]}\|\varphi\|_{p,q,r}^2\|\psi\|_{p,q,r}^2,
\end{aligned}$$

and (3.17) follows. ∎

Corollary 3.6 *Both $(E)_{\mathbb{C}}^{-\beta}$ and $(E)_{\mathbb{C}}^{\beta}$ are closed under Wick product. Moreover, the map $\{\varphi, \psi\} \longmapsto \varphi \diamond \psi$ is a continuous bilinear form from $(E)_{\mathbb{C}}^{-\beta} \times (E)_{\mathbb{C}}^{-\beta}$ to $(E)_{\mathbb{C}}^{-\beta}$ and from $(E)_{\mathbb{C}}^{\beta} \times (E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$.*

The following theorem gives a more precise result for the case $(E)_{\mathbb{C}}^{-1}$.

Theorem 3.7 *Let $p \in \mathbb{Z}, q \in \mathbb{R}$. Then $\forall \epsilon > 0$,*

$$\|\varphi \diamond \psi\|_{p,q,-1} \le (1 - 2^{-\epsilon})^{-1/2}\|\varphi\|_{p,q+\epsilon,-1}\|\psi\|_{p,q,-1}. \tag{3.18}$$

Proof. Let $\varphi \sim \{f_n\}, \psi \sim \{g_n\}, \varphi \diamond \psi \sim \{h_n\}$. Then by (3.16),

$$\begin{aligned}
\|\varphi \diamond \psi\|_{p,q,-1}^2 &= \sum_{n=0}^\infty 2^{nq}|h_n|_p^2 \\
&\le \sum_{n=0}^\infty 2^{nq}\Big(\sum_{k=0}^n |f_k|_p|g_{n-k}|_p\Big)^2 \\
&= \sum_{n=0}^\infty 2^{nq}\sum_{k,j=0}^n |f_k|_p|f_j|_p|g_{n-k}|_p|g_{n-j}|_p \\
&= \sum_{k,j=0}^\infty 2^{\frac{k+j}{2}q}|f_k|_p|f_j|_p\sum_{n\ge k\vee j}2^{(n-\frac{k+j}{2})q}|g_{n-k}|_p|g_{n-j}|_p \\
&\le \sum_{k,j=0}^\infty 2^{\frac{k+j}{2}q}|f_k|_p|f_j|_p\sum_{n\ge k}2^{(n-k)q}|g_{n-k}|_p^2 \\
&= \Big(\sum_{k=0}^\infty 2^{kq/2}|f_k|_p\Big)^2\|\psi\|_{p,q,-1}^2 \\
&\le \Big(\sum_{k=0}^\infty 2^{-k\epsilon}\Big)\|\varphi\|_{p,q+\epsilon,-1}^2\|\psi\|_{p,q,-1}^2.
\end{aligned}$$

Hence (3.18) follows. ∎

Remark. In the classical framework for white noise analysis (see section 1.3), by (1.25), (3.18) can be rewritten as

$$\|\varphi \diamond \psi\|_{p,0,-1} \le (1 - \|A^{-1}\|^{2\epsilon})^{-1/2}\|\varphi\|_{p+\epsilon,0,-1}\|\psi\|_{p,0,-1}.$$

This is an improved form of an inequality due to Vage[1] (see also Holden-Øksendal-Ubøe-Zhang[1]).

3.3 Application to Feynman integrals

Consider the Schrödinger equation

$$\frac{\partial\psi}{\partial t} = i\Big(\frac{\Delta}{2} - V\Big)\psi, \quad \psi(0,x) = f(x), \tag{3.19}$$

where Δ is the Laplace operator on $\mathbb{R}^d$, V is a real Borel function on $\mathbb{R}^d$ (representing the potential). In 1948, Feynman conjectured that the fundamental solution of (3.19) (called propagator) can be expressed as

$$G(t,x,y) = N^{-1}\int_{\Gamma_{x,y}}\exp\Big\{i\int_0^t\Big[\frac{1}{2}|\dot\gamma(s)|^2 - V(\gamma(s))\Big]ds\Big\}\mathcal{D}(\gamma), \tag{3.20}$$

where $\Gamma_{x,y}$ is the path space connecting x and y, $\mathcal{D}(\gamma)$ is a "flat measure" on $\Gamma_{x,y}$ and N is a renormalization factor. The "integral" in (3.20) is usually called Feynman (path) integral. But the "flat measure" $\mathcal{D}(\gamma)$ makes no sense in the infinite dimensional space $\Gamma_{x,y}$, thus the integral in (3.20) has no mathematical meaning. How to give the Feynman integral a rigorous mathematical definition is a long standing open problem.

We now study Feynman integral from the viewpoint of white noise analysis. For this purpose, consider first the following heat equation:

$$\frac{\partial u}{\partial t} = \Big(\frac{\lambda}{2}\Delta - iV\Big)u, \quad u(0,x) = f(x), \tag{3.21}$$

where $\lambda > 0$. Under certain conditions, the solution of equation (3.21) is expressed by the following Feynman-Kac formula:

$$u(t,x) = \mathbb{E}\Big[f(\sqrt{\lambda}B_t + x)\exp\Big\{-i\int_0^t V(\sqrt{\lambda}B_s + x)ds\Big\}\Big], \tag{3.22}$$

where $\{B_t\}$ is the standard Brownian motion starting from 0. In terms of the Hida distribution $F(\sqrt{\lambda})$ (see Example 2.22), we can rewrite (3.22) as

$$\begin{aligned} u(t,x) &= \int f(B_t + x)e^{-i\int_0^t V(B_s+x)ds}d\mu^{(\sqrt{\lambda})} \\ &= \langle\langle F(\sqrt{\lambda})f(B_t + x)e^{-i\int_0^t V(B_s+x)ds}, 1\rangle\rangle, \end{aligned} \tag{3.23}$$

where the first equality is formal. In order to make sense the second equality, we must give the meaning of the product of the three terms therein. Let $f(x) = \delta_y(x)$. Then by (3.23) we obtain the fundamental solution of (3.21):

$$q^\lambda(t,x,y) = \langle\langle F(\sqrt{\lambda})\delta(B_t - y + x)e^{-i\int_0^t V(B_s+x)ds}, 1\rangle\rangle.$$

Suppose that q^λ has an analytic continuation in λ. Then it is natural to guess that the fundamental solution of (3.19) can be expressed as

$$G(t,x,y) = \langle\langle F(\sqrt{i})\delta(B_t - y + x)e^{-i\int_0^t V(B_s+x)ds}, 1\rangle\rangle. \tag{3.24}$$

In order to make sense of the r.h.s. of (3.24), we must prove that under some conditions imposed on V, the product appeared in (3.24) is a Hida distribution. First, express $V(B_s + x)$ as the following Bochner integral:

$$V(B_s + x) = \int_{\mathbb{R}^d} dz\, V(z)\delta(B_s + x - z).$$

For $n \in \mathbb{N}$, put $\Delta_n = \{(t_1, \cdots, t_n) : 0 < t_1 < \cdots < t_n < t\}$. Then

$$\begin{aligned} \exp\Big\{-i\int_0^t V(B_s + x)ds\Big\} &= \sum_{n=0}^{\infty}(-i)^n\int_{\Delta_n} d^nt\prod_{i=1}^n V(B_{t_i} + x) \\ &= \sum_{n=0}^{\infty}(-i)^n\int_{\Delta_n} d^nt\prod_{i=1}^n\int_{\mathbb{R}^d} dz_i V(z_i)\delta(B_{t_i} + x - z_i). \end{aligned}$$

In Khandekar-Streit[1], the authors have given a reasonable meaning of the following product as a Hida distribution:

$$F(\sqrt{i})\delta(B_t - y + x)\prod_{i=1}^n \delta(B_{t_i} + x - z_i), \tag{3.25}$$

and proved that when $d = 1$ and $V(y)dy$ is a compactly supported signed measure, the product of the three terms in the r.h.s. of (3.24) is a Hida distribution. Here $d = 1$ is essential, for in the calculation the following integral is involved:

$$M_n = \int_{\Delta_n} d^nt\prod_{i=1}^{n+1}|t_i - t_{i-1}|^{-d/2}.$$

This integral exists only when $d = 1$, and in this case, $\sum_n M_n < \infty$.

Concerning the reasonable definition of the product in (3.25), consult also Yan[10]. That G defined by (3.4) is indeed the fundamental solution of (3.19) has been proved by means of analytic Feynman integrals (see Yan[9], Theorem 5.4).

§4. Moment characterizations of distributious and positive distributions

We shall give a unified "moment characterization" of distribution space $(E)^{-\beta}$ $(0 \le \beta < +\infty)$, by means of this characterization, we shall prove that any positive distribution in $(E)_{\mathbb{F}}^{-\beta}$ $(0 \le \beta \le 1)$ can be represented by some measure.

4.1 The renormalization operator

The renormalization operator defined below will play a crucial role in the moment characterization of distribution spaces. The notion of "renormalization" has its origin in quantum physics. In the context of white noise analysis, the so-called renormalization means replacing the ordinary product by Wick product. Thus for $\varphi = \prod_{i=1}^n W_{\xi_i}, \xi_i \in H_{\mathbb{C}}$, the renormalization of φ (denoted by $R\varphi$) is

$$W_{\xi_1} \diamond \cdots \diamond W_{\xi_n} = I_n(\hat{\otimes}_{i=1}^n \xi_i).$$

We shall extend the operator R to $(E)_{\mathbb{C}}^{\beta}$ $(0 \le \beta < \infty)$. We first observe the fact: if let $\xi_i \in E_{\mathbb{C}}$, $1 \le i \le n$, $\varphi = \prod_{i=1}^n W_{\xi_i}$, and $\mathcal{E}_x$ the exponential functional defined in §4, then

$$\begin{aligned} \langle\langle R\varphi, \mathcal{E}_x\rangle\rangle &= \langle\langle I_n(\hat{\otimes}_{i=1}^n \xi_i), \frac{1}{n!}I_n(x^{\otimes n})\rangle\rangle \\ &= \prod_{i=1}^n \langle \xi_i, x\rangle = \varphi(x)\ , \ \forall x \in E^*\ . \end{aligned} \tag{4.1}$$

On the other hand, by Example 2.23 and the example following Theorem 3.4, we have

$$\langle\langle \varphi, \mathcal{E}_x \diamond F(0)\rangle\rangle = \langle\langle \varphi, \delta_x\rangle\rangle = \varphi(x)\ , \ \forall x \in E^*\ . \tag{4.2}$$

Comparing (4.1) and (4.2), we have

$$\langle\langle R\varphi, \mathcal{E}_x\rangle\rangle = \langle\langle \varphi, \mathcal{E}_x \diamond F(0)\rangle\rangle\ . \tag{4.3}$$

Since $\{\mathcal{E}_x, x \in E^*\}$ is total in $(E)^{-\beta}$ $(0 \leq \beta < \infty)$, by Theorem 3.5,

$$\langle\langle R\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond F(0)\rangle\rangle\ , \ \forall G \in (E)_{\mathbb{C}}^{-\beta}\ . \tag{4.4}$$

Furthermore, consider the functional

$$\varphi = f(W_{\xi_1}, \cdots, W_{\xi_n}),\ n \in \mathbb{N},\ \xi_1, \cdots, \xi_n \in E_{\mathbb{C}}\ , \tag{4.5}$$

where f is a polynomial, i.e., φ is a polynomial functional. We denote the space of all these functionals by $\mathcal{P}$. Clearly, $\mathcal{P} \subset \bigcap_{\beta \geq 0}(E)_{\mathbb{C}}^{\beta}$, and $\mathcal{P}$ is dense in $(E)_{\mathbb{C}}^{\beta}$. The operator R can be naturally extended to $\mathcal{P}$ by linearity. Thus (4.4) holds for any $\varphi \in \mathcal{P}$.

The following theorem shows that $\forall \beta \in [0, \infty)$, R can be extended to a continuous operator from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$. We call R the *renormalization operator*.

Theorem 4.1 *Let $0 \leq \beta < \infty$. The linear operator R defined by (4.4) can be extended to a continuous operator from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$ such that (4.4) holds. The inverse R^{-1} is also continuous. Moreover,*

$$\langle\langle R^{-1}\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond F(\sqrt{2})\rangle\rangle\ , \ \forall \varphi \in (E)_{\mathbb{C}}^{\beta},\ G \in (E)_{\mathbb{C}}^{-\beta}\ . \tag{4.6}$$

Proof. Take $p > 0$, $q \geq 0$ such that $2^q > \|I_{0p}\|_{\mathrm{HS}}^2$. Let $\epsilon > (1-\beta)^+$. Then for any $\varphi \in \mathcal{P}$, by (4.4) and (3.17),

$$\begin{aligned}
\|R\varphi\|_{p,q,\beta} &= \sup_{\|G\|_{-p,-q,-\beta}=1} \left|\langle\langle R\varphi, G\rangle\rangle\right| \\
&\leq \|\varphi\|_{p,q+\epsilon,\beta} \sup_{\|G\|_{-p,-q,-\beta}=1} \|G \diamond F(0)\|_{-p,-(q+\epsilon),-\beta} \\
&\leq \|\varphi\|_{p,q+\epsilon,\beta}(1 - 2^{-\epsilon+(1-\beta)^+})^{-1/2}\|F(0)\|_{-p,-q,-\beta}\ .
\end{aligned}$$

Hence R is continuous from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$.

We now show that R is injective from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$, and (4.6) holds. By (1.3) and (1.6), R is injective from $\mathcal{P}$ to $\mathcal{P}$. Moreover, let $\varphi = I_n(\otimes_{i=1}^n \xi_i)$, $\xi_i \in E_{\mathbb{C}}$, $1 \leq i \leq n$. Then by (4.4) (note $F(\sqrt{2}) \diamond F(0) = 1$),

$$\begin{aligned}
\langle\langle R^{-1}\varphi, \mathcal{E}_x\rangle\rangle &= \langle\langle R^{-1}\varphi, \mathcal{E}_x \diamond F(\sqrt{2}) \diamond F(0)\rangle\rangle \\
&= \langle\langle \varphi, \mathcal{E}_x \diamond F(\sqrt{2})\rangle\rangle\ .
\end{aligned}$$

Since $\{\mathcal{E}_x, x \in E_{\mathbb{C}}^*\}$ is total in $(E)_{\mathbb{C}}^{-\beta}$ $(0 \leq \beta < \infty)$, by Theorem 3.5, (4.6) holds for any $\varphi \in \mathcal{P}$ and $G \in (E)_{\mathbb{C}}^{-\beta}$. R^{-1} can be treated similarly as R. Moreover, (4.6) holds. ∎

Remark 1. Let $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $\varphi \sim \{f_n\}$, $R\varphi \sim \{g_n\}$, $R^{-1}\varphi \sim \{h_n\}$. The reader may verify the following two formulas:

$$g_n = \frac{1}{n!}\sum_{k=0}^{\infty}(-1)^k \frac{(n+2k)!}{k!2^k}\langle\tau^{\otimes k}, f_{n+2k}\rangle\ , \tag{4.7}$$

$$h_n = \frac{1}{n!}\sum_{k=0}^{\infty} \frac{(n+2k)!}{k!2^k}\langle\tau^{\otimes k}, f_{n+2k}\rangle\ . \tag{4.8}$$

Here both series converge in $E_{\mathbb{C}}^{\widehat{\otimes} n}$, $\langle\tau^{\otimes k}, f_{n+2k}\rangle$ is an element of $E_{\mathbb{C}}^{\widehat{\otimes} n}$ such that $\forall w_n \in E^{*\widehat{\otimes} n}_{\mathbb{C}}$,

$$\langle\langle\tau^{\otimes k}, f_{n+2k}\rangle, w_n\rangle = \langle\tau^{\otimes k} \otimes w_n, f_{n+2k}\rangle\ .$$

Remark 2. Let $\varphi \in (E)_{\mathbb{C}}^{\beta}$. Take $p_0 \in \mathbb{N}$ such that the embedding from H_{p_0} to H is a trace class operator. Then by (1.4), (1.5) and Theorem 2.18, we may prove

$$R\varphi(x) = \int_{E^*} \varphi(x + iy)\mu(dy)\ ,\ x \in H_{-p_0}\ . \tag{4.9}$$

$$R^{-1}\varphi(x) = \int_{E^*} \varphi(x + y)\mu(dy)\ ,\ x \in H_{-p_0}\ . \tag{4.10}$$

Remark 3. Let $p > 0$, $q \geq 0$ such that $2^q > \|I_{0p}\|_{\mathrm{HS}}^2$. If $\epsilon > (1-\beta)^+$, then by the proof of Theorem 4.1, R can be extended to a bounded operator from $(H_{p,q+\epsilon,\beta})_{\mathbb{C}}$ to $(H_{p,q,\beta})_{\mathbb{C}}$.

4.2 Moment characterization of distribution spaces

We assume $0 \leq \beta < \infty$ in this subsection.

Definition 4.2 Let $\Phi \in (E)_{\mathbb{C}}^{-\beta}$, $M_0^{\Phi} = \langle\langle\Phi, 1\rangle\rangle$. For $n \geq 1$, let M_n^{Φ} be the symmetric n-linear form on E^n defined as

$$M_n^{\Phi}(f_1, \cdots, f_n) = \langle\langle\Phi, W_{f_1}\cdots W_{f_n}\rangle\rangle\ ,\ f_1, \cdots, f_n \in E\ . \tag{4.11}$$

M_n^{Φ} is called the *n-th moment* of Φ.

The following theorem gives the moment characterization of $(E)_{\mathbb{C}}^{-\beta}$.

Theorem 4.3 *Let $0 \leq \beta < \infty$, $M_0 \in \mathbb{C}$, $\forall n \geq 1$, M_n be a symmetric n-linear form on E^n, $\widehat{M}_n$ be the n-homogeneous polynomial corresponding to M_n. In order that there exists a $\Phi \in (E)_{\mathbb{C}}^{-\beta}$ such that $\forall n \geq 0$, M_n is the n-th moment of Φ, it is necessary and sufficient that there exist $p \geq 0$, $C > 0, K > 0$, such that $\forall n \in \mathbb{N}$, $\forall f \in E$,*

$$|\,\widehat{M}_n(f)\,| \leq KC^n(n!)^{\frac{1+\beta}{2}}\,|\,f\,|_p^n\ . \tag{4.12}$$

Proof. Necessity. If there exists $\Phi \in (E)_{\mathbb{C}}^{-\beta}$ such that $M_n = M_n^{\Phi}$, then there exist $p \geq 0, q \geq 0$, such that $\Phi \diamond F(\sqrt{2}) \in (H_{-p,-q,-\beta})_{\mathbb{C}}$. Hence by (4.11) and (4.6),

$$\begin{aligned}
|\,\widehat{M}_n(f)\,| &= \left|\langle\langle\Phi, R^{-1}\left(I_n(f^{\otimes n})\right)\rangle\rangle\right| \\
&= \left|\langle\langle\Phi \diamond F(\sqrt{2}), I_n(f^{\otimes n})\rangle\rangle\right| \\
&\leq \|\Phi \diamond F(\sqrt{2})\|_{-p,-q,-\beta}\|I_n(f^{\otimes n})\|_{p,q,\beta} \\
&\leq \|\Phi \diamond F(\sqrt{2})\|_{-p,-q,-\beta}(n!)^{\frac{1+\beta}{2}}2^{\frac{nq}{2}}|f|_p^n\ .
\end{aligned}$$

and (4.12) follows.

Sufficiency. Assume that there exist $p \geq 0, C > 0, K > 0$ such that (4.12) holds. Take $p' > p$ such that the embedding $I_{pp'}$ from $H_{p'}$ to H_p is an H-S operator. Then by the nuclear theorem (Theorem 3.17 in Chapter 1), there exists $M^{(n)} \in H_{-p',\mathbb{C}}^{\hat{\otimes} n}$ such that

$$M_n(f_1, \cdots, f_n) = \langle M^{(n)}, f_1 \otimes \cdots \otimes f_n \rangle , \ n \geq 1 , \tag{4.13}$$

and by (4.12) (see the proof of lemma 2.7 and 2.8),

$$|M^{(n)}|_{-p'}^2 \leq K^2 e^{2n} C^{2n} (n!)^{1+\beta} \|I_{pp'}\|_{\mathrm{HS}}^{2n} . \tag{4.14}$$

Let $\Psi \sim \{(n!)^{-1} M^{(n)}\}$ with $M^{(0)} \equiv M_0$. Then $\Psi \in (E)_{\mathbb{C}}^{-\beta}$. In fact, by (4.14), when $q > 0$ and $2^q > e^2 C^2 \|I_{pp'}\|_{\mathrm{HS}}^2$, $\Psi \in (H_{-p',-q,-\beta})_{\mathbb{C}}$. Let $\Phi = \Psi \circ F(0)$. Then $\langle\langle \Phi, 1 \rangle\rangle = M_0$. Now (4.13) and (4.4) imply that $\forall n \geq 1, f_k \in E_{\mathbb{C}}, 1 \leq k \leq n$,

$$\begin{aligned} \langle\langle \Phi, W_{f_1} \cdots W_{f_n} \rangle\rangle &= \langle\langle \Psi \circ F(0), R^{-1} I_n(\hat{\otimes}_{k=1}^n f_k) \rangle\rangle \\ &= \langle\langle \Psi, I_n(\hat{\otimes}_{k=1}^n f_k) \rangle\rangle \\ &= \langle M^{(n)}, \hat{\otimes}_{k=1}^n f_k \rangle \\ &= M_n(f_1, \cdots, f_n) . \end{aligned}$$

This means that M_n is the n-th moment of Φ. ∎

Remark. Let M_n be a symmetric n-linear form on E^n. By the nuclear theorem, there exists $M^{(n)} \in E^{*\hat{\otimes} n}$ such that (4.13) holds. $M^{(n)}$ is called the *kernel* of M_n.

The following theorem gives an estimate of the norm of a distribution by means of the moment estimate (4.12).

Theorem 4.4 *Let $0 \leq \beta < \infty$, $\Phi \in (E)_{\mathbb{C}}^{-\beta}$, $\{M_n, n \geq 0\}$ the moment sequence of Φ. Assume that (4.12) holds and take $p' > p$ such that $\|I_{pp'}\|_{\mathrm{HS}} < \infty$. Then when $\epsilon > (1-\beta)^{-1}$ and $q > \log_2[\|I_{0p}\|_{\mathrm{HS}}^2 \vee e^2C^2\|I_{pp'}\|_{\mathrm{HS}}^2]$, we have*

$$\begin{aligned} &\|\Phi\|_{-p',-(q+\epsilon),-\beta} \\ &\leq K[(1-2^{-\epsilon(1-\beta)})(1-2^{-q}\|I_{0p}\|_{\mathrm{HS}}^2)(1-2^{-q}e^2C^2\|I_{pp'}\|_{\mathrm{HS}}^2)]^{-1/2}. \end{aligned} \tag{4.15}$$

Proof. Let $M^{(n)}$ be the kernel of M_n, $\Psi \sim \{(n!)^{-1}M^{(n)}\}$. Then by (4.14),

$$\|\Psi\|_{-p',-q,-\beta} \leq K(1-2^{-q}e^2C^2\|I_{pp'}\|_{\mathrm{HS}}^2)^{-1/2} . \tag{4.16}$$

By theorem 4.3, $\Phi = \Psi \circ F(0)$. Hence (4.15) follows from (4.16) and (3.17). ∎

The following two results are important corollaries of Theorem 4.4. We omit their proof (see the proofs of Theorems 2.10 and 2.11).

Theorem 4.5 *Let $0 \leq \beta < \infty$, $\{\Phi_k, k \in \mathbb{N}\}$ be a sequence in $(E)_{\mathbb{C}}^{-\beta}$. If the following conditions are satisfied:*

1) $\forall f \in E, \forall n \geq 0$, $\{\langle\langle \Phi_k, W_f^n \rangle\rangle, k \in \mathbb{N}\}$ is a Cauchy sequence in $\mathbb{C}$;

2) there exist $p \in \mathbb{N}_0$, $K > 0$ and $C > 0$ such that

$$|\langle\langle \Phi_k, W_f^n \rangle\rangle| \leq K C^n (n!)^{\frac{1+\beta}{2}} |f|_p^n , \ \forall f \in E, n \geq 0, k \geq 1, \tag{4.17}$$

then $\{\Phi_k, k \in \mathbb{N}\}$ converges strongly in $(E)_{\mathbb{C}}^{-\beta}$.

Theorem 4.6 *Let $0 \leq \beta < \infty$, $(\Omega, \mathcal{F}, \nu)$ be a measure space, $\omega \mapsto \Phi_\omega$ a map from Ω to $(E)_{\mathbb{C}}^{-\beta}$. If the following are satisfied:*

1) $\forall f \in E, n \geq 0$, $\omega \longmapsto \langle\langle \Phi_\omega, W_f^n \rangle\rangle$ is a measurable map on $(\Omega, \mathcal{F})$;

2) there exist constants $K > 0, C > 0, p \in \mathbb{N}_0$ and a ν-integrable function $G(\omega)$ on Ω such that for ν-a.e. ω,

$$|\langle\langle \Phi_\omega, W_f^n \rangle\rangle| \leq K G(\omega) C^n (n!)^{\frac{1+\beta}{2}} |f|_p^n , \forall f \in E, n \geq 0 , \tag{4.18}$$

then there exist $p', q \in \mathbb{N}$ such that $\omega \longmapsto \Phi_\omega$ is Bochner integrable on $(H_{-p',-q,-\beta})$ and

$$\langle\langle \int_\Omega \Phi_\omega d\nu(\omega), W_f^n \rangle\rangle = \int_\Omega \langle\langle \Phi_\omega, W_f^n \rangle\rangle d\nu(\omega), \forall f \in E, n \geq 0 . \tag{4.19}$$

4.3 Measure representation of positive distributions

In classical theory of Schwartz distributions, positive distributions are in one-to one correspondence to certain class of Borel measures. We shall generalize this result to the infinite dimensional case and prove that : for any $\beta \in [0,1]$, any positive distribution $\Phi \in (E)^{-\beta}$ corresponds to a finite measure ν on $(E^*, \mathcal{B}(E^*))$ such that

$$\int_{E^*} \varphi(x)\nu(dx) = \langle\langle \Phi, \varphi \rangle\rangle , \ \forall \varphi \in (E)^\beta . \tag{4.20}$$

Moreover, for $0 \leq \beta < -\infty$, we shall give the characterization of the measures ν which correspond to positive distributions through (4.20).

Definition 4.7 Let $\Phi \in (E)^{-\beta}$. If $\langle\langle \Phi, \varphi \rangle\rangle \geq 0$ for any non-negative testing functional $\varphi \in (E)^\beta$, then Φ is called a *positive distribution*.

In what follows, we denote by $(E)_+^\beta$ the set of non-negative elements in $(E)^\beta$ and by $(E)_+^{-\beta}$ the set of positive distribution in $(E)^{-\beta}$.

Example. Let $y \in E^*$. Then $\delta_y \in (E)_+^*$ since $\forall \varphi \in (E)_+$, $\langle\langle \varphi, \delta_y \rangle\rangle = \varphi(y) \geq 0$.

The following lemma follows easily from Theorem 2.2 of Chapter 5 in Berezansky-Kondratiev[1], we omit the proof.

Lemma 4.8 *Let $0 \leq \beta \leq 1$, $M_0 \in \mathbb{R}$, $\forall n \geq 1$, let M_n be a symmetric n-linear form on E^n, and $\widehat{M}_n$ the corresponding n-polynomial associated with M_n such that $\forall f \in E$, $\widehat{M}_n(f)$ satisfies (4.12). If the sequence $\{M_n, n \geq 0\}$ is positive definite, i.e., $M_0 \geq 0$, and for any $n \geq 1$, $g_k \in E^{\hat{\otimes} k}, 1 \leq k \leq n$,*

$$\sum_{k,j=1}^{n} \langle M^{(k+j)}, g_k \otimes g_j \rangle \geq 0 , \tag{4.21}$$

where $M^{(n)}$ is the kernel of M_n, then there exists a unique finite Borel measure ν on $(E^*, \mathcal{B}(E^*))$ such that $\forall n \geq 0$, M_n is the n th moment of ν, i.e.,

$$\int_{E^*} W_{f_1} \cdots W_{f_n} \nu(dx) = M_n(f_1, \cdots, f_n) \ , \ \forall f_1, \cdots, f_n \in E \ . \tag{4.22}$$

The following theorem shows that for $0 \leq \beta \leq 1$ any positive distribution in $(E)^{-\beta}$ can be represented by a finite measure on $(E^*, \mathcal{B}(E^*))$.

Theorem 4.9 *Let $0 \leq \beta \leq 1$, $\Phi \in (E)_+^{-\beta}$. Then there exists a unique finite measure ν on $(E^*, \mathcal{B}(E^*))$ such that (4.20) holds.*

Proof. Let

$$\varphi(x) = \sum_{k=0}^{n} \langle g_k, x^{\otimes k} \rangle \ , \ x \in E^* \ .$$

Then by Theorem 2.18, $\varphi \in (E)^{\beta}$. By the positivity of Φ,

$$\sum_{k,j=1}^{n} \langle M^{(k+j)}, g_k \otimes g_j \rangle = \langle\langle \Phi, \varphi^2 \rangle\rangle \geq 0 \ ,$$

where $M^{(n)}$ is the kernel of M_n^{Φ}. By Lemma 4.8, there exists a unique finite measure ν on $(E^*, \mathcal{B}(E^*))$ such that $\forall n > 0$, M_n^{Φ} is the n-th moment of ν, i.e., (4.22) holds. Now let $\varphi \in (E)^{\beta}$ and take a sequence $\{\varphi_n, n \geq 1\}$ in $\mathcal{P}$ such that $\{\varphi_n\}$ converges strongly to φ in $(E)^{\beta}$. By Corollary 3.3, $\{\varphi_n^2\}$ converges strongly to φ^2 in $(E)^{\beta}$. Assume $\Phi \in (H_{-p,-q,-\beta})$. Since $\{\varphi_n^2\}$ converges to φ^2 in $(H_{p,q,\beta})$, we have

$$\begin{aligned} \lim_{n\to\infty} \int_{E^*} \varphi_n^2 d\nu &= \lim_{n\to\infty} \langle\langle \Phi, \varphi_n^2 \rangle\rangle \\ &= \langle\langle \Phi, \varphi^2 \rangle\rangle < \infty \ . \end{aligned}$$

This means that the sequence $\{\varphi_n, n \geq 1\}$ is uniformly integrable with respect to ν. On the other hand,

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \langle\langle \varphi_n, \delta_x \rangle\rangle = \langle\langle \varphi, \delta_x \rangle\rangle = \varphi(x) \ , \forall x \in E^* \ .$$

Thus by (4.22), we have

$$\int_{E^*} \varphi d\nu = \lim_{n\to\infty} \int_{E^*} \varphi_n d\nu = \lim_{n\to\infty} \langle\langle \Phi, \varphi_n \rangle\rangle = \langle\langle \Phi, \varphi \rangle\rangle \ .$$

The theorem is proved. ∎

Remark. If $0 \leq \beta < 1$, then $\forall f \in E$, the exponential functional $\mathcal{E}_f \in (E)^{\beta}$, and we can also prove Theorem 4.9 by using the Minlos theorem. But for $\beta = 1$, we must use Lemma 4.8. The proof presented here applies to both cases.

In practical applications, it is important to characterize the measures on $(E^*, \mathcal{B}(E^*))$ which correspond to positive distributions through (4.20). The following theorem gives such a characterization.

Theorem 4.10 *Let $0 \leq \beta < +\infty$, ν be a finite Borel measure on $(E^*, \mathcal{B}(E^*))$. In order that there exists $\Phi \in (E)_+^{-\beta}$ such that (4.20) holds, it is necessary and sufficient that ν satisfies*

(C.1) *there exist $p \geq 0, C > 0, K > 0$ such that*

$$\int_{E^*} \langle f, x \rangle^{2n} \nu(dx) \leq K C^{2n} ((2n)!)^{\frac{1+\beta}{2}} |f|_p^{2n} \ , \ \forall f \in E \ , \forall n \geq 0. \tag{4.23}$$

Proof. By Theorem 4.3, the necessity is obvious. We now prove the sufficiency. Assume (C.1) holds. By the Schwarz inequality and the inequality $(2n)! \leq 2^{2n}(n!)^2$, we have (note $\nu(E^*) \leq K$)

$$\begin{aligned} \int_{E^*} |\langle f, x \rangle|^n \nu(dx) &\leq \nu(E^*)^{1/2} \Big[\int_{E^*} \langle f, x \rangle^{2n} \nu(dx) \Big]^{1/2} \\ &\leq K (C 2^{\frac{1+\beta}{2}})^n (n!)^{\frac{1+\beta}{2}} |f|_p^n \ . \end{aligned} \tag{4.24}$$

Let M_n be the n-th moment of ν, i.e.

$$M_n(f_1, \cdots, f_n) = \int_{E^*} \prod_{i=1}^{n} \langle f_i, x \rangle \nu(dx) \ , \ \forall f_1, \cdots, f_n \in E \ .$$

Then by (4.24) and the polarization formula, $\forall f_1, \cdots, f_n \in E$,

$$\begin{aligned} |M_n(f_1, \cdots, f_n)| &\leq K C_1^n \frac{n^n}{n!} (n!)^{\frac{1+\beta}{2}} \prod_{j=1}^{n} |f_j|_p \\ &< K (eC_1)^n (n!)^{\frac{1+\beta}{2}} \prod_{j=1}^{n} |f_j|_p \ , \end{aligned} \tag{4.25}$$

where $C_1 = C 2^{\frac{1+\beta}{2}}$. By Theorem 4.3, there exists $\Phi \in (E)^{-\beta}$ such that M_n is the n-th moment of Φ, i.e.

$$\langle\langle \Phi, \prod_{i=1}^{n} \langle f_i, \cdot \rangle \rangle\rangle = \int_{E^*} \prod_{i=1}^{n} \langle f_i, x \rangle \nu(dx) \ .$$

Thus we further have

$$\langle\langle \Phi, \varphi \rangle\rangle = \int_{E^*} \varphi d\nu \ , \ \forall \varphi \in \mathcal{P} \ , \tag{4.26}$$

where $\mathcal{P}$ is the set of all polynomial smooth functionals. Now (4.20) follows from the proof of Theorem 4.9. In particular, this implies $\Phi \in (E)_+^{-\beta}$. ∎

The following theorem gives an equivalent form of the condition (C.1) which seems to be more convenient for the practical use.

Theorem 4.11 *Take any $p_0 \in \mathbb{N}$ such that the embedding from H_{p_0} to H is of trace class ($\mu(H_{-p_0}) = 1$). Let ν be a finite Borel measure on $(E^*, \mathcal{B}(E^*))$. Then condition (C.1) in Theorem 4.10 is equivalent to*

(C.1)$'$ *there exist $p\ge p_0, \epsilon>0$ such that*

$$\int_{E^*}\exp\Big\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\Big\}\nu(dx)<\infty. \tag{4.27}$$

Proof. Assume (C.1) holds. Take $p'>p_0, p'>p$ such that the embedding $I_{pp'}$ from $H_{p'}$ to H_p is a Hilbert-Schmidt operator. Let $\{e_j, j\ge 1\}\subset E$ be an orthonormal base of $H_{p'}$. Then by (4.23),

$$\begin{aligned}\int_{E^*}|x|_{-p'}^{2n}\nu(dx)&=\int_{E^*}\Big(\sum_{j=1}^\infty\langle x,e_j\rangle^2\Big)^n\nu(dx)\\&=\sum_{k_1=1}^\infty\cdots\sum_{k_n=1}^\infty\int_{E^*}\langle x,e_{k_1}\rangle^2\cdots\langle x,e_{k_n}\rangle^2\nu(dx)\\&\le K(eC_1)^{2n}[(2n)!]^{\frac{1+\beta}{2}}\sum_{k_1=1}^\infty\cdots\sum_{k_n=1}^\infty\prod_{j=1}^n|e_{k_j}|_p^2\\&=K(eC_1)^{2n}[(2n)!]^{\frac{1+\beta}{2}}\|I_{pp'}\|_{HS}^{2n}.\end{aligned} \tag{4.28}$$

Hence when $0<\epsilon<\frac{1}{2}(eC\|I_{pp'}\|_{HS})^{\frac{2}{1+\beta}}$, from (4.28) we get

$$\begin{aligned}\int_{E^*}\exp\Big\{\epsilon|x|_{-p'}^{\frac{2}{1+\beta}}\Big\}\nu(dx)&=\sum_{n=0}^\infty\frac{\epsilon^n}{n!}\int_{E^*}|x|_{-p'}^{\frac{2n}{1+\beta}}\nu(dx)\\&\le\sum_{n=0}^\infty\frac{\epsilon^n}{n!}[\nu(E^*)]^{\frac{\beta}{1+\beta}}\Big(\int_{E^*}|x|_{-p'}^{2n}\nu(dx)\Big)^{\frac{1}{1+\beta}}\\&\le[\nu(E^*)]^{\frac{\beta}{1+\beta}}\sum_{n=0}^\infty\frac{\epsilon^n}{n!}\Big(eC\|I_{pp'}\|_{HS}\Big)^{\frac{2n}{1+\beta}}[(2n)!]^{\frac{1}{2}}\\&\le[\nu(E^*)]^{\frac{\beta}{1+\beta}}\sum_{n=0}^\infty[2\epsilon(eC\|I_{pp'}\|_{HS})^{\frac{2}{1+\beta}}]^n<\infty.\end{aligned}$$

Thus (C.1)$\Rightarrow$(C.1)$'$.

Conversely, assume (C.1)$'$ holds. By Theorem 2.18, the following functional L defined on $(E)^\beta$ is continuous:

$$L(\varphi)=\int_{E^*}\varphi(x)\nu(dx),\ \varphi\in(E)^\beta.$$

Thus there exists $\Phi\in(E)_+^{-\beta}$ such that (4.20) holds. Now (C.1) follows from Theorem 4.10. ∎

Definition 4.12 Let $0\le\beta<+\infty$, ν a finite Borel measure on $(E^*,\mathcal{B}(E^*))$ satisfying (C.1)$'$ (or (C.1) of Theorem 4.10). We call the positive distribution Φ associated with ν (i.e., Φ satisfies (4.20)) the *generalized Radon-Nikodym derivative* of ν with respect to the Gaussian measure μ and denote it by $d\nu/d\mu$.

Henceforth we denote by $\mathcal{M}^\beta(E^*)$ the set of all measures on $(E^*,\mathcal{B}(E^*))$ satisfying (C.1) or (C.1)$'$.

Theorem 4.13 *Let $0\le\beta<+\infty$, ν a finite Borel measure on $(E^*,\mathcal{B}(E^*))$.*

1) *Assume ν satisfies* (C.1). *Let $p'>p$ be such that $\|I_{pp'}\|_{HS}<\infty$. If $\epsilon>(1-\beta)^+$ and*

$$q>\log_2(\|I_{0p'}\|_{HS}^2)\vee e^2C^22^{1+\beta}\|I_{pp'}\|_{HS}^2,$$

then

$$\begin{aligned}\|\frac{d\nu}{d\mu}\|_{-p',-(q+\epsilon),-\beta}&\le K\nu(E^*)^{1/2}(1-2^{-\epsilon+(1-\beta)^+})^{-1/2}\\&\times\Big[(1-2^{-q}\|I_{0p'}\|_{HS}^2)(1-2^{1+\beta-q}e^2C^2\|I_{pp'}\|_{HS}^2)\Big]^{-1/2}.\end{aligned} \tag{4.29}$$

2) *Assume ν satisfies* (C.1)$'$. *If q is sufficiently large such that* (2.42) *holds and $\frac{1+\beta}{2}2^{\frac{(1-\beta)^+}{1+\beta}-q}\le\epsilon$, then*

$$\begin{aligned}\|\frac{d\nu}{d\mu}\|_{-p,-q,-\beta}&\le\int_{E^*}\exp\Big\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\Big\}\nu(dx)\\&\times\Big(\int_{E^*}\exp\Big\{(1+\beta)2^{\frac{(1-\beta)^+-q}{1+\beta}}|x|_{-p}^{\frac{2}{1+\beta}}\Big\}\mu(dx)\Big)^{1/2}.\end{aligned} \tag{4.30}$$

Proof. 1) follows from (4.24) and Theorem 4.4. It remains to prove 2). Let $\varphi\in(E)^\beta$. By (2.43),

$$|\varphi(x)|\le\|\varphi\|_{p,q,\beta}C_1\exp\Big\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\Big\},$$

where C_1 is given by (2.42). Thus we have

$$\begin{aligned}\Big|\langle\langle\frac{d\nu}{d\mu},\varphi\rangle\rangle\Big|&=\Big|\int_{E^*}\varphi(x)\nu(dx)\Big|\\&\le\|\varphi\|_{p,q,\beta}C_1\int_{E^*}\exp\Big\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\Big\}\nu(dx),\end{aligned}$$

and (4.30) follows. ∎

As a consequence of Theorem 4.13 and 4.9, we have

Theorem 4.14 *Let $0\le\beta\le1$, $\{\nu_n,n\ge1\}$ a sequence of finite measures on $(E^*,\mathcal{B}(E^*))$. If*

(1) *$\forall f\in E,k\in\mathbb{N}_0$, $\{\nu_n(W_f^k),n\ge1\}$ is a Cauchy sequence in $\mathbb{R}$;*

(2) *any of the following equivalent condition holds:*

(2a) *there exist $p\ge0,K>0,C>0$ such that*

$$\Big|\int_{E^*}\langle f,x\rangle^{2k}\nu_n(dx)\Big|\le K^2C^{2k}\Big((2k)!\Big)^{\frac{1+\beta}{2}}|f|_p^{2k},$$
$$\forall f\in E,k\ge1,n\ge1, \tag{4.31}$$

(2b) *there exist $p > 0, \epsilon > 0, K > 0$ such that*

$$\int_{E^*} \exp\{\epsilon|x|_{-p}^{\frac{2}{1+\beta}}\}\nu_n(dx) \le K, \ \forall n \ge 1, \tag{4.32}$$

then there exists a finite measure ν on $(E^*, \mathcal{B}(E^*))$ such that

$$\lim_{n\to\infty} \int_{E^*} \varphi d\nu_n = \int_{E^*} \varphi d\nu, \ \forall \varphi \in (E)^\beta. \tag{4.33}$$

Proof. On the one hand, condition (1) implies that (4.33) holds for any $\varphi \in \mathcal{P}$. On the other hand, by Theorem 4.13, condition (2a) or (2b) implies that the sequence $\{d\nu_n/d\mu, n \ge 1\}$ is bounded in $(E)^{-\beta}$. Thus $\{d\nu_n/d\mu, n \ge 1\}$ converges strongly in $(E)^{-\beta}$ to a limit Φ. Clearly Φ is positive, hence by Theorem 4.9, Φ corresponds to a finite measure ν on $(E^*, \mathcal{B}(E^*))$, i.e., $\Phi = d\nu/d\mu$. Thus (4.33) holds. ∎

Let ν_1 and ν_2 be two finite measures on $(E^*, \mathcal{B}(E^*))$. Put

$$\nu_1 * \nu_2(B) = \int_{E^*} \nu_1(B - x)\nu_2(dx), \ \forall B \in \mathcal{B}(E^*). \tag{4.34}$$

$\nu_1 * \nu_2$ is called the *convolution* of ν_1 and ν_2.

The following theorem shows that $\mathcal{M}^\beta(E^*)$ is closed under convolution.

Theorem 4.15 *Let $0 \le \beta < +\infty$, $\nu_1, \nu_2 \in \mathcal{M}^\beta(E^*)$. Then $\nu_1 * \nu_2 \in \mathcal{M}^\beta(E^*)$, and*

$$\frac{d(\nu_1 * \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} \diamond \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), \tag{4.35}$$

where $F(\sqrt{2})$ is defined in Example 2.22.

Proof. Let $\Phi \in (E)^{-\beta}$ and M_n^Φ be its n-th moment. Then $\forall f \in E$,

$$\widehat{M}_n^\Phi(f) = \langle\langle \Phi, W_f^n \rangle\rangle = \langle\langle \Phi \diamond F(\sqrt{2}), I_n(f^{\otimes n})\rangle\rangle. \tag{4.36}$$

We denote by M_n^1, M_n^2 and M_n the n-th moments of ν_1, ν_2 and $\nu_1 * \nu_2$, respectively.

Now $\forall f \in E$, by (4.36) and (3.16),

$$\begin{aligned}
\widehat{M}_n(f) &= \int_{E^*} \langle f, x\rangle^n (\nu_1 * \nu_2)(dx) \\
&= \int_{E^*}\int_{E^*} \langle f, x + y\rangle^n \nu_1(dx)\nu_2(dy) \\
&= \int_{E^*}\int_{E^*} (\langle f, x\rangle + \langle f, y\rangle)^n \nu_1(dx)\nu_2(dy) \\
&= \sum_{k=0}^n \binom{n}{k} \widehat{M}_k^1(f)\widehat{M}_{n-k}^2(f) \\
&= \sum_{k=0}^n \binom{n}{k} \langle\langle \frac{d\nu_1}{d\mu} \diamond F(\sqrt{2}), I_k(f^{\otimes k})\rangle\rangle \\
&\qquad \cdot \langle\langle \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), I_{n-k}(f^{\otimes n-k})\rangle\rangle \\
&= \langle\langle \frac{d\nu_1}{d\mu} \diamond F(\sqrt{2}) \diamond \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), I_n(f^{\otimes n})\rangle\rangle \\
&= \langle\langle \frac{d\nu_1}{d\mu} \diamond \frac{d\nu_2}{d\mu} \diamond F(\sqrt{2}), W_f^n\rangle\rangle,
\end{aligned}$$

from which we conclude the theorem. ∎

Theorem 4.16 *Let $0 \le \beta < +\infty$, $\nu \in \mathcal{M}^\beta(E^*)$, $y \in E^*$, and*

$$\nu_y(B) = \nu(B - y), \ \forall B \in \mathcal{B}(E^*).$$

Then $\nu_y \in \mathcal{M}^\beta(E^*)$ and

$$\frac{d\nu_y}{d\mu} = \mathcal{E}_y \diamond \frac{d\nu}{d\mu}. \tag{4.37}$$

In particular, we have $d\mu_y/d\mu = \mathcal{E}_y$.

Proof. We have

$$\begin{aligned}
\widehat{M}_n^{\nu_y}(f) &= \int_{E^*} \langle f, x\rangle^n \nu_y(dx) = \int_{E^*} \langle f, x + y\rangle^n \nu(dx) \\
&= \int_{E^*} \Big(\langle f, x\rangle + \langle f, y\rangle\Big)^n \nu(dx) \\
&= \sum_{k=0}^n \binom{n}{k} \widehat{M}_k^\nu(f)\langle f, y\rangle^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k} \langle\langle \frac{d\nu}{d\mu} \diamond F(\sqrt{2}), I_k(f^{\otimes k})\rangle\rangle \langle\langle \mathcal{E}_y, I_{n-k}(f^{\otimes n-k})\rangle\rangle \\
&= \langle\langle \frac{d\nu}{d\mu} \diamond F(\sqrt{2}) \diamond \mathcal{E}_y, I_n(f^{\otimes n})\rangle\rangle.
\end{aligned}$$

Thus (4.37) follows. ∎

4.4 Application to $P(\phi)_2$-quantum fields

In this section, we give an application of white noise analysis in quantum fields——we show that a $P(\phi)_2$ quantum field can be expressed as a positive Hida distribution (for more details see Potthoff-Streit[2]).

Let $\mathcal{H}$ be a complex Hilbert space with inner product $(\cdot,\cdot)$. By a canonical field on $\mathbb{R}^d$ we mean a pair of linear maps (ϕ,π) from $S(\mathbb{R}^d)$ to the space of self-adjoint operators on $\mathcal{H}$ satisfying the *canonical commutation relations*:

$$\begin{aligned} &[\phi(f),\phi(g)] = [\pi(f),\pi(g)] = 0, \\ &[\pi(f),\phi(g)] = \frac{1}{i}(f,g)_0. \end{aligned} \tag{4.38}$$

Here $f,g \in S(\mathbb{R}^d)$, $(\cdot,\cdot)_0$ denotes the inner product on $L^2(\mathbb{R}^d)$. The dynamics of the field is described by a positive self-adjoint operator H (called the *Hamiltonian*) on $\mathcal{H}$. The following commuation relation is usually postulated:

$$[H,\phi(f)] = \frac{1}{i}\pi(f), \quad f \in S(\mathbb{R}^d). \tag{4.39}$$

In terms of *Weyl's operators* $U(f) = e^{i\phi(f)}$ and $V(f) = e^{i\pi(f)}$, (4.38) can be rewritten as the following *Weyl relations*:

$$\begin{aligned} U(f)U(g) = U(f+g), \quad V(f)V(g) = V(f+g), \\ U(f)V(g) = V(g)U(f)e^{-i(f,g)_0}. \end{aligned} \tag{4.40}$$

And consequently, (4.39) can be rewritten as

$$[H,U(f)] = U(f)\pi(f) + \frac{1}{2}(f,f)_0U(f), \tag{4.41}$$

or equivalently,

$$[U(f),[H,U(g)]] = (f,g)_0U(f+g). \tag{4.42}$$

Furthermore, we assume that there exists a unique (normalized) real eigenvector Ω of H such that $H\Omega = 0$. Ω is called the vacuum. Finally, suppose that there exists a unique anti-unitary operator (representing time inversion) T (satisfying $T^2 = I$) such that

$$\begin{aligned} TU(f)T^{-1} = U(-f), \quad TV(f)T^{-1} = V(f), \\ THT^{-1} = H, \quad T\Omega = \Omega. \end{aligned} \tag{4.43}$$

The triplet (ϕ,H,Ω) is called a *canonical quantum field*.

If we regard ϕ and π as operator-valued distributions $\phi(x),\pi(x)$, i.e., representating ϕ and π formally as

$$\phi(f) = \int \phi(x)f(x)dx, \quad \pi(f) = \int \pi(x)f(x)dx,$$

then we can define the time evolution of the field as

$$\begin{aligned} \phi(t,x) = \exp(itH)\phi(x)\exp(-itH), \quad t \in \mathbb{R}, \\ \pi(t,x) = \exp(itH)\pi(x)\exp(-itH), \quad t \in \mathbb{R}. \end{aligned}$$

The above triplet (ϕ,H,Ω) together with the Poincaré group on the algebra generated by the evoluted field constitute a relativistic invariant canonical quantum field. The construction of non-trivial 4-dimensional quantum field remains an open problem.

Definition 4.17 Let (ϕ,H,Ω) be a canonical quantum field. If for any $f,g \in S(\mathbb{R}^d)$, $U(f)\Omega \in \mathcal{D}(H^{1/2})$, then by (4.40) and (4.41), we have

$$(U(f)\Omega, HU(g)\Omega) = \frac{1}{2}(f,g)_0(U(f)\Omega,U(g)\Omega). \tag{4.44}$$

(4.44) is called the *Araki relation*.

Definition 4.18 If there exist constants $\alpha,\beta,\gamma \geq 0$ and $p \in \mathbb{N}_0$ such that $\forall f \in S(\mathbb{R}^d)$, as bilinear forms on $\mathcal{D}_f = \{F(\phi(f))\Omega : F \in S(\mathbb{R})\}$, it holds that

$$\pm\phi(f) \leq \alpha H + \beta|f|_p^2 + \gamma, \tag{4.45}$$

where $|\cdot|_p$ is the norm defined by the d-dimensional oscillator on $L^2(\mathbb{R}^d)$ (see Section 3.2 of Chapter 1), then we say that the ϕ-bound holds.

Theorem 4.19 Assume that (ϕ,H,Ω) satisfies the Araki relation and the ϕ-bound. Then there exists a (positive) Hida distribution $\Xi \in \left(S(\mathbb{R}^d)\right)^*$ such that

$$(\Omega, e^{i\phi(f)}\Omega) = \langle\langle \Xi, e^{iW_f}\rangle\rangle, \quad f \in S(\mathbb{R}^d). \tag{4.46}$$

The key step of the proof is to obtain the following estimate by (4.45): there exists a constant K (depending on α,β,γ) such that $\forall n \in \mathbb{N}$,

$$|(\Omega,\phi(f)^n\Omega)| \leq K^n\sqrt{n!}|f|_p^n, \quad f \in S(\mathbb{R}^d), \tag{4.47}$$

from which we see that $f \mapsto (\Omega, e^{i\phi(f)}\Omega)e^{\frac{1}{2}(f,f)_0}$ is a U_0-functional. Hence by Theorem 2.9, there exists $\Xi \in (S(\mathbb{R}^d))^*$ such that (4.46) holds. The positiviness of Ξ follows from the positive definiteness of its moment sequence (see Lemma 4.8). The existence of Ξ also follows from Theorem 3.3 and the relation (4.47).

In what follows, using Theorem 4.19 we show that in the framework of white noise analysis, the $P(\phi)_2$-quantum field can be represented by a positive Hida distribution. Let $m^2 > 0$, ω_0 be the self-adjoint operator determined by the pseudo-differential operator $\sqrt{-\Delta + m^2}$ on $L^2(\mathbb{R})$. Let $l > 0$ and Δ_D be the Laplace operator on $L^2([-l,l])$ with the Dirichlet boundary condition, and let $\omega_D = \sqrt{-\Delta_D + m^2}$. We still denote by ω_D its natural extension to $L^2(\mathbb{R})$. In the sequel, ω denotes either ω_0 or ω_D. Let μ be the standard Gaussian measure

on $\mathbb{R}$. $\forall f \in S(\mathbb{R})$, we define a self-adjoint operator on $L^2(\mu)_{\mathbb{C}}$ as follows:

$$\phi_l(f) = D^*_{\omega^{-1/2}f} + D_{\omega^{-1/2}f},$$
$$H_0 = \frac{1}{2}\sum_k D^*_{\omega^{1/2}e_k} D_{\omega^{1/2}e_k},$$

where $\{e_k, k \in \mathbb{N}\} \subset S(\mathbb{R})$ is an orthonormal base of $L^2(\mathbb{R})$. We define the Wick power of the field ϕ_l with respect to ω as

$$:\phi_l(f)^n := \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} (2k-1)!!\phi_l(f)^{n-2k} |\omega^{-1/2} f|_0^{2k}.$$

Let P be a real polynomial satisfying one of the following conditions:

(H1) $P(u) = \lambda \sum_{k=0}^{2n} a_k u^k$, $n \in \mathbb{N}$, $\lambda, a_{2n} > 0$, λ/m^2 sufficiently small,

(H2) $P(u) = \sum_{k=0}^{n} a_k u^{2k} + bu$, $a_n > 0$.

Let $\{\delta_{t,n}, n \in \mathbb{N}\} \subset S(\mathbb{R})$ be a sequence converging to the Dirac δ-function δ_t in $S'(\mathbb{R})$. Put

$$V_{l,n} = \int_{-l}^{l} :P\Big(\phi_l(\delta_{t,n})\Big): dt.$$

It can be prove that (see Simon[1]): $\lim_{n\to\infty} V_{l,n} = V_l$, and $V_l \in L^p(\mu), \forall p > 1$. Thus as a multiplication operator, V_l defines a self-adjoint operator on $L^2(\mu)$ (still denoted by V_l). We denote by $\widehat{H}_l$ the self-adjoint extension of the essentially self-adjoint operator $H_0 + V_l$ on $L^2(\mu)$ and put

$$H_l = \widehat{H}_l - \inf \operatorname{spec} \widehat{H}_l.$$

Then H has a unique vacuum $\Omega_l \in L^2(\mu)$. It can be verified that $\forall l > 0$, (ϕ_l, H_l, Ω_l) satisfies the Araki relation and the ϕ-bound:

$$\pm\phi_l(f) \le \alpha H_l + \beta |f|_1^2 + \gamma,$$

where α, β and γ do not depend on $l > 0$. Hence by Theorem 4.19, there exists $\Xi_l \in (S(\mathbb{R}))^*_+$ such that

$$(\Omega_l, e^{i\phi_l(f)}\Omega_l) = \langle\langle \Xi_l, e^{iW_f}\rangle\rangle, \ f \in S(\mathbb{R}),$$

and the K in the estimate corresponding to (4.47) does not depend on l. Moreover, by the known results in quantum fields, there exist a complex Hilbert space $\mathcal{K}$ and a linear map ϕ from $S(\mathbb{R})$ to the space of self-adjoint operators on $\mathcal{K}$ and a real unit vector Ω such that $\forall f \in S(\mathbb{R})$,

$$\lim_{l\to\infty} (\Omega_l, e^{i\phi_l(f)}\Omega_l) = (\Omega, e^{i\phi(f)}\Omega)_{\mathcal{K}}.$$

We call $(\phi, \mathcal{K}, \Omega)$ the *$P(\phi)_2$-quantum field*. By Theorem 2.10, there exists $\Xi \in (S(\mathbb{R}))^*_+$ such that

$$\left(\Omega, e^{i\phi(f)}\Omega\right)_{\mathcal{K}} = \langle\langle \Xi, e^{iW_f}\rangle\rangle. \tag{4.48}$$

Chapter V

Linear Operators on Distribution Space

§1. Analytic calculus for distributions

We have established a general framework for white noise analysis in the last chapter. Now we shall introduce several analytic calculus for distributions which will be crucial for the practical applications of white noise analysis. These are : scaling transformations, shift operators, Sobolev differentiations, gradient and divergence operators. In the sequel we shall assume $0 \le \beta < \infty$.

1.1 Scaling transformations

According to Theorem 2.18 in Chapter 4, $(E)^{\beta}_{\mathbb{C}} = \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$. For $\varphi \in (E)^{\beta}_{\mathbb{C}}$. We denote by $\tilde{\varphi}$ the entire analytic continuation of φ to $E^*_{\mathbb{C}}$. Let $\lambda \in \mathbb{C}$, we define the operator σ_λ on $(E)^{\beta}_{\mathbb{C}}$ as

$$\sigma_\lambda \varphi(x) = \tilde{\varphi}(\lambda x), \quad x \in E^* \tag{1.1}$$

Obviously, $\sigma_\lambda \varphi \in \mathcal{A}^{\frac{2}{1+\beta}}(E^*)$. Hence σ_λ is a continuous linear operator from $(E)^{\beta}_{\mathbb{C}}$ to $(E)^{\beta}_{\mathbb{C}}$. We call σ_λ the *λ-scaling transformation.*

In order to study the scaling transformations, we introduce the second quantization $\Gamma(\lambda)$ of the multiplication by λ. It will be frequently used later.

For $\lambda \in \mathbb{C}, \varphi \in (E)^{-\beta}_{\mathbb{C}}, \varphi \sim \{f_n\}$, put

$$\Gamma(\lambda)\varphi \sim \{\lambda^n f_n\}. \tag{1.2}$$

It is easily seen that $\Gamma(\lambda)$ is a continuous linear operator on $(E)_{\mathbb{C}}^{-\beta}$ and on $(E)^{\beta}_{\mathbb{C}}$. In fact, $\forall \lambda \neq 0$, we have

$$\|\Gamma(\lambda)\varphi\|_{p,q-2\log_2|\lambda|,\pm\beta} = \|\varphi\|_{p,q,\pm\beta}, \quad \forall p \in \mathbb{Z}, q \in \mathbb{R}. \tag{1.3}$$

The next theorem shows that σ_λ is a continuous linear operator on $(E)^{\beta}_{\mathbb{C}}$ and gives the chaos decomposition of $\sigma_\lambda\varphi$. The adjoint σ^*_λ of σ_λ is also given explicitly. Henceforth, we denote by A^* the adjoint of the continuous linear operator A from one topological linear space to another (see Appendix B).

Theorem 1.1 *For $\lambda \in \mathbb{C}, \varphi \in (E)^{\beta}_{\mathbb{C}}$, $\varphi \sim \{f_n\}$, let $\sigma_\lambda\varphi \sim \{h_n\}$. Then*

$$h_n = \frac{\lambda^n}{n!}\sum_{l=0}^{\infty}(\lambda^2 - 1)^l \frac{(n+2l)!}{l!2^l}\langle \tau^{\hat{\otimes} l}, f_{n+2l}\rangle, \tag{1.4}$$

where the series converges absolutely in $E^{\hat{\otimes} n}_{\mathbb{C}}$, and we have

$$\langle\langle \sigma_\lambda\varphi, G\rangle\rangle = \langle\langle \varphi, \Gamma(\lambda)G \diamond F(\lambda)\rangle\rangle, \ \forall G \in (E)^{-\beta}_{\mathbb{C}}. \tag{1.5}$$

Here $F(\lambda)$ is defined as in Example 2.22 of Chapter 4. In particular, σ_λ is a continuous linear operator from $(E)^{\beta}_{\mathbb{C}}$ to $(E)^{\beta}_{\mathbb{C}}$, whose adjoint σ^*_λ is given by

$$\sigma^*_\lambda G = \Gamma(\lambda)G \diamond F(\lambda), \quad \forall G \in (E)^{-\beta}_{\mathbb{C}}. \tag{1.6}$$

Moreover, σ^*_λ is continuous from $(E)^{-\beta}_{\mathbb{C}}$ to $(E)^{-\beta}_{\mathbb{C}}$.

Proof. First we prove the following formula: $\forall x \in E^*$,

$$:(\lambda x)^{\otimes n}: = n!\sum_{k=0}^{[n/2]}\frac{(\lambda^2-1)^k\lambda^{n-2k}}{2^k k!(n-2k)!} :x^{\otimes(n-2k)}: \hat{\otimes}\tau^{\otimes k}. \tag{1.7}$$

The proof of this formula is analogous to that of (A.9) in the Appendix. In fact, for any $t \in \mathbb{R}$, $f \in E$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{n!}\langle f^{\otimes n}, :(\lambda x)^{\otimes n}:\rangle = \mathcal{E}_{tf}(\lambda x) \\
&= \mathcal{E}_{t\lambda f}(x)\exp\left\{\tfrac{(\lambda^2-1)t^2}{2}\langle f, f\rangle\right\} \\
&= \sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\langle f^{\otimes j}, :x^{\otimes j}:\rangle\sum_{k=0}^{\infty}\frac{(\lambda^2-1)^k t^{2k}}{2^k k!}\langle f^{\otimes 2k}, \tau^{\otimes k}\rangle,
\end{aligned}$$

comparing the coefficients of t^n we obtain (1.7).

Take $p \in \mathbb{N}$ such that the imbedding I_{0p} from H_p to H is a Hilbert-Schmidt operator. Then by (2.53) in Chapter 4,

$$|\langle \tau^{\hat{\otimes} l}, f_{n+2l}\rangle|_p \le \|I_{0p}\|^{2l}_{\mathrm{HS}}\,|f_{n+2l}|_p,$$

from which it is readily verified that the series in (1.4) converges absolutely in $E^{\hat{\otimes} n}_{\mathbb{C}}$. Consequently, by (1.7),

$$\sum_{n=0}^{\infty}\langle h_n, :x^{\otimes n}:\rangle = \tilde{\varphi}(\lambda x).$$

Hence $\sigma_\lambda\varphi \sim \{h_n\}$.

Now suppose that $G \in (E)^{-\beta}_{\mathbb{C}}$, $G \sim \{g_n\}$. Then by (1.4),

$$n!\langle h_n, g_n\rangle = \sum_{l=0}^{\infty}(n+2l)!\langle \tfrac{(\lambda^2-1)^l}{2^l l!}\tau^{\hat{\otimes} l}\hat{\otimes}\lambda^n g_n, f_{n+2l}\rangle.$$

In view of (2.52) and the expression (3.16) in Chapter 4 we obtain (1.5).

From (1.5), (1.3) and (3.17) in Chapter 4 it is easily proved that (see the proof of Theorem 4.1 in Chapter 4) if $p \in \mathbb{N}$, $q \geq 0$ and $2^q > |\lambda^2 - 1|\|I_{0p}\|_{HS}^2$, then $\forall r > (1-\beta)^+$, it holds that

$$\begin{aligned}&\|\sigma_\lambda\varphi\|_{p,q-2\log_2|\lambda|,\beta}\\ &\leq (1-2^{-r+(1-\beta)^+})^{-1/2}\|\varphi\|_{p,q+r,\beta}\|F(\lambda)\|_{-p,-q,-\beta}.\end{aligned} \tag{1.8}$$

Hence σ_λ is a continuous linear operator from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$. By (1.5), the adjoint σ_λ^* of σ_λ is given by (1.6). ∎

Remark 1. By (1.8), if $p \in \mathbb{N}$, $q \geq 0$ and $2^q > |\lambda^2 - 1|\|I_{0p}\|_{HS}^2$, then $\forall \epsilon > (1-\beta)^+$, σ_λ can be extended to a bounded linear operator from $(H_{p,q+\epsilon,\beta})_{\mathbb{C}}$ to $(H_{p,q-2\log_2|\lambda|,\beta})_{\mathbb{C}}$.

Remark 2. Comparing (4.6) in Chapter 4 and (1.5) in this section, we have

$$R^{-1} = \Gamma((\sqrt{2})^{-1})\sigma_{\sqrt{2}}. \tag{1.9}$$

Consequently,

$$R = \sigma_{(\sqrt{2})^{-1}}\Gamma(\sqrt{2}). \tag{1.10}$$

Remark 3. By Theorem 3.5 in Chapter 4 it can be easily proved that $\forall \varphi \in (E)_{\mathbb{C}}^{\beta}$, the mapping $\lambda \longmapsto \sigma_\lambda\varphi$ is continuous from $\mathbb{C}$ to $(E)_{\mathbb{C}}^{\beta}$ and $\forall G \in (E)_{\mathbb{C}}^{-\beta}$, the mapping $\lambda \longmapsto \langle\langle \sigma_\lambda\varphi, G\rangle\rangle$ is entire analytic on $\mathbb{C}$.

1.2 Shift operators and Sobolev differentiations

For $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $y \in E_{\mathbb{C}}^*$, put

$$\tau_y\varphi(x) = \tilde{\varphi}(y + x), \quad x \in E^*. \tag{1.11}$$

Clearly, $\tau_y\varphi \in (E)_{\mathbb{C}}^{\beta}$. We call τ_y a *shift operator*. By (1.4) in Chapter 4, we have

$$\begin{aligned}:(x+y)^{\otimes n}: &= \int_{E^*}(x+y+iz)^{\otimes n}\mu(dz)\\ &= \sum_{k=0}^{n}\binom{n}{k}\int_{E^*}(x+iz)^{\otimes k}\hat{\otimes}y^{\otimes n-k}\mu(dz)\\ &= \sum_{k=0}^{n}\binom{n}{k}:x^{\otimes k}:\hat{\otimes}y^{\otimes n-k}.\end{aligned} \tag{1.12}$$

Hence, similar to the proof of Theorem 1.1, we obtain

Theorem 1.2 *For $y \in E_{\mathbb{C}}^*$, $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $\varphi \sim \{f_n\}$, let $\tau_y\varphi \sim \{h_n\}$. Then*

$$h_n = \sum_{k=0}^{\infty}\binom{k+n}{n}\langle y^{\otimes k}, f_{n+k}\rangle, \tag{1.13}$$

where the series convegers absolutely in $E_{\mathbb{C}}^{\hat{\otimes}n}$, and we have

$$\langle\langle \tau_y\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond \mathcal{E}_y\rangle\rangle, \ \forall G \in (E)_{\mathbb{C}}^{-\beta}. \tag{1.14}$$

In particular, τ_y is a continuous operator from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$ whose adjoint τ_y^* is given by

$$\tau_y^*G = G \diamond \mathcal{E}_y, \quad \forall G \in (E)_{\mathbb{C}}^{-\beta}. \tag{1.15}$$

Moreover, for any $\varphi \in (E)_{\mathbb{C}}^{\beta}$, the mapping $y \longmapsto \tau_y\varphi$ is continuous from $E_{\mathbb{C}}^*$ to $(E)_{\mathbb{C}}^{\beta}$.

Remarks. (1) If $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $r > -1$, then by (1.14) and (3.17) in Chapter 4, we know that $\forall y \in H_{-p,\mathbb{C}}$, $\epsilon > (1-r)^+$,

$$\|\tau_y\varphi\|_{p,q,r} \leq (1-2^{-\epsilon+(1-r)^+})^{-1/2}\|\varphi\|_{p,q+\epsilon,r}\|\mathcal{E}_y\|_{-p,-q,-r}. \tag{1.16}$$

In this case, τ_y can be extended to a continuous linear operator from $(H_{p,q+\epsilon,r})_{\mathbb{C}}$ to $(H_{p,q,r})_{\mathbb{C}}$. In particular, if $y \in E_{\mathbb{C}}$, then τ_y can be extended to a continuous linear operator from $(E)_{\mathbb{C}}^{-\beta}$ to $(E)_{\mathbb{C}}^{-\beta}$. Moreover,

$$\langle\langle \tau_y\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond \mathcal{E}_y\rangle\rangle, \forall \varphi \in (E)_{\mathbb{C}}^{-\beta}, G \in (E)_{\mathbb{C}}^{\beta}. \tag{1.14$'$}$$

(2) If $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $\epsilon > 0$, $y \in H_{-p,\mathbb{C}}$ and $|y|_{-p}^2 < 2^{-q}$, then

$$\|\tau_y\varphi\|_{p,q-\epsilon,1} \leq (1-2^{-\epsilon})^{-1/2}\|\varphi\|_{p,q,1}\|\mathcal{E}_y\|_{-p,-q,-1},$$

and τ_y can be extended to a continuous linear operator from $(H_{p,q,1})_{\mathbb{C}}$ to $(H_{p,q-\epsilon,1})_{\mathbb{C}}$.

(3) If $0 \leq \beta < 1$, $\varphi \in (E)_{\mathbb{C}}^{-\beta}$, $\eta \in E_{\mathbb{C}}$, then by (1.14)$'$,

$$S(\tau_\eta\varphi)(\xi) = S\varphi(\eta + \xi).$$

Next we define differentiations in $(E)_{\mathbb{C}}^{\beta}$ using shift operators.

If $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $y \in E_{\mathbb{C}}^*$, then by (1.14) we know that $\forall G \in (E)_{\mathbb{C}}^{-\beta}$,

$$\begin{aligned}\lim_{t\downarrow 0}\langle\langle \frac{\tau_{ty}\varphi - \varphi}{t}, G\rangle\rangle &= \lim_{t\downarrow 0}\langle\langle \varphi, G \diamond \frac{\mathcal{E}_{ty} - 1}{t}\rangle\rangle\\ &= \langle\langle \varphi, G \diamond I_1(y)\rangle\rangle.\end{aligned}$$

Thus we can define a continuous linear operator D_y from $(E)_{\mathbb{C}}^{\beta}$ to $(E)_{\mathbb{C}}^{\beta}$ by the relation

$$\langle\langle D_y\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond I_1(y)\rangle\rangle. \tag{1.17}$$

We call D_y a *Sobolev differentiation (operator)*. By (1.17), it is easily seen that if $\varphi \sim \{f_n\}$, then $D_y\varphi \sim \{h_n\}$ with

$$h_n = (n+1)\langle y, f_{n+1}\rangle, \quad n \in \mathbb{N}_0. \tag{1.18}$$

By (1.17) and (3.17) in Chapter 4, we can easily prove

Theorem 1.3 *If $p \in \mathbb{Z}$, $q, r \in \mathbb{R}$, $y \in H_{-p,\mathfrak{c}}$, then for any $\epsilon > (1-r)^+$, D_y can be extended to a continuous linear operator from $(H_{p,q+\epsilon,r})_{\mathfrak{c}}$ to $(H_{p,q,r})_{\mathfrak{c}}$. Furthermore,*

$$\|D_y\varphi\|_{p,q,r} \le C_{r,\epsilon} 2^{-q/2} |y|_{-p} \|\varphi\|_{p,q+\epsilon,r}. \tag{1.19}$$

Here

$$C_{r,\epsilon} = (1 - 2^{-\epsilon+(1-r)^+})^{-1/2}.$$

In particular, for any $\varphi \in (E)^{\beta}_{\mathfrak{c}}$, the mapping $y \longmapsto D_y\varphi$ is continuous from $E^*_{\mathfrak{c}}$ to $(E)^{\beta}_{\mathfrak{c}}$; for any $\varphi \in (E)^{-\beta}_{\mathfrak{c}}$, the mapping $y \longmapsto D_y\varphi$ is continuous from $E_{\mathfrak{c}}$ to $(E)^{-\beta}_{\mathfrak{c}}$; for any $y \in E^*_{\mathfrak{c}}$, the mapping $\varphi \longmapsto D_y\varphi$ is continuous from $(E)^{\beta}_{\mathfrak{c}}$ to $(E)^{\beta}_{\mathfrak{c}}$; for any $y \in E_{\mathfrak{c}}$, the mapping $\varphi \longmapsto D_y\varphi$ from $(E)^{-\beta}_{\mathfrak{c}}$ to $(E)^{-\beta}_{\mathfrak{c}}$ is continuous.

Theorem 1.4 (1) *If $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $r > -1$, $\epsilon > (1-r)^+$, $\varphi \in (H_{p,q+\epsilon,r})_{\mathfrak{c}}$, $y \in H_{-p,\mathfrak{c}}$, then*

$$\lim_{t\to 0} \left\| \frac{\tau_{ty}\varphi - \varphi}{t} - D_y\varphi \right\|_{p,q,r} = 0. \tag{1.20}$$

(2) *If $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $\epsilon > 2$, $\varphi \in (H_{p,q+\epsilon,r})_{\mathfrak{c}}$, $y \in H_{-p,\mathfrak{c}}$, then*

$$\lim_{t\to 0} \left\| \frac{\tau_{ty}\varphi - \varphi}{t} - D_y\varphi \right\|_{p,q,-1} = 0. \tag{1.21}$$

(3) *If $0 \le \beta < 1$, $\varphi \in (E)^{-\beta}_{\mathfrak{c}}$, $\eta \in E_{\mathfrak{c}}$, then*

$$SD_\eta\varphi(\xi) = \lim_{t\downarrow 0} \frac{1}{t}\{S\varphi(\xi + t\eta) - S\varphi(\xi)\}. \tag{1.22}$$

Proof. (1) It follows from (1.14) and (1.17) that $\forall G \in (E)^{|r|}_{\mathfrak{c}}$,

$$\left\langle\left\langle \frac{\tau_{ty}\varphi - \varphi}{t} - D_y\varphi, G \right\rangle\right\rangle = \left\langle\left\langle \varphi, G \diamond \left(\frac{\mathcal{E}_{ty} - 1}{t} - I_1(y)\right)\right\rangle\right\rangle.$$

Now by (3.17) in Chapter 4,

$$\begin{aligned} &\left| \left\langle\left\langle \frac{\tau_{ty}\varphi - \varphi}{t} - D_y\varphi, G \right\rangle\right\rangle \right| \\ &\le C_{r,\epsilon} \|\varphi\|_{p,q+\epsilon,r} \|G\|_{-p,-q,-r} \left\| \frac{\mathcal{E}_{ty} - 1}{t} - I_1(y) \right\|_{-p,-q,-r}. \end{aligned}$$

But since $r + 1 > 0$, we have

$$\lim_{t\downarrow 0} \left\| \frac{\mathcal{E}_{ty} - 1}{t} - I_1(y) \right\|^2_{-p,-q,-r} = \lim_{t\downarrow 0} \sum_{n=1}^{\infty} \frac{t^{n-1} 2^{-nq}}{(n!)^{1+r}} |y|^{2n}_{-p} = 0.$$

Hence we obtain (1.20).

(2) If t is sufficiently small, then $t^2|y|^2_{-p} < 2^{-q}$. Thus $\tau_{ty}\varphi$ is well-defined. The remaining part of the proof is similar to (1) (use Theorem 1.2 and Remark (2)).

(3) It follows from Remark (3) of Theorem 1.2. ∎

Corollary 1.5 *If $\varphi \in (E)^{\beta}_{\mathfrak{c}}$, $y \in E^*$, then the Gâteaux differentiation of $\tilde{\varphi}$ along the direction y exists everywhere and equals $\widetilde{D_y\varphi}$.*

Proof. By Theorem 1.4, $(\tau_{ty}\varphi - \varphi)/t$ converges to $D_y\varphi$ in $(E)^{\beta}_{\mathfrak{c}}$. Consequently, $\forall y \in E^*$,

$$\begin{aligned} \lim_{t\downarrow 0} \frac{\tilde{\varphi}(x + ty) - \tilde{\varphi}(x)}{t} &= \lim_{t\downarrow 0} \left\langle\left\langle \frac{\tau_{ty}\varphi - \varphi}{t}, \delta_x \right\rangle\right\rangle \\ &= \langle\langle D_y\varphi, \delta_x \rangle\rangle = \widetilde{D_y\varphi}(x). \end{aligned}$$ ∎

The Taylor expansion of a distribution is given by

Theorem 1.6 (1) *Let $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $r > -1$, $\varphi \in (H_{p,q,r})_{\mathfrak{c}}$. If $y \in H_{-p,\mathfrak{c}}$, then*

$$\tau_y\varphi = \sum_{k=0}^{\infty} \frac{1}{k!} D^k_y\varphi, \tag{1.23}$$

where the series converges absolutely in $(H_{p,q-\epsilon,r})_{\mathfrak{c}}$, $\forall \epsilon > (1-r)^+$. In particular, if $\varphi \in (E)^{\beta}_{\mathfrak{c}}$, then for any $x, y \in E^*$,

$$\tilde{\varphi}(x + y) = \sum_{k=0}^{\infty} \frac{1}{k!} \widetilde{D^k_y\varphi}(x). \tag{1.24}$$

(2) *Let $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $\varphi \in (H_{p,q,-1})_{\mathfrak{c}}$. If $y \in H_{-p,\mathfrak{c}}$ and $|y|^2_{-p} < 2^{-q}$, then (1.23) holds, where the series converges absolutely in $(H_{p,q-\epsilon,-1})_{\mathfrak{c}}$ for any $\epsilon > 2$.*

Proof. We only prove (1), the proof of (2) being similar. By (1.17),

$$\langle\langle D^k_y\varphi, G\rangle\rangle = \langle\langle \varphi, G \diamond I_k(y^{\otimes k})\rangle\rangle, \ \forall G \in (E)^{|r|}_{\mathfrak{c}}. \tag{1.25}$$

Thus $\forall \epsilon > (1-r)^+$,

$$\|D^k_y\varphi\|_{p,q-\epsilon,r} \le C_{r,\epsilon} \|\varphi\|_{p,q,r} \|I_k(y^{\otimes k})\|_{-p,-q,-r}.$$

From this we know that the series in the r.h.s. of (1.24) converges in $(H_{p,q-\epsilon,r})_{\mathfrak{c}}$. Further, it follows from (1.25) that $\forall G \in (E)^{|r|}_{\mathfrak{c}}$,

$$\langle\langle \tau_y\varphi, G\rangle\rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \langle\langle D^k_y\varphi, G\rangle\rangle.$$

Thus (1.23) follows. (1.24) is a consequence of (1.23). ∎

The main properties of differentiation are summarized in

Theorem 1.7 *If $y, z \in E^*_{\mathfrak{c}}$, $\xi, \eta \in E_{\mathfrak{c}}$, $\varphi, \psi \in (E)^{\beta}_{\mathfrak{c}}$, $F, G \in (E)^{-\beta}_{\mathfrak{c}}$, then*

$$D_y(\varphi\psi) = \psi D_y\varphi + \varphi D_y\psi, \tag{1.26}$$

$$D_\xi(\varphi F) = F D_\xi\varphi + \varphi D_\xi F, \tag{1.27}$$

$$D_y(\varphi \diamond \psi) = \psi \diamond D_y\varphi + \varphi \diamond D_y\psi, \tag{1.28}$$

$$D_\xi(F\diamond G)=G\diamond D_\xi F+F\diamond D_\xi G, \tag{1.29}$$
$$(D_y+D_y^*)\varphi=I_1(y)\varphi, \tag{1.30}$$
$$(D_\xi+D_\xi^*)F=I_1(\xi)F, \tag{1.31}$$
$$D_yD_x\varphi=D_xD_y\varphi, \tag{1.32}$$
$$D_\xi D_\eta F=D_\eta D_\xi F, \tag{1.33}$$
$$D_y^*D_x^*F=D_x^*D_y^*F, \tag{1.34}$$
$$(D_yD_\xi^*-D_\xi^*D_y)\varphi=\langle y,\xi\rangle\varphi, \tag{1.35}$$
$$(D_\xi D_y^*-D_y^*D_\xi)F=\langle y,\xi\rangle F, \tag{1.36}$$
$$D_\xi^*(\varphi F)+FD_\xi\varphi=\varphi D_\xi^*F, \tag{1.37}$$
$$D_\xi^*(\varphi F)+\varphi D_\xi F=FD_\xi^*\varphi, \tag{1.38}$$
$$D_y^*(\varphi\psi)+\varphi D_y\psi=\psi D_y^*\varphi. \tag{1.39}$$

Proof. Let $\xi_1,\xi_2\in E_{\mathbb{C}}$, $m,n\in \mathbb{N}_0$. It is easily verified that (1.26) holds for $\varphi=W_{\xi_1}^m$ and $\psi=W_{\xi_2}^n$ and (1.28) holds for $\varphi=I_m(\xi_1^{\otimes m})$ and $\psi=I_n(\xi_2^{\otimes n})$. Thus by the linearity and continuity of product (or Wick product) and differentiation, (1.26) and (1.28) holds for all $\varphi,\psi\in(E)_{\mathbb{C}}^\beta$. In (1.26) and (1.28) taking $\psi_n\in(E)_{\mathbb{C}}^\beta,\varphi_n\in(E)_{\mathbb{C}}^\beta$ such that $\psi_n\to F,\varphi_n\to G$, we obtain (1.27) and (1.29).

According to the generalized form of (3.10) in Chapter 4, it is easily seen that (1.30) holds true for $\varphi=I_n(f^{\otimes n})$, $f\in E_{\mathbb{C}}$. Hence by the same reason, (1.30) holds true for any $\varphi\in(E)_{\mathbb{C}}^\beta$ and (1.31) follows. (1.34) is obvious. Consequently, we obtain (1.32) and (1.33). By (1.28),

$$\begin{aligned}D_yD_\xi^*\varphi&=D_y(I_1(\xi)\diamond\varphi)=\varphi\diamond D_yI_1(\xi)+I_1(\xi)\diamond D_y\varphi\\&=\langle y,\xi\rangle\varphi+D_\xi^*D_y\varphi,\end{aligned}$$

this is just (1.35). From (1.29) we can prove (1.36) by the similar method.

Finally by (1.26),

$$\begin{aligned}&\langle\langle D_\xi^*(\varphi F),\psi\rangle\rangle+\langle\langle FD_\xi\varphi,\psi\rangle\rangle\\&=\langle\langle\varphi F,D_\xi\psi\rangle\rangle+\langle\langle\psi D_\xi\varphi,F\rangle\rangle\\&=\langle\langle\varphi D_\xi\psi+\psi D_\xi\varphi,F\rangle\rangle=\langle\langle D_\xi(\varphi\psi),F\rangle\rangle\\&=\langle\langle\varphi\psi,D_\xi^*F\rangle\rangle=\langle\langle\varphi D_\xi^*F,\psi\rangle\rangle,\end{aligned}$$

hence (1.37) follows. The proof of (1.38) and (1.39) is similar. ∎

1.3 Gradient and divergence operators

We have already introduced gradient and divergence operators in Chapter 3. Now we shall define similarly these operators in the context of white noise analysis. By Theorem 1.6 in Chapter 4, the testing functional space in this context can be continuously and densely imbedded into the space of Meyer-Watanabe's

testing functionals. Thus the gradient and divergence operators defined below are extensions of the corresponding operators in Chapter 2.

Definition 1.8 Let $\varphi\in(E)_{\mathbb{C}}^{-\beta}$. By Theorem 1.3 there exists a unique $D\varphi\in E_{\mathbb{C}}^*\otimes(E)_{\mathbb{C}}^{-\beta}$ such that

$$\langle\langle D\varphi,\xi\otimes\psi\rangle\rangle=\langle\langle D_\xi\varphi,\psi\rangle\rangle,\quad \xi\in E_{\mathbb{C}},\psi\in(E)_{\mathbb{C}}^\beta. \tag{1.40}$$

We call $D\varphi$ the *gradient* of φ.

If $\varphi\in(E)_{\mathbb{C}}^\beta$, then by Theorem 1.3 we have $D\varphi\in E_{\mathbb{C}}\otimes(E)_{\mathbb{C}}^\beta$ and

$$\langle\langle D\varphi,y\otimes\psi\rangle\rangle=\langle\langle D_y\varphi,\psi\rangle\rangle,\ y\in E_{\mathbb{C}}^*,\psi\in(E)_{\mathbb{C}}^{-\beta}. \tag{1.41}$$

Definition 1.9 For $F\in E_{\mathbb{C}}^*\otimes(E)_{\mathbb{C}}^{-\beta}$, the unique element $D^*F\in(E)_{\mathbb{C}}^{-\beta}$ such that

$$\langle\langle D^*F,\varphi\rangle\rangle=\langle\langle F,D\varphi\rangle\rangle,\ \varphi\in(E)_{\mathbb{C}}^\beta \tag{1.42}$$

is called the *divergence* of F.

If $F\in E_{\mathbb{C}}\otimes(E)_{\mathbb{C}}^\beta$, then by Theorem 1.3, we have $D^*F\in(E)_{\mathbb{C}}^\beta$, and

$$\langle\langle D^*F,\varphi\rangle\rangle=\langle\langle F,D\varphi\rangle\rangle,\ \varphi\in(E)_{\mathbb{C}}^{-\beta}. \tag{1.43}$$

Theorem 1.10 *D is a continuous linear operator from $(E)_{\mathbb{C}}^{-\beta}$ to $E_{\mathbb{C}}^*\otimes(E)_{\mathbb{C}}^{-\beta}$, D^* defined by (1.43) is the adjoint of D and is continuous from $E_{\mathbb{C}}\otimes(E)_{\mathbb{C}}^\beta$ to $(E)_{\mathbb{C}}^\beta$. The restriction of D to $(E)_{\mathbb{C}}^\beta$ is continuous from $(E)_{\mathbb{C}}^\beta$ to $E_{\mathbb{C}}\otimes(E)_{\mathbb{C}}^\beta$. D^* defined by (1.42) is a continuous extension of the one defined by (1.43) and is continuous from $E_{\mathbb{C}}^*\otimes(E)_{\mathbb{C}}^{-\beta}$ to $(E)_{\mathbb{C}}^{-\beta}$.*

Proof. Immediately follows from the definition of operator D and Theorem 1.3. ∎

We call D the *gradient operator* and D^* the *divergence operator*. The next theorem gives the expressions of D_y and D_y^* in terms of D and D^*.

Theorem 1.11 *If $y\in E_{\mathbb{C}}^*,\xi\in E_{\mathbb{C}},\varphi\in(E)_{\mathbb{C}}^{-\beta},\psi\in(E)_{\mathbb{C}}^\beta$, then*

$$D_y\psi=(D\psi,y),\quad D_\xi\varphi=(D\varphi,\xi), \tag{1.44}$$
$$D_y^*\varphi=D^*(y\otimes\varphi). \tag{1.45}$$

Proof. Since

$$\begin{aligned}\langle\langle D_y\psi,\varphi\rangle\rangle&=\langle\langle D\psi,y\otimes\varphi\rangle\rangle\\&=\langle\langle(D\psi,y),\varphi\rangle\rangle,\quad\forall\varphi\in(E)_{\mathbb{C}}^{-\beta},\end{aligned}$$

the first formula of (1.44) follows. The second can be proved similarly.

On the other hand,

$$\begin{aligned}&\langle\langle D^*(y\otimes\varphi),\psi\rangle\rangle=\langle\langle y\otimes\varphi,D\psi\rangle\rangle\\&=\langle\langle\varphi,D_y\psi\rangle\rangle=\langle\langle D_y^*\varphi,\psi\rangle\rangle,\ \forall\psi\in(E)_{\mathbb{C}}^\beta,\end{aligned}$$

thus (1.45) follows. ∎

In the context of white noise analysis, the number operator N can also be defined as $N\varphi \sim \{nf_n\}$ for $\varphi \in (E)_{\mathbb{C}}^{\beta}$, $\varphi \sim \{f_n\}$. It is easy to prove that N is continuous linear operator on $(E)_{\mathbb{C}}^{\beta}$ and on $(E)_{\mathbb{C}}^{\beta}$.

Theorem 1.12 *It holds that*

$$N = D^*D. \tag{1.46}$$

Proof. By Theorem 1.10, D^*D is a continuous linear operator from $(E)_{\mathbb{C}}^{-\beta}$ to $(E)_{\mathbb{C}}^{\beta}$. Moreover, N and D^*D coincide on a dense subset of $(E)_{\mathbb{C}}^{-\beta}$ by vertue of Proposition 3.2 in Chapter 2. Hence (1.46) holds. ∎

We shall study the gradient and divergence in a classical framework of white noise analysis. We adopt the notations in Section 1.3 of Chapter 4. Let $\{e_i, i \geq 0\}$ be eigenvectors of A which form an orthonormal base for H. The corresponding eigenvalues are $\{\lambda_i, i \geq 0\}$. By assumption, there exists $p_0 > 0$ such that $\sum_{j=0}^{\infty} \lambda_j^{-2p_0} < \infty$.

Theorem 1.13 *If $\varphi \in (E)_{\mathbb{C}}$ ($\varphi \in (E)_{\mathbb{C}}^*$), then*

$$D\varphi = \sum_{j=0}^{\infty} e_j \otimes D_{e_j}\varphi, \tag{1.47}$$

where the series converges in $E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}$ (respectively $E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^*$).

Proof. Since $|e_j|_p = \lambda_j^p, p \in \mathbb{R}$, the convergence of the series in (1.47) follows from (1.10). Moreover, $\forall f, g \in E$,

$$\begin{aligned}\langle\langle \sum_{j=0}^{\infty} e_j \otimes D_{e_j}\varphi, f \otimes \mathcal{E}_g\rangle\rangle &= \sum_{j=0}^{\infty} (f, e_j)\langle\langle D_{e_j}\varphi, \mathcal{E}_g\rangle\rangle \\ &= \sum_{j=0}^{\infty} (f, e_j)\langle\langle D\varphi, e_j \otimes \mathcal{E}_g\rangle\rangle = \langle\langle D\varphi, f \otimes \mathcal{E}_g\rangle\rangle.\end{aligned}$$

Hence we have (1.47). ∎

Corollary 1.14 *If $\varphi \in (E)_{\mathbb{C}}, y \in E_{\mathbb{C}}^*$ ($\varphi \in (E)_{\mathbb{C}}^*, y \in E_{\mathbb{C}}$), then*

$$D_y\varphi = \sum_{j=0}^{\infty} (y, e_j) D_{e_j}\varphi, \tag{1.48}$$

where the series converges in $(E)_{\mathbb{C}}$ (respectively $(E)_{\mathbb{C}}^*$).

Proof. Immediately follows from (1.47) and (1.44). ∎

Theorem 1.15 *If $\varphi \in (E)_{\mathbb{C}}, y \in E_{\mathbb{C}}$ ($\varphi \in (E)_{\mathbb{C}}^*, y \in E_{\mathbb{C}}^*$), then*

$$D_y^*\varphi = \sum_{j=0}^{\infty} (y, e_j) D_{e_j}^*\varphi, \tag{1.49}$$

where the series converges in $(E)_{\mathbb{C}}$ (respectively $(E)_{\mathbb{C}}^*$).

Proof. Since

$$y \otimes \varphi = \sum_{j=0}^{\infty} (y, e_j) e_j \otimes \varphi, \tag{1.50}$$

the series converges in $E_{\mathbb{C}} \otimes (E)_{\mathbb{C}}$ (respectively $E_{\mathbb{C}}^* \otimes (E)_{\mathbb{C}}^*$). Now the conclusion follows from (1.50) and (1.45). ∎

§2. Continuous linear operators on distribution spaces

Let $E \hookrightarrow H \hookrightarrow E^*$, $F \hookrightarrow K \hookrightarrow F^*$ be two Gel'fand triplets. By the remark following Theorem 1.5 in Chapter 4, for $p \in \mathbb{Z}, q, \beta \in \mathbb{R}$, we can define the Hilbert spaces $(H_{p,q,\beta})$ and $(K_{p,q,\beta})$. In this section, we shall study continuous linear operators from $(H_{p_1,q_1,\beta_1})_{\mathbb{C}}$ to $(K_{-p_2,-q_2,-\beta_2})_{\mathbb{C}}$ or from $(E)_{\mathbb{C}}^{\beta_1}$ to $(F)_{\mathbb{C}}^{\beta_2}$. For Hilbert spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the set of all continuous linear operators from X to Y. In what follows we also denote by $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{\beta_2})$ the space of continuous operators from $(E)_{\mathbb{C}}^{\beta_1}$ to $(F)_{\mathbb{C}}^{\beta_2}$.

2.1 Symbols and chaos decompositions for operators

Let $0 < \beta_1, \beta_2 < 1$ and $A \in \mathcal{L}((H_{p_1,q_1,\beta_1})_{\mathbb{C}}, (K_{-p_2,-q_2,-\beta_2})_{\mathbb{C}})$. For $f \in E_{\mathbb{C}}, g \in F_{\mathbb{C}}$, we have $\mathcal{E}_f \in (H_{p_1,q_1,\beta_1})_{\mathbb{C}}$, $\mathcal{E}_g \in (K_{p_2,q_2,\beta_2})_{\mathbb{C}}$. Thus we can define a function on $E_{\mathbb{C}} \times F_{\mathbb{C}}$ by

$$\widehat{A}(f, g) = \langle\langle A\mathcal{E}_f, \mathcal{E}_g\rangle\rangle, \quad f \in E_{\mathbb{C}}, g \in F_{\mathbb{C}}. \tag{2.1}$$

We call the restriction of $\widehat{A}$ to $E \times F$ the *symbol* of A. Clearly, any operator is uniquely determined by its symbol.

Remark. Let $0 \leq \beta_1, \beta_2 < 1$. If $A \in \mathcal{L}((E)^{\beta_1}, (F)^{\beta_2})$, then $\widehat{A}$ can be uniquely extended to $E_{\mathbb{C}} \times F_{\mathbb{C}}^*$. If $A \in \mathcal{L}((E)^{-\beta_1}, (F)^{-\beta_2})$, then $\widehat{A}$ can be uniquely extended to $E_{\mathbb{C}}^* \times F_{\mathbb{C}}$. We stilll denote this extension by $\widehat{A}$.

The following result is a generalization of Lemma 2.8 in Chapter 4.

Lemma 2.1 Let $0 < \beta_1, \beta_2 < 1$ and G be a complex function on $E \times F$ satisfying

(C1) $\forall f_1, g_1 \in E$, $f_2, g_2 \in F$, the map $(z, w) \longmapsto G(g_1 + zf_1, g_2 + wf_2)$ has an entire analytic continuation on $\mathbb{C} \times \mathbb{C}$ (still denoted by G);

(C2) there exist constants $C, K_1, K_2 > 0, p_1, p_2 \in \mathbb{Z}$, such that

$$|G(zf, wg)| \leq C \exp\{K_1(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + K_2(|w||g|_{p_2})^{\frac{2}{1-\beta_2}}\},$$
$$f \in E, g \in F, z, w \in \mathbb{C}. \tag{2.2}$$

If $p_1' > p_1, p_2' > p_2$ satisfy $\|I_{p_1,p_1'}\|_{HS} < \infty$, $\|I_{p_2,p_2'}\|_{HS} < \infty$, and q_1, q_2 satisfy

$$\begin{aligned}2^{q_1} &> e^2\left(\tfrac{2K_1}{1-\beta_1}\right)^{1-\beta_1}\|I_{p_1,p_1'}\|_{HS}^2, \\ 2^{q_2} &> e^2\left(\tfrac{2K_2}{1-\beta_2}\right)^{1-\beta_2}\|I_{p_2,p_2'}\|_{HS}^2,\end{aligned} \tag{2.3}$$

then there exists $A \in \mathcal{L}((H_{p_1',q_1,\beta_1})_{\mathbb{C}}, (K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}})$ such that $\widehat{A}(f,g) = G(f,g)$. Moreover, we have

$$\|A\varphi\|_{-p_2',-q_2,-\beta_2} \le C'\|\varphi\|_{p_1',q_1,\beta_1}, \tag{2.4}$$

where

$$\begin{aligned} C' = C\Big[\big(1 - 2^{-q_1}e^2\big(\tfrac{2K_1}{1-\beta_1}\big)^{1-\beta_1}\|I_{p_1p_1'}\|_{HS}^2\big) \\ \times \big(1 - 2^{-q_2}e^2\big(\tfrac{2K_2}{1-\beta_2}\big)^{1-\beta_2}\|I_{p_2p_2'}\|_{HS}^2\big)\Big]^{-1/2}. \end{aligned} \tag{2.5}$$

Proof. First, similar to the proof of Lemma 2.7 in Chapter 4, G can be uniquely extended to an entire function on $E_{\mathbb{C}} \times F_{\mathbb{C}}$. Thus we have the Taylor expansion:

$$G(zf, wg) = \sum_{l,m=0}^{\infty} G_{l,m}(g,f)w^l z^m, \quad f \in E, g \in F, w, z \in \mathbb{C}, \tag{2.6}$$

where $G_{l,m}(g,f)$ is determined by the Cauchy formula:

$$G_{l,m}(g,f) = \Big(\frac{1}{2\pi i}\Big)^2 \int_{|z|=R_1}\int_{|w|=R_2} \frac{G(zf,wg)}{z^{m+1}w^{l+1}}\,dz\,dw. \tag{2.7}$$

Since $G_{l,m}(g,f)$ is a homogeneous polynomial with respect to f and g on E and F respectively, by Schwartz nuclear theorem, there exists $a_{l,m} \in F_{\mathbb{C}}'^{\hat{\otimes} l} \otimes E_{\mathbb{C}}'^{\hat{\otimes} m}$ such that

$$\langle a_{l,m}, g^{\otimes l} \otimes f^{\otimes m}\rangle = G_{l,m}(g,f), \quad f \in E, g \in F. \tag{2.8}$$

Similar to the proof of Lemma 2.8 in Chapter 4, we obtain the following estimate of $a_{l,m}$:

$$\begin{aligned} |a_{l,m}|_{-p_2',-p_1'}^2 \le{}& C^2(m!)^{\beta_1-1}(l!)^{\beta_2-1}\Big[e^2\Big(\frac{2K_1}{1-\beta_1}\Big)^{1-\beta_1}\Big]^m \\ &\times \Big[e^2\Big(\frac{2K_2}{1-\beta_2}\Big)^{1-\beta_2}\Big]^l \|I_{p_1p_1'}\|_{HS}^{2m}\|I_{p_2p_2'}\|_{HS}^{2l}. \end{aligned} \tag{2.9}$$

We define an operator $A \in \mathcal{L}((H_{p_1',q_1,\beta_1})_{\mathbb{C}}, (K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}})$ as follows: for $\varphi \in (H_{p_1',q_1,\beta_1})_{\mathbb{C}}$ with $\varphi \sim \{f_n\}$, put

$$h_l = \sum_{m=0}^{\infty} m!\langle a_{l,m}, f_m\rangle. \tag{2.10}$$

Then

$$\begin{aligned} |h_l|_{-p_2'}^2 &\le \Big(\sum_{m=0}^{\infty} m!|a_{l,m}|_{-p_2',-p_1'}|f_m|_{p_1'}\Big)^2 \\ &\le \|\varphi\|_{p_1',q_1,\beta_1}^2 \sum_{m=0}^{\infty}(m!)^{1-\beta_1}2^{-mq_1}|a_{l,m}|_{-p_2',-p_1'}^2. \end{aligned} \tag{2.11}$$

Thus if we put $A\varphi \sim \{h_l\}$, then (2.4) follows immediately from (2.11) and (2.9), and $\widehat{A} = G$ follows from (2.10),(2.8) and (2.6). ∎

Remark. Under the conditions of the above lemma, for $\varphi \in (H_{p_1',q_1,\beta_1})_{\mathbb{C}}$ with $\varphi \sim \{f_n\}$ and $\forall l, m \in \mathbb{N}_0$, put

$$A_{l,m}\varphi = m! I_l(\langle a_{l,m}, f_m\rangle). \tag{2.12}$$

Then we have

$$A_{l,m} \in \mathcal{L}((H_{p_1',q_1,\beta_1})_{\mathbb{C}}, (K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}}),$$

and

$$A = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} A_{l,m}. \tag{2.13}$$

We call (2.13) the *chaos decomposition* of A. We have

$$\|A\|^2 = \sum_{l=0}^{\infty}\Big\|\sum_{m=0}^{\infty} A_{l,m}\Big\|^2, \quad \sum_{m=0}^{\infty}\|A_{l,m}\| < \infty,$$

where $\|\cdot\|$ denotes the norm of a bounded operator. Moreover, let $\varphi \in (H_{p_1',q_1,\beta_1})_{\mathbb{C}}, \varphi \sim \{f_n\}$, $\psi \in (K_{-p_2',-q_2,-\beta_2})_{\mathbb{C}}$, $\psi \sim \{g_n\}$. Then

$$\langle\langle A\varphi, \psi\rangle\rangle = \sum_{l,m=0}^{\infty} l!m!\langle a_{l,m}, g_l \otimes f_m\rangle. \tag{2.14}$$

the series in the r.h.s. converges absolutely.

Definition 2.2 Let $0 \le \beta_1, \beta_2 < 1$, G be a complex function on $E \times F$. If G satisfies (C1) and (C2) in Lemma 2.1, then G is called a *U_{β_1,β_2}-functional*.

The following theorem gives the symbol characterization of operators.

Theorem 2.3 Let $0 \le \beta_1, \beta_2 < 1$ and G be a complex function on $E \times F$. Then G is the symbol of some $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$ if and only if G is a U_{β_1,β_2}-functional.

Proof. Let $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$. Then $A^* \in \mathcal{L}((F)_{\mathbb{C}}^{\beta_2}, (E)_{\mathbb{C}}^{-\beta_1})$. Since

$$\widehat{A}(f,g) = \langle\langle A\mathcal{E}_f, \mathcal{E}_g\rangle\rangle = S(A^*\mathcal{E}_g)(f),$$

$\widehat{A}$ is a U_{β_1,β_2}-functional. Conversely, let G be a U_{β_1,β_2}-functional. Then similar to the proof of Lemma 2.7 in Chapter 4, we have

$$|G(\xi,\eta)| \le C'\exp\{K'(|\xi|_p^{\frac{2}{1-\beta_1}} + |\eta|_p^{\frac{2}{1-\beta_2}})\}, \xi \in E_{\mathbb{C}}, \eta \in F_{\mathbb{C}},$$

where C', K' are constants determined as follows: take any $0 < \rho < 1$,

$$C' = C(1-\rho)^{-(1+\frac{\beta_1+\beta_2}{2})}, \; K' = (2e^2)^{(\frac{1}{1-\beta_1}+\frac{1}{1-\beta_2})}K\rho^{-(\frac{1+\beta_1}{1-\beta_1}+\frac{1+\beta_2}{1-\beta_2})}.$$

In particular, G satisfies (C2) in Lemma 2.1. Thus by Lemma 2.1, G is the symbol of some $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$. ∎

Theorem 2.4 *Let $0 \le \beta_1, \beta_2 < 1$ and G be a complex function on $E \times F$. Then G is the symbol of some $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{\beta_2})$ if and only if it satisfies the condition* **(C1)** *in Lemma 2.1 and*

(C3) $\forall p_2 \ge 0, \forall \varepsilon > 0$, *there exists $C > 0, p_1 \ge 0$ such that*

$$|G(zf, wg)| \le C \exp\{\varepsilon[(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + (|w||g|_{-p_2})^{\frac{2}{1+\beta_2}}]\},$$
$$f \in E,\ g \in F,\ z, w \in \mathbb{C}. \tag{2.15}$$

Proof. Necessity can be verified easily, we shall prove sufficiency. For any $p_2' \ge 0, q_2 \ge 0$, take $p_2 > p_2'$ such that $\|I_{p_2', p_2}\|_{\mathrm{HS}} < \infty$, and take $\varepsilon > 0$ sufficiently small such that

$$2^{-q_2} > e^2\left(\tfrac{2\varepsilon}{1+\beta_2}\right)^{1+\beta_2}\|I_{p_2' p_2}\|_{\mathrm{HS}}^2 .$$

Take $C > 0, p_1 \ge 0$ such that (2.15) holds. Furthermore take $p_1' > p_1$ and $q_1 \ge 0$ such that

$$2^{q_1} > e^2\left(\tfrac{2\varepsilon}{1-\beta_1}\right)^{1-\beta_1}\|I_{p_1 p_1'}\|_{\mathrm{HS}}^2 .$$

By Lemma 2.1, the operator A corresponding to G satisfies

$$\|A\varphi\|_{-p_2', -q_2, -\beta_2} \le C'\|\varphi\|_{p_1', q_1, \beta_1},$$

where

$$C' = C\Big[\big(1 - 2^{-q_1}e^2\big(\tfrac{2K_1}{1-\beta_1}\big)^{1-\beta_1}\|I_{p_1 p_1'}\|_{\mathrm{HS}}^2\big)$$
$$\times \big(1 - 2^{q_2}e^2\big(\tfrac{2K_2}{1+\beta_2}\big)^{1+\beta_2}\|I_{p_2' p_2}\|_{\mathrm{HS}}^2\big)\Big]^{-1/2}.$$

This means $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$. ∎

Just as the case of distributions, we have the following two important corollaries of Theorem 2.3.

Theorem 2.5 *Let $0 \le \beta_1, \beta_2 < 1$, $\{G_n, n \in \mathbb{N}\}$ be a sequence of U_{β_1, β_2}-functionals on $E \times F$ satisfying*

(1) $\forall f \in E, g \in F$, $\{G_n(f,g), n \in \mathbb{N}\}$ *is a Cauchy sequence in $\mathbb{C}$;*

(2) *there exist $p_1, p_2 \ge 0, C, K > 0$ such that*

$$|G_n(zf, wg)| \le C \exp K[(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + (|w||g|_{-p_2})^{\frac{2}{1+\beta_2}}],\quad z, w \in \mathbb{C}. \tag{2.16}$$

Let A_n be the operator with symbol G_n. Then $\forall \varphi \in (E)_{\mathbb{C}}^{\beta_1}$, $\{A_n\varphi, n \in \mathbb{N}\}$ converges strongly in $(F)_{\mathbb{C}}^{-\beta_2}$.

Theorem 2.6 *Let $0 \le \beta_1, \beta_2 < 1$, $(\Omega, \mathcal{F}, \nu)$ be a measure space, $\omega \longmapsto A_\omega$ a map from Ω to $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (E)_{\mathbb{C}}^{-\beta_2})$, $\widehat{A}_\omega$ the symbol of A_ω. If the following conditions are satisfied:*

(1) $\forall f \in E, g \in F$, $\widehat{A}_\omega(f,g)$ *is a measurable map on $(\Omega, \mathcal{F})$;*

(2) *there exists $K > 0, p_1, p_2 \ge 0$ and ν-integrable function $C(\omega)$ on Ω such that for ν-a.e.ω,*

$$|\widehat{A}_\omega(zf, wg)| \le C(\omega)\exp\{K[(|z||f|_{p_1})^{\frac{2}{1-\beta_1}} + (|w||g|_{-p_2})^{\frac{2}{1+\beta_2}}]\},$$
$$\forall f \in E,\ g \in F,\ z, w \in \mathbb{C}, \tag{2.17}$$

then there exist $p', q \ge 0$ such that $\forall \varphi \in (E)_{\mathbb{C}}^{\beta_1}$, the map $\omega \longmapsto A_\omega\varphi$ is Bochner integrable in $(K_{-p', -q, -\beta_2})_{\mathbb{C}}$ and

$$S\Big(\int_\Omega A_\omega\varphi\, d\nu(\omega)\Big)(g) = \int_\Omega S(A_\omega\varphi)(g)\, d\nu(\omega). \tag{2.18}$$

Remark. There are two similar corollaries of Theorem 2.4.

We now turn to study $\mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$. Let $A \in \mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$. Then $A^* \in \mathcal{L}((F)_{\mathbb{C}}^1, (E)_{\mathbb{C}}^{-1})$. Clearly, there exists some neighbourhood U of $0 \in E_{\mathbb{C}} \times F_{\mathbb{C}}$ such that the following function is well defined on U:

$$\widehat{A}(\xi, \eta) = \langle\langle A\mathcal{E}_\xi, \mathcal{E}_\eta\rangle\rangle,\quad (\xi, \eta) \in U. \tag{2.19}$$

For fixed ξ, $\widehat{A}(\xi, \cdot)$ is the local S-transform of $A\mathcal{E}_\xi$, thus $\widehat{A}(\xi, \cdot) \in \mathrm{Hol}_0(F_{\mathbb{C}})$; for fixed η, $\widehat{A}(\cdot, \eta)$ is the local S-transform of $A^*\mathcal{E}_\eta$, thus $\widehat{A}(\cdot, \eta) \in \mathrm{Hol}_0(E_{\mathbb{C}})$. In summary, $\widehat{A}$ is holomorphic on U. We call $\widehat{A}$ the *local symbol* of A.

The following theorem characterizes $\mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$ by means of symbols. Since the proof is similar to that of Theorem 2.13 in Chapter 4, we omit it.

Theorem 2.7. *Let $G \in \mathrm{Hol}_0(E_{\mathbb{C}} \times F_{\mathbb{C}})$. Then there exists a unique $A \in \mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$ such that the local S-transform $\widehat{A}$ of A coincides with G on some neighbourhood of 0.*

By virtue of Theorem 2.14 and 2.15 in Chapter 4, we obtain the following two corollaries of Theorem 2.7.

Theorem 2.8 *Let $A_n \in \mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$, $n \ge 1$. If there exists a neighbourhood U of $0 \in E_{\mathbb{C}} \times F_{\mathbb{C}}$ such that every $\widehat{A}_n$ is well-defined on U, and $\{\widehat{A}_n, n \ge 1\}$ is uniformly bounded and converges everywhere on U, then for any $\varphi \in (E)_{\mathbb{C}}^1$, $\{A_n\varphi, n \ge 1\}$ converges strongly in $(F)_{\mathbb{C}}^{-1}$.*

Theorem 2.9 *Let $\omega \longmapsto A_\omega$ be a measurable map from $(\Omega, \mathcal{F}, \nu)$ to $\mathcal{L}((E)_{\mathbb{C}}^1, (F)_{\mathbb{C}}^{-1})$. Assume that there exists some neighbourhood U of $0 \in E_{\mathbb{C}} \times F_{\mathbb{C}}$ such that every $\widehat{A}_\omega$ is well defined on U and satisfies*

(1) $\forall (\xi, \eta) \in U, \omega \longmapsto \widehat{A}_\omega(\xi, \eta)$ *is measurable;*

(2) *there exists a non-negative ν-integrable function $C(\omega)$ such that for ν-a.e. ω,*

$$|\widehat{A}_\omega(\xi, \eta)| \le C(\omega),\ \forall (\xi, \eta) \in U.$$

Then there exist $p \in \mathbb{N}_0, q \ge 0$ such that $\forall \varphi \in (E)_{\mathbb{C}}^1$, $\omega \longmapsto A_\omega\varphi$ is Bochner integrable in $(K_{-p,-q,-1})_{\mathbb{C}}$ and there exists a neighbourhood V_φ of $0 \in F_{\mathbb{C}}$ such that

$$\Big\langle\Big\langle \int_\Omega A_\omega\varphi\, d\nu(\omega), \mathcal{E}_\eta\Big\rangle\Big\rangle = \int_\Omega \langle\langle A_\omega\varphi, \mathcal{E}_\eta\rangle\rangle\, d\nu(\omega),\ \eta \in V_\varphi.$$

We now study the characterization of $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$. Just as the case of distributions, when $\beta > 1$, since we cannot define the S-transforms or local S-transforms for distributions, we can not define the symbols or local symbols for elements of $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$. But by virtue of the moment characterization of

distributions studied in §4 of Chapter 4, we can give a unified characterization of $\mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$.

In what follows we denote by $\mathcal{P}_E$ and $\mathcal{P}_F$ the polynomial functionals on E^* and F^*, respectively.

Theorem 2.10 *Let $0 \le \beta_1, \beta_2 < \infty$, G be a bilinear form on $\mathcal{P}_E \times \mathcal{P}_F$. In order that there exist $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta_1}, (F)_{\mathbb{C}}^{-\beta_2})$ such that*

$$\langle\langle A\varphi, \psi\rangle\rangle = G(\varphi, \psi), \ \varphi \in \mathcal{P}_E, \psi \in \mathcal{P}_F, \tag{2.20}$$

it is necessary and sufficient that there exist $p_1, p_2 \ge 0, C > 0, K > 0$ such that $\forall j, k \in \mathbb{N}_0$,

$$|G(W_f^j, W_g^k)| \le KC^{j+k}(j!)^{\frac{1+\beta_1}{2}}(k!)^{\frac{1+\beta_2}{2}}|f|_{p_1}^j |g|_{p_2}^k, \ f \in E, g \in F. \tag{2.21}$$

Proof. Left to the reader as an exercise.

2.2 S-transforms and Wick products of generalized operators

Assume $0 \le \beta < 1$ in this section. Elements of $\mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{-\beta})$ are called *generalized operators.* For $y \in E_{\mathbb{C}}^*$, $D_y \in \mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{\beta})$, $D_y^* \in \mathcal{L}((E)_{\mathbb{C}}^{-\beta}, (E)_{\mathbb{C}}^{-\beta})$. In quantum physics, D_y and D_y^* are called *annihilation* and *creation operator* respectively. For $x \in E_{\mathbb{C}}^*$, $D_y^* D_x$ is still a generalized operator, but $D_x D_y^*$ makes no sense in general (except for $y \in E_{\mathbb{C}}$). If $y \in E_{\mathbb{C}}$, then by (1.35),

$$D_y^* D_x = D_x D_y^* - \langle x, y\rangle.$$

When $x, y \in E_{\mathbb{C}}^*$, $D_y^* D_x$ can be formally regarded as the difference of two ill-defined objects in the r.h.s. of the above equality. $D_y^* D_x$ is called the Wick renormalization or *Wick ordering* of $D_x D_y^*$. This section is devoted to generalizing this Wick ordering to Wick calculus of generalized operators.

We have defined the symbols of generalized operators in the last section. For the notational convenience, we shall define the S-transform of generalized operators which is the natural generalization of the S-transform of distributions. To this end first consider a distribution $F \in (E)_{\mathbb{C}}^{-\beta}$. Multiplying testing functionals by F can be viewed as a generalized operator (called multiplication operator) acting on the testing functional space. Let $f, g \in E_{\mathbb{C}}$. Then

$$\begin{aligned} SF(f+g) &= \langle\langle F, \mathcal{E}_{f+g}\rangle\rangle = \langle\langle F, \mathcal{E}_f \mathcal{E}_g\rangle\rangle e^{-\langle f, g\rangle} \\ &= \langle\langle F\mathcal{E}_f, \mathcal{E}_g\rangle\rangle e^{-\langle f, g\rangle} = \widehat{F}(f, g) e^{-\langle f, g\rangle}. \end{aligned}$$

Thus for a general $A \in \mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{-\beta})$, we define its S-transform as

$$\widetilde{A}(f, g) = \widehat{A}(f, g) e^{-\langle f, g\rangle}, \ f, g \in E. \tag{2.22}$$

Now we interprete the Wick ordering from the viewpoint of S-transforms of generalized operators. Since

$$\begin{aligned} \widehat{D_y^* D_x}(f, g) &= \langle\langle D_y^* D_x \mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\ &= \langle\langle D_x \mathcal{E}_f, D_y \mathcal{E}_g\rangle\rangle \\ &= \langle x, f\rangle\langle y, g\rangle\langle\langle \mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\ &= \langle x, f\rangle\langle y, g\rangle e^{\langle f, g\rangle}, \end{aligned}$$

we have $\widetilde{D_y^* D_x}(f, g) = \langle x, f\rangle\langle y, g\rangle$. On the other hand,

$$\widehat{D}_y^*(f, g) = \langle y, g\rangle e^{\langle f, g\rangle}, \quad \widehat{D}_x(f, g) = \langle x, f\rangle e^{\langle f, g\rangle}.$$

Thus $\widetilde{D_y^*}(f, g) = \langle y, g\rangle$, $\widetilde{D_x}(f, g) = \langle x, f\rangle$. Finally we have

$$\widetilde{D_y^* D_x} = \widetilde{D_y^*}\widetilde{D_x}. \tag{2.23}$$

Next we interpret Wick products of distributions from the viewpoint of S-transforms of generalized operators. Let $F, G \in (E)_{\mathbb{C}}^{-\beta}$. F, G and $F \diamond G$ can be regarded as multiplication operators with S-transforms

$$\begin{aligned} \widetilde{F}(f, g) &= \langle\langle F\mathcal{E}_f, \mathcal{E}_g\rangle\rangle e^{-\langle f, g\rangle} = SF(f+g), \\ \widetilde{G}(f, g) &= SG(f+g), \\ \widetilde{F \diamond G}(f, g) &= S(F \diamond G)(f+g) = SF(f+g) SG(f+g), \end{aligned}$$

respectively. Hence

$$\widetilde{F \diamond G} = \widetilde{F}\widetilde{G}. \tag{2.24}$$

Finally, let $\mathcal{G}_{\beta,\beta}$ and $\mathcal{G}_{\beta,-\beta}$ be the spaces of S-transforms of $\mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{\beta})$ and $\mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{-\beta})$ respectively. Then by Theorems 2.3 and 2.4, it is easily verified that both $\mathcal{G}_{\beta,\beta}$ and $\mathcal{G}_{\beta,-\beta}$ are closed under product.

Motivated by the above observations, we introduce

Definition 2.11 Let $A, B \in \mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{-\beta})$. The operator with S-transform $\widetilde{A}\widetilde{B}$ is called the *Wick product* of A and B, denoted by $A \diamond B$, i.e.,

$$\widetilde{A \diamond B} = \widetilde{A} \cdot \widetilde{B}. \tag{2.25}$$

By the discussion above, Wick product of generalized operators is a generalization of both Wick ordering and Wick product of distributions. By definition, Wick product is commutative and associative. Moreover, both $\mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{\beta})$ and $\mathcal{L}((E)_{\mathbb{C}}^{\beta}, (E)_{\mathbb{C}}^{-\beta})$ are algebras under Wick product with the identity operator as unit.

The following lemma shows that Wick ordering in quantum physics can be interpreted as Wick product of generalized operators.

Lemma 2.12 *Let $A, B \in \mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{-\beta}_{\mathscr{E}})$, $y \in E^*_{\mathscr{E}}$. Then*

(i) $(A \diamond B)^* = A^* \diamond B^*$;

(ii) $D^*_y \diamond A = D^*_y A, \quad D_y \diamond A = A D_y$.

Proof. (i) Let $K \in \mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{-\beta}_{\mathscr{E}})$. Then $\widetilde{K^*}(f,g) = \widetilde{K}(g,f)$, from which (i) follows.

(ii) For $f, g \in E$, we have

$$\begin{aligned}\widetilde{D^*_y A}(f,g) &= e^{-(f,g)}\langle\langle D^*_y A\mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\ &= e^{-(f,g)}\langle\langle A\mathcal{E}_f, D_y\mathcal{E}_g\rangle\rangle \\ &= (y,g)e^{-(f,g)}\widetilde{A}(f,g) = \widetilde{D^*_y}(f,g)\widetilde{A}(f,g),\end{aligned}$$

from which the first formula of (ii) follows. The second formula can be proved similarly. ∎

Definition 2.13 Let

$$\mathcal{G}^{\beta} = \{S\varphi : \varphi \in (E)^{\beta}_{\mathscr{E}}\}, \ \mathcal{G}^{-\beta} = \{S\psi : \psi \in (E)^{-\beta}_{\mathscr{E}}\}.$$

Define a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{G}^{-\beta} \times \mathcal{G}^{\beta}$ as

$$\langle S\psi, S\varphi\rangle = \langle\langle \psi, \varphi\rangle\rangle, \ \varphi \in (E)^{\beta}_{\mathscr{E}}, \psi \in (E)^{-\beta}_{\mathscr{E}}. \tag{2.26}$$

Lemma 2.14 *Let $F \in \mathcal{G}^{\beta}, G \in \mathcal{G}^{-\beta}$, $A \in \mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{\beta}_{\mathscr{E}})$, $B \in \mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{-\beta}_{\mathscr{E}})$. Then $\forall f \in E_{\mathscr{E}}$,*

$$F(f+\cdot) \in \mathcal{G}^{\beta}, G(f+\cdot) \in \mathcal{G}^{-\beta}; \widetilde{A}(f,\cdot) \in \mathcal{G}^{\beta}, \widetilde{B}(f,\cdot) \in \mathcal{G}^{-\beta}.$$

Proof. Immediately follows from Theorem 2.9 and 2.16 in Chapter 4. ∎

Lemma 2.15 *Let $F \in \mathcal{G}^{\beta}, G \in \mathcal{G}^{-\beta}, f \in E_{\mathscr{E}}$. Then*

$$G(f) = \langle G, e^{(f,\cdot)}\rangle, \tag{2.27}$$

$$\langle G, Fe^{(f,\cdot)}\rangle = \langle G(f+\cdot), F\rangle, \tag{2.28}$$

$$\langle e^{(f,\cdot)}G, F\rangle = \langle G, F(f+\cdot)\rangle. \tag{2.29}$$

Proof. Let $G = S\psi$. Then

$$\begin{aligned}G(f) &= \langle\langle \psi, \mathcal{E}_f\rangle\rangle = \langle S\psi, S\mathcal{E}_f\rangle \\ &= \langle G, e^{(f,\cdot)}\rangle,\end{aligned}$$

which gives (2.27). In order to prove (2.28), it suffices to consider the case $F = e^{(g,\cdot)}$, $g \in E_{\mathscr{E}}$. By (2.27),

$$\langle G, e^{(g,\cdot)}e^{(f,\cdot)}\rangle = G(f+g) = \langle G(f+\cdot), e^{(g,\cdot)}\rangle.$$

Thus (2.28) is proved. (2.29) can be proved similarly. ∎

By (2.26) and (2.28), we have

Theorem 2.16 *Let $\varphi \in (E)^{-\beta}_{\mathscr{E}}, \psi \in (E)^{\beta}_{\mathscr{E}}$. Then $\forall f \in E_{\mathscr{E}}$, we have*

$$S(\varphi\psi)(f) = \langle S\varphi(f+\cdot), S\psi(f+\cdot)\rangle. \tag{2.30}$$

Proof. First, $\forall g \in E_{\mathscr{E}}$, we have

$$\langle\langle \psi\mathcal{E}_f, \mathcal{E}_g\rangle\rangle = e^{(f,g)}S\psi(f+g).$$

Thus

$$\begin{aligned}S(\varphi\psi)(f) &= \langle\langle \varphi\psi, \mathcal{E}_f\rangle\rangle \\ &= \langle\langle \varphi, \psi\mathcal{E}_f\rangle\rangle = \langle S\varphi, S(\psi\mathcal{E}_f)\rangle \\ &= \langle S\varphi, e^{(f,\cdot)}S\psi(f+\cdot)\rangle \\ &= \langle S\varphi(f+\cdot), S\psi(f+\cdot)\rangle.\end{aligned}$$ ∎

The following theorem is a generalization of the above theorem (see the Remark below).

Theorem 2.17 *Let $A \in \mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{-\beta}_{\mathscr{E}})$, $B \in \mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{\beta}_{\mathscr{E}})$. Then $\forall f, g \in E_{\mathscr{E}}$, we have*

$$\widetilde{AB}(f,g) = \langle \widetilde{A}(f+\cdot, g), \widetilde{B}(f, g+\cdot)\rangle. \tag{2.31}$$

Proof. We have

$$\begin{aligned}\widetilde{AB}(f,g) &= e^{-(f,g)}\langle\langle AB\mathcal{E}_f, \mathcal{E}_g\rangle\rangle \\ &= e^{-(f,g)}\langle\langle A^*\mathcal{E}_g, B\mathcal{E}_f\rangle\rangle \\ &= e^{-(f,g)}\langle S(A^*\mathcal{E}_g), S(B\mathcal{E}_f)\rangle \\ &= e^{-(f,g)}\langle e^{(g,\cdot)}\widetilde{A^*}(g,\cdot), e^{(f,\cdot)}\widetilde{B}(f,\cdot)\rangle \\ &= e^{-(f,g)}\langle e^{(g,\cdot)}\widetilde{A}(\cdot,g), e^{(f,\cdot)}\widetilde{B}(f,\cdot)\rangle \\ &= e^{-(f,g)}\langle e^{(g,f+\cdot)}\widetilde{A}(f+\cdot,g), \widetilde{B}(f,\cdot)\rangle (\text{by } (2.28)) \\ &= \langle e^{(g,\cdot)}\widetilde{A}(f+\cdot,g), \widetilde{B}(f,\cdot)\rangle \\ &= \langle \widetilde{A}(f+\cdot,g), \widetilde{B}(f,g+\cdot)\rangle (\text{by } (2.29)).\end{aligned}$$ ∎

Remark. If we interprete φ, ψ and $\varphi\psi$ in Theorem 2.16 as multiplication operators, then by (2.30),

$$\begin{aligned}\widetilde{\varphi\psi}(f,g) &= S(\varphi\psi)(f+g) \\ &= \langle S\varphi(f+g+\cdot), S\psi(f+g+\cdot)\rangle \\ &= \langle \widetilde{\varphi}(f+\cdot,g), \widetilde{\psi}(f,g+\cdot)\rangle.\end{aligned}$$

Thus, Theorem 2.17 indeed generalizes Theorem 2.16.

The following theorem gives a characterization for a class of operators in $\mathcal{L}((E)^{\beta}_{\mathscr{E}}, (E)^{\beta}_{\mathscr{E}})$.

Theorem 2.18 *Let $B \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{\beta}_{\mathbb{C}})$. Then the following assertions are equivalent:*

(i) $BD_\xi = D_\xi B$, $\forall \xi \in E_{\mathbb{C}}$;

(ii) $\widetilde{B}(f,g) = \widetilde{B}(f,0)$, $\forall f,g \in E_{\mathbb{C}}$;

(iii) $A \diamond B = AB$, $\forall A \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{-\beta}_{\mathbb{C}})$.

Proof. (i)$\Rightarrow$(ii). $\forall f,g \in E_{\mathbb{C}}$, we have

$$\begin{aligned}\widetilde{BD_\xi}(f,g) &= e^{-(f,g)}\widehat{BD_\xi}(f,g) - (f,\xi)e^{-(f,g)}\widehat{B}(f,g)\\ &= (f,\xi)\widetilde{B}(f,g),\end{aligned} \tag{2.32}$$

$$\begin{aligned}\widetilde{D_\xi B}(f,g) &= e^{-(f,g)}\langle\langle D_\xi B\mathcal{E}_f, \mathcal{E}_g\rangle\rangle\\ &- e^{-(f,g)}\langle\langle B\mathcal{E}_f, D_\xi^*\mathcal{E}_g\rangle\rangle\\ &- e^{-(f,g)}\langle\langle B\mathcal{E}_f, I_1(\xi)\diamond\mathcal{E}_g\rangle\rangle\\ &= e^{-(f,g)}\lim_{\epsilon\downarrow 0}\langle\langle B\mathcal{E}_f, \frac{1}{\epsilon}(\mathcal{E}_{g+\epsilon\xi} - \mathcal{E}_g)\rangle\rangle\\ &- \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}(\widetilde{B}(f,g+\epsilon\xi) - \widetilde{B}(f,g))\\ &+ \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}(e^{-(f,g)} - e^{-(f,g+\epsilon\xi)})\langle\langle B\mathcal{E}_f, \mathcal{E}_{g+\epsilon\xi}\rangle\rangle\\ &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}(\widetilde{B}(f,g+\epsilon\xi) - \widetilde{B}(f,g)) + (f,\xi)\widetilde{B}(f,g).\end{aligned}$$

If (i) holds, then

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}(\widetilde{B}(f,g+\epsilon\xi) - \widetilde{B}(f,g)) = 0, \ \forall g,\xi \in E_{\mathbb{C}}.$$

Thus $\widetilde{B}(f,g)$ does not depend on g, i.e., (ii) holds.

(ii)$\Rightarrow$(iii). If (ii) holds, then by (2.31) and (ii), $\forall f,g \in E_{\mathbb{C}}$,

$$\begin{aligned}\widetilde{AB}(f,g) &= (\widetilde{A}(f+\cdot,g), \widetilde{B}(f,0))\\ &= \widetilde{B}(f,0)(\widetilde{A}(f+\cdot,g),1)\\ &= \widetilde{B}(f,g)\widetilde{A}(f,g) = \widetilde{A\diamond B}(f,g).\end{aligned}$$

(iii)$\Rightarrow$(i). By (2.32), we have $BD_\xi = B\diamond D_\xi$, thus (iii) $\Rightarrow$(i). ∎

As a dual form of Theorem 2.18, we have

Theorem 2.19 *Let $A \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{-\beta}_{\mathbb{C}})$. Then the following assertions are equivalent:*

(i) $AD_\xi^* = D_\xi^* A$, $\forall \xi \in E_{\mathbb{C}}$;

(ii) $\widetilde{A}(f,g) = \widetilde{A}(0,g)$, $\forall f,g \in E_{\mathbb{C}}$;

(iii) $A \diamond B = AB$, $\forall B \in \mathcal{L}((E)^{\beta}_{\mathbb{C}}, (E)^{\beta}_{\mathbb{C}})$.

Proof. Similar to the proof of Theorem 2.18. ∎

We should mention that all results in this section can be adapted for the case $\beta = 1$, provided we replace S-transforms by local S-transforms. We suggest the reader to formulate the results and give their proofs.

§3. Integral kernel operators and integral kernel representation for operators

In this section we shall study continuous operators from $(E)_{\mathbb{C}}$ to $(E)^*_{\mathbb{C}}$ in the classical framework of white noise analysis. We assume that the Gel'fand triplet $E \hookrightarrow H \hookrightarrow E^*$ is generated by a separable Hilbert space H and a positive self-adjoint operator on it (see Section 1.3 of Chapter 4), $\{|\cdot|_p, p \geq 0\}$ is the standard sequence of norms determined by A. For convenience, we assume that $\|A^{-1}\|_{\mathrm{HS}} < \infty$ (otherwise replacing A by A^m). From Section 1.3 of Chapter 4 we know that (E) is the projective limit of $\{(E)_p, p \in \mathbb{N}_0\}$ (where $(E)_p = D(\Gamma(A)^p)$), and $(E)^*$ is the inductive limit of $\{(E)_{-p}, p \in \mathbb{N}_0\}$.

Henceforth, let ρ denote $\|A^{-1}\|$, δ denote $\|A^{-1}\|_{\mathrm{HS}}$. By assumption, $0 < \rho < 1$.

3.1 Contraction of tensor products

We shall generalize the definition of contraction of tensor products defined in Section 3.1 of Chapter 4. Let $\{e_j, j \geq 0\}$ be an orthonormal base of H consisting of the eigenvectors of A. Let $1 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \cdots$ be the corresponding eigenvalues, i.e., $Ae_j = \lambda_j e_j$, $\forall j \geq 0$. Clearly $\{e_j, j \geq 0\}$ also constitutes an orthonormal base of $H_{\mathbb{C}}$. When A^{-1} is viewed as a self-adjoint operator on $H_{\mathbb{C}}$, we have

$$\|A^{-1}\| = \sup\{\lambda_j^{-1}, j \in \mathbb{N}_0\} = \lambda_0^{-1} = \rho,$$

$$\|A^{-1}\|^2_{\mathrm{HS}} = \sum_{j=0}^{\infty}\lambda_j^{-2} = \delta^2.$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_n$, $e_\alpha = \otimes_{j=1}^n e_{\alpha_j}$. Then $e_\alpha \in E^{\otimes n}$, $\{e_\alpha, \alpha \in \mathbb{N}_n\}$ constitutes an orthonormal base of both $H^{\otimes n}$ and $H_{\mathbb{C}}^{\otimes n}$.

Lemma 3.1 *We have*

$$|e_\alpha|_p |e_\alpha|_{-p} = 1, \quad p \in \mathbb{R}, \tag{3.1}$$

$$|e_\alpha|_p \geq \rho^{-pn}, \quad p \geq 0, \alpha \in \mathbb{N}_n, \tag{3.2}$$

$$|f|_p^2 = \sum_{\alpha \in \mathbb{N}_n} |(f,e_\alpha)|^2 |e_\alpha|_p^2, \ f \in H^{\otimes n}_{p,\mathbb{C}}, \ p \in \mathbb{R}. \tag{3.3}$$

Proof. Let $\lambda_\alpha^p = \prod_i \lambda_{\alpha_i}^p$. Since $|e_\alpha|_p = \lambda_\alpha^p$, (3.1) and (3.2) obviously hold. But $\{\lambda_\alpha^{-p} e_\alpha, \alpha \in \mathbb{N}_n\}$ is an orthonormal base of $H^{\otimes n}_{p,\mathbb{C}}$, thus (3.3) holds. ∎

Let $m, n \in \mathbb{N}$, $p, q \in \mathbb{R}$. Denote by $|\cdot|_{m,n;p,q}$ the norm on $H_{p,\mathbb{C}}^{\otimes m} \otimes H_{q,\mathbb{C}}^{\otimes n}$, i.e.,

$$|f|_{m,n;p,q} = |(A^p)^{\otimes m} \otimes (A^q)^{\otimes n} f|_0 .$$

Similarl to the proof of (3.3), we have

$$|f|^2_{m,n;p,q} = \sum_{\alpha \in \mathbb{N}_m, \sigma \in \mathbb{N}_n} |(f, e_\alpha \otimes e_\sigma)|^2 |e_\alpha|_p^2 |e_\sigma|_q^2. \tag{3.4}$$

Thus by (3.4) we have

$$|f|_p = |f|_{m,n;p,p}, \quad f \in H_{p,\mathbb{C}}^{\otimes m+n}, \tag{3.5}$$

$$|f|_{m,n;p,q} \leq \rho^{mr+ns} |f|_{m,n;p+r,q+s}, \; p, q \in \mathbb{R}, \; r, s \geq 0. \tag{3.6}$$

Lemma 3.2 *For $f \in E_{\mathbb{C}}^{\otimes l+m}$, $g \in E_{\mathbb{C}}^{\otimes l+n}$, define $f \otimes_l g$ by (3.4) in Chapter 4. Then*

$$|f \otimes_l g|_{m,n;p,q} \leq |f|_{m,l;p,r} |g|_{n,l;q,-r}, \; p, q, r \in \mathbb{R}, \tag{3.7}$$

$$|f \otimes_l g|_p \leq \rho^{2pl} |f|_p |g|_p, \quad p \geq 0, \tag{3.8}$$

$$|f \otimes_l g|_{-p} \leq \rho^{2pm} |f|_{-p} |g|_p, \quad p \geq 0, \tag{3.9}$$

$$|f \otimes_l g|_p \leq \rho^{qn} |f|_{m,l;p,-(p+q)} |g|_{p+q}, \; p \in \mathbb{R}, \; q \geq 0 \tag{3.10}$$

Proof. Since $|e_\alpha|_{-r} |e_\alpha|_r = 1$, by (3.4) above and (3.4) in Chapter 4, we have

$$\begin{aligned} |f \otimes_l g|^2_{m,n;p,q} &= \sum_{\sigma \in \mathbb{N}_m, \delta \in \mathbb{N}_n} \Big| \sum_{\alpha \in \mathbb{N}_l} (f, e_\sigma \otimes e_\alpha)(g, e_\delta \otimes e_\alpha) \Big|^2 |e_\sigma|_p^2 |e_\delta|_q^2 \\ &\leq \sum_{\alpha \in \mathbb{N}_l, \sigma \in \mathbb{N}_m} |(f, e_\sigma \otimes e_\alpha)|^2 |e_\alpha|_r^2 |e_\sigma|_p^2 \\ &\quad \times \sum_{\alpha \in \mathbb{N}_l, \delta \in \mathbb{N}_n} |(g, e_\delta \otimes e_\alpha)|^2 |e_\alpha|_{-r}^2 |e_\delta|_q^2, \end{aligned}$$

this is just (3.7). Now put $r = q = p$ and $r = q = -p$ respectively in (3.7) and use the inequalities

$$|g|_{n,l;p,-p} \leq \rho^{2pl} |g|_p, \; |g|_{n,l;-p,p} \leq \rho^{2pn} |g|_p, \; p \geq 0,$$

we obtain (3.8) and (3.9). Finally, put $r = -(p-q)$ in (3.7) we obtain (3.10). ∎

Remark 1. By (3.8), $\{f, g\} \longrightarrow f \otimes_l g$ is a continuous bilinear form from $E_{\mathbb{C}}^{\otimes l+m} \times E_{\mathbb{C}}^{\otimes l+n}$ to $E_{\mathbb{C}}^{\otimes m+n}$. By (3.9), $\{f, g\} \longmapsto f \otimes_l g$ can be extended to a separatively continuous bilinear form from $E_{\mathbb{C}}^{*\otimes l+m} \times E_{\mathbb{C}}^{\otimes l+n}$ to $E_{\mathbb{C}}^{*\otimes m+n}$. Thus, in (3.9) and (3.10), f can be taken from $E_{\mathbb{C}}^{*\otimes l+m}$.

Remark 2. Since symmetrization does not increase norm, (3.7)–(3.10) hold for $f \hat{\otimes}_l g$ (as an element of $E_{\mathbb{C}}^{\hat{\otimes} m+n}$ or $E_{\mathbb{C}}^{*\hat{\otimes} m+n}$).

Lemma 3.3 *Let $F \in E_{\mathbb{C}}^{*\hat{\otimes} k}$, $G \in E_{\mathbb{C}}^{*\hat{\otimes} l}$, $h \in E_{\mathbb{C}}^{\hat{\otimes} k+l+m}$. Then*

$$(F \hat{\otimes} G) \hat{\otimes}_{k+l} h = F \hat{\otimes}_k (G \hat{\otimes}_l h). \tag{3.11}$$

Proof. Put $m = 0$ in (3.10). Then

$$|f \otimes_l g|_p \leq \rho^{qn} |f|_{-(p+q)} |g|_{p+q}, \; p \in \mathbb{R}, q > 0. \tag{3.12}$$

From this we easily conclude that for fixed F and G, both sides of (3.11) are continuous with respect to h. But (3.11) obviously holds for $h = \xi^{\otimes k+l+m}$ ($\xi \in E_{\mathbb{C}}$), hence also holds for any $h \in E_{\mathbb{C}}^{\hat{\otimes} k+l+m}$. ∎

Lemma 3.4 *Let $F \in E_{\mathbb{C}}^{*\hat{\otimes} l}$, $G \in E_{\mathbb{C}}^{*\hat{\otimes} m}$. Then for any $f \in E_{\mathbb{C}}^{\hat{\otimes} l+n}$, $g \in E_{\mathbb{C}}^{\hat{\otimes} m+n}$,*

$$\langle F \otimes_l f, G \otimes_m g \rangle = \langle F \otimes G, f \otimes_n g \rangle. \tag{3.13}$$

Proof. By (3.9) and (3.12), both sides of (3.13) are continuous bilinear form with respect to f and g. Moreover, let $\xi, \eta \in E_{\mathbb{C}}$. Then (3.13) holds for $f = \xi^{\otimes l+n}$ and $g = \eta^{\otimes m+n}$, hence still holds for general f and g. ∎

Remark. Put $m = 0$, $G = 1$ in the above lemma. Then

$$\langle F \otimes_l f, g \rangle = \langle F, f \otimes_n g \rangle. \tag{3.14}$$

3.2 Integral kernel operators

In this and next subsections, we further assume that

$$H = L^2(T, \mathcal{B}(T), \nu),$$

where T is a Hausdorff space, ν is a σ-finite Borel measure on $\mathcal{B}(T)$. We identify every element of H with its ν-equivalent class and assume that H is a separable Hilbert space. For example, if the Borel σ-algebra $\mathcal{B}(T)$ is separable, then the above assumption holds. Furthermore, we assume that

(H1) every element of E has a continuous version on T, i.e. for any $\xi \in E$, there exists a continuous function $\tilde{\xi}$ on T such $\xi(t) = \tilde{\xi}(t)$, ν-a.e. t. In what follows we always take this continuous version so long as an element of E is concerned;

(H2) for any $t \in T$, the Dirac δ-functional $\delta_t : \xi \longmapsto \xi(t)$ is continuous on E, i.e., $\delta_t \in E^*, \forall t \in T$;

(H3) the map $t \mapsto \delta_t$ is continuous from T to E^*.

In the sequel, we shall denote D_{δ_t} and $D_{\delta_t}^*$ by ∂_t and ∂_t^* respectively (∂_t is called *Hida's differential operator*), and denote $\nu(dt)$ simply by dt and $\|\cdot\|_{p,0,0}$ by $\|\cdot\|_p$.

Lemma 3.5 *For $\varphi, \psi \in (E)_{\mathbb{C}}$, put*

$$\eta_{\varphi,\psi}(s_1, \cdots, s_l, t_1, \cdots, t_m) = \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \varphi, \psi \rangle\rangle. \tag{3.15}$$

Then $\forall p > 0$,

$$|\eta_{\varphi,\psi}|_p \leq (1 - \rho^{2p})^{-\frac{l+m-2}{2}} \sqrt{l!m!} \|\varphi\|_p \|\psi\|_p. \tag{3.16}$$

In particular, $\eta_{\varphi,\psi} \in E_{\mathbb{C}}^{\hat{\otimes} l} \otimes E_{\mathbb{C}}^{\hat{\otimes} m}$.

Proof. Let $\varphi \sim \{f_n\}, \psi \sim \{g_n\}$. Since

$$\eta_{\varphi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m) = \langle\langle \partial_{t_1}\cdots\partial_{t_m}\varphi, \partial_{s_1}\cdots\partial_{s_l}\psi\rangle\rangle,$$

by (1.16), we have

$$\partial_{t_1}\cdots\partial_{t_m}\varphi \sim \left\{\frac{(n+m)!}{n!}\delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_m}\widehat{\otimes}_m f_{n+m}\right\},$$
$$\partial_{s_1}\cdots\partial_{s_l}\psi \sim \left\{\frac{(n+l)!}{n!}\delta_{s_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{s_l}\widehat{\otimes}_l g_{n+l}\right\}.$$

Hence by (3.13),

$$\begin{aligned}
&\eta_{\varphi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m)\\
&=\sum_{n=0}^{\infty} n!\frac{(n+m)!}{n!}\frac{(n+l)!}{n!}\langle \delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_m}\otimes_m f_{n+m}, \delta_{s_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{s_l}\otimes_l g_{n+l}\rangle\\
&=\sum_{n=0}^{\infty}\frac{(n+m)!(n+l)!}{n!}\langle \delta_{s_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{s_l}\otimes\delta_{t_1}\widehat{\otimes}\cdots\widehat{\otimes}\delta_{t_m}, g_{n+l}\otimes_n f_{n+m}\rangle,
\end{aligned}$$

i.e.,

$$\eta_{\varphi,\psi} = \sum_{n=0}^{\infty}\frac{(n+m)!(n+l)!}{n!} g_{n+l}\otimes_n f_{n+m}. \tag{3.17}$$

By (3.8),

$$|g_{n+l}\otimes_n f_{n+m}|_p \le \rho^{2pn}|f_{n+m}|_p|g_{n+l}|_p,$$

we have

$$\begin{aligned}
|\eta_{\varphi,\psi}|_p &\le \sum_{n=0}^{\infty}\frac{\sqrt{(n+m)!(n+l)!}}{n!}\rho^{2pn}\sqrt{(n+m)!}|f_{n+m}|_p\sqrt{(n+l)!}|g_{n+l}|_p\\
&\le C_{l,m,\rho,p}\|\varphi\|_p\|\psi\|_p,
\end{aligned} \tag{3.18}$$

where

$$C_{l,m,\rho,p} = \sup_{n\ge 0}\frac{\sqrt{(n+m)!(n+l)!}}{n!}\rho^{2pn}. \tag{3.19}$$

Since

$$\sum_{n=0}^{\infty}\frac{(n+m)!}{n!}\rho^{2pn} = (1-\rho^{2p})^{-(m+1)}m!,$$

we have

$$C_{l,m,\rho,p} \le (1-\rho^{2p})^{-\frac{l+m+2}{2}}\sqrt{l!m!}. \tag{3.20}$$

Thus (3.16) is proved. ∎

Remark. $\eta_{\varphi,\psi}$ defined by (3.15) depends on (l,m) and is sometimes denoted by $\eta_{\varphi,\psi}^{(l,m)}$. When (l,m) is clear from the context, we shall denote it simply by $\eta_{\varphi,\psi}$.

In the following definition, some useful notations are introduced.

Definition 3.6 (1) Let $\kappa \in E_{\mathbb{C}}^{*\otimes n}$ and σ be a permutation of $\{1,2,\cdots,n\}$. Then there exists a unique $\kappa^\sigma \in E_{\mathbb{C}}^{*\otimes n}$ such that

$$\langle\kappa^\sigma, \xi_1\otimes\cdots\otimes\xi_n\rangle = \langle\kappa, \xi_{\sigma^{-1}(1)}\otimes\cdots\otimes\xi_{\sigma^{-1}(n)}\rangle, \xi_1,\cdots,\xi_n \in E_{\mathbb{C}}.$$

For $\kappa \in E_{\mathbb{C}}^{*\otimes l+m}$, put

$$s_{l,m}(\kappa) = \frac{1}{l!m!}\sum_{\sigma\in\mathfrak{S}_l\times\mathfrak{S}_m}\kappa^\sigma,$$

where $\mathfrak{S}_l$ is the set of all permutations of $\{1,2,\cdots,l\}$. Then $s_{l,m}(\kappa)$ is the unique element of $E_{\mathbb{C}}^{*\widehat{\otimes} l}\otimes E_{\mathbb{C}}^{*\widehat{\otimes} m}$ such that

$$\langle s_{l,m}(\kappa), \xi^{\otimes l}\otimes\eta^{\otimes m}\rangle = \langle\kappa, \xi^{\otimes l}\otimes\eta^{\otimes m}\rangle, \xi,\eta\in E_{\mathbb{C}}.$$

$s_{l,m}(\kappa)$ is called the (l,m)-*symmetrization* of κ.

(2) For $\kappa \in E_{\mathbb{C}}^{*\otimes l}\otimes E_{\mathbb{C}}^{*\otimes m}$, there exists a unique $t_{m,l}(\kappa) \in E_{\mathbb{C}}^{*\otimes m}\otimes E_{\mathbb{C}}^{*\otimes l}$ such that

$$\langle t_{m,l}(\kappa), \eta\otimes\xi\rangle = \langle\kappa, \xi\otimes\eta\rangle, \ \eta\in E_{\mathbb{C}}^{\otimes m}, \xi\in E_{\mathbb{C}}^{\otimes l}. \tag{3.21}$$

$t_{m,l}(\kappa)$ is called the (m,l)-*interchange* of κ.

Theorem 3.7 *Let $\kappa \in E_{\mathbb{C}}^{*\otimes l+m}$. Then there exists a unique $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}}^*)$ such that*

$$\langle\langle \Xi_{l,m}(\kappa)\varphi,\psi\rangle\rangle = \langle\kappa,\eta_{\varphi,\psi}\rangle, \quad \varphi,\psi\in(E)_{\mathbb{C}}, \tag{3.22}$$

where $\eta_{\varphi,\psi}$ is given by (3.15). Moreover, $\Xi_{l,m}(\kappa) = \Xi_{l,m}(s_{l,m}(\kappa))$,

$$\|\Xi_{l,m}(\kappa)\varphi\|_{-p} \le C_{l,m,\rho,p}|\kappa|_{-p}\|\varphi\|_p. \tag{3.23}$$

If $\varphi\in(E)_{\mathbb{C}}, \varphi\sim\{f_n\}$, then $\Xi_{l,m}(\kappa)\varphi\sim\{h_n\}$, where

$$\begin{aligned}
h_n &= 0, \ n\le l-1,\\
h_n &= \frac{(m+n-l)!}{(n-l)!}s_{l,m}(\kappa)\widehat{\otimes}_m f_{n+m-l}, \ n\ge l.
\end{aligned} \tag{3.24}$$

If $\kappa\in E_{\mathbb{C}}^{*\widehat{\otimes} l}\otimes E_{\mathbb{C}}^{*\widehat{\otimes} m}$, then $\Xi_{l,m}(\kappa)^* = \Xi_{m,l}(t_{m,l}(\kappa))$.

Proof. By (3.16), $(\varphi,\psi)\longmapsto\langle\kappa,\eta_{\varphi,\psi}\rangle$ is a continuous bilinear form on $(E)_{\mathbb{C}}\times(E)_{\mathbb{C}}$. Thus by Theorem 3.17 in Chapter 1, there exist a unique $\Xi_{l,m}(\kappa)\in\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}}^*)$ such that (3.22) holds. (3.23) follows from (3.22) and (3.18). Since $\eta_{\varphi,\psi}\in E_{\mathbb{C}}^{\widehat{\otimes} l}\otimes E_{\mathbb{C}}^{\widehat{\otimes} m}$, we have $\langle\kappa,\eta_{\varphi,\psi}\rangle = \langle s_{l,m}(\kappa),\eta_{\varphi,\psi}\rangle$. Thus $\Xi_{l,m}(\kappa) = \Xi_{l,m}(s_{l,m}(\kappa))$. But $\langle t_{m,l}(\kappa),\eta_{\psi,\varphi}\rangle = \langle\kappa,\eta_{\varphi,\psi}\rangle$, hence $\Xi_{l,m}(\kappa)^* = \Xi_{m,l}(t_{m,l}(\kappa))$. Finally, we prove (3.24). Let $\varphi\sim\{f_n\}, \psi\sim\{g_n\}$. Then by (3.17) and (3.14),

$$\begin{aligned}
\langle\langle \Xi_{l,m}(\kappa)\varphi,\psi\rangle\rangle &= \sum_{n=0}^{\infty}\frac{(m+n)!(l+n)!}{n!}\langle\kappa, g_{l+n}\otimes_n f_{m+n}\rangle\\
&= \sum_{n=0}^{\infty}\frac{(m+n)!(l+n)!}{n!}\langle s_{l,m}(\kappa)\otimes_m f_{n+m}, g_{l+n}\rangle,
\end{aligned} \tag{3.25}$$

from which (3.24) follows. ∎

Remark 1. Let $\kappa \in E_{\mathbb{C}}^{*\widehat{\otimes} l} \otimes E_{\mathbb{C}}^{*\widehat{\otimes} m}$. Then

$$
\begin{aligned}
\langle\langle \Xi_{l,m}(\kappa)\mathcal{E}_f, \mathcal{E}_g\rangle\rangle &= \langle \kappa, \eta_{\mathcal{E}_f,\mathcal{E}_g}\rangle \\
&= \langle \kappa, g^{\otimes l} \otimes f^{\otimes m}\rangle e^{\langle f, g\rangle}.
\end{aligned} \tag{3.26}
$$

Consequently, $\Xi_{l,m}(\kappa)$ is the null operator if and only if $\kappa = 0$. Moreover, from (3.26), we have for $w \in E_{\mathbb{C}}^{*\widehat{\otimes} j} \otimes E_{\mathbb{C}}^{*\widehat{\otimes} k}$,

$$\Xi_{l,m}(\kappa) \circ \Xi_{k,j}(w) = \Xi_{l+k,m+j}(s_{l+k,m+j}(\kappa \otimes w)).$$

Remark 2. From (3.22) and (3.15), $\forall \varphi, \psi \in (E)_{\mathbb{C}}$,

$$\langle\langle \Xi_{l,m}(\kappa)\varphi, \psi\rangle\rangle = \langle \kappa, \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \varphi, \psi\rangle\rangle\rangle.$$

Thus $\Xi_{l,m}(\kappa)$ and $\Xi_{l,m}(\kappa)^*$ can be formally expressed as

$$
\begin{aligned}
&\Xi_{l,m}(\kappa) \\
&= \int_{T^{l+m}} \kappa(s_1, \cdots s_l, t_1, \cdots, t_m)\partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m, \\
&\Xi_{l,m}(\kappa)^* \\
&= \int_{T^{l+m}} \kappa(s_1, \cdots s_l, t_1, \cdots, t_m)\partial_{t_1}^* \cdots \partial_{t_m}^* \partial_{s_1} \cdots \partial_{s_l} ds_1 \cdots ds_l dt_1 \cdots dt_m.
\end{aligned}
$$

$\Xi_{l,m}(\kappa)$ is called the *integral kernel operator* with kernel κ.

We shall study the problem that under what conditions on κ one has $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$. The following lemma is the key to solve this problem.

Lemma 3.8 *Let $\kappa \in E_{\mathbb{C}}^{*\otimes l} \otimes E_{\mathbb{C}}^{*\otimes m}$ and $K \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{*\otimes l})$ be made in correspondence by the following relation:*

$$\langle Kf, g\rangle = \langle \kappa, g \otimes f\rangle = \langle \kappa \otimes_m f, g\rangle, g \in E_{\mathbb{C}}^{\otimes l}, f \in E_{\mathbb{C}}^{\otimes m}. \tag{3.27}$$

Then the following assertions are equivalent:

(i) $\kappa \in E_{\mathbb{C}}^{\otimes l} \otimes E_{\mathbb{C}}^{*\otimes m}$;

(ii) $K \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{\otimes l})$;

(iii) $\forall p \geq 0, \exists C \geq 0, q \geq 0$, such that

$$|\langle \kappa, g \otimes f\rangle| \leq C|g|_{-p}|f|_{p+q}, \quad \forall g \in E_{\mathbb{C}}^{\otimes l}, f \in E_{\mathbb{C}}^{\otimes m}; \tag{3.28}$$

(iv) $\forall p \geq 0, \exists q \geq 0$ *such that* $|\kappa|_{l,m;p,-(p+q)} < \infty$.

Proof. (i)⇔(ii). It follows from the fact that $\mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{\otimes l}) \cong E_{\mathbb{C}}^{\otimes l} \otimes E_{\mathbb{C}}^{*\otimes m}$ (see I.3.34).

(ii)⇒(iii). By the continuity of K, $\forall p \geq 0, \exists C \geq 0, q \geq 0$, such that

$$|Kf|_p \leq C|f|_{p+q}.$$

Thus (3.28) follows from (3.17).

(iii)⇒(iv). Let $p \geq 0$. By assumption, there exist $C \geq 0, q \geq 1$ such that

$$|\langle \kappa, g \otimes f\rangle| \leq C|g|_{-(p+1)}|f|_{p+q}, g \in E_{\mathbb{C}}^{\otimes l}, f \in E_{\mathbb{C}}^{\otimes m}.$$

Hence

$$
\begin{aligned}
|\kappa|_{l,m;p,-(p+q+1)}^2 &= \sum_{\alpha,\beta} |\langle \kappa, e_\alpha \otimes e_\beta\rangle|^2 |e_\alpha|_p^2 |e_\beta|_{-(p+q+1)}^2 \\
&\leq C^2 \sum_{\alpha,\beta} |e_\alpha|_{-(p+1)}^2 |e_\beta|_{p+q}^2 |e_\alpha|_p^2 |e_\beta|_{-(p+q+1)}^2 \\
&= C^2 \sum_{\alpha,\beta} |e_\alpha|_{-1}^2 |e_\beta|_{-1}^2 \\
&= C^2 \delta^{2(l+m)} < \infty.
\end{aligned}
$$

(iv)⇒(ii). Since $Kf = \kappa \otimes_m f$, from (3.10) we know that $\forall p \geq 0, \exists q \geq 0$ such that

$$|Kf|_p \leq \rho^{qm}|\kappa|_{l,m;p,-(p+q)}|f|_{p+q}, \quad f \in E_{\mathbb{C}}^{\otimes m}.$$

Thus $K \in \mathcal{L}(E_{\mathbb{C}}^{\otimes m}, E_{\mathbb{C}}^{\otimes l})$. ∎

Theorem 3.9 *Let $\kappa \in E_{\mathbb{C}}^{*\widehat{\otimes} l+m}$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$ if and only if $\kappa \in E_{\mathbb{C}}^{\otimes l} \otimes E_{\mathbb{C}}^{*\otimes m}$. Moreover, $\forall q > 0$, we have*

$$\|\Xi_{l,m}(\kappa)\varphi\|_p \leq C_{l,m,\sqrt{p},q}|s_{l,m}(\kappa)|_{l,m;p,-(p+q)}\|\varphi\|_{p+q}, \tag{3.29}$$

where $C_{l,m,\sqrt{p},q}$ is defined by (3.19).

Proof. Let $\varphi \in (E)_{\mathbb{C}}$, $\varphi \sim \{f_n\}$. Then $\Xi_{l,m}(\kappa)\varphi \sim \{h_n\}$, where h_n is given by (3.24). Thus for $p \geq 0$, by (3.10),

$$
\begin{aligned}
\|\Xi_{l,m}(\kappa)\varphi\|_p^2 &= \sum_{n=0}^{\infty}(l+n)!\left(\frac{(n+m)!}{n!}\right)^2 |s_{l,m}(\kappa)\widehat{\otimes}_m f_{n+m}|_p^2 \\
&\leq \sum_{n=0}^{\infty}(l+n)!\left(\frac{(n+m)!}{n!}\right)^2 \rho^{2qm}|s_{l,m}(\kappa)|_{l,m;p,-(p+q)}^2 |f_{n+m}|_{p+q}^2 \\
&= |s_{l,m}(\kappa)|_{l,m;p,-(p+q)}^2 \sum_{n=0}^{\infty}(n+m)!|f_{n+m}|_{p+q}^2 \frac{(l+n)!(n+m)!}{(n!)^2}\rho^{2qm} \\
&\leq C_{l,m,\sqrt{p},q}^2 |s_{l,m}(\kappa)|_{l,m;p,-(p+q)}^2 \|\varphi\|_{p+q}^2.
\end{aligned}
$$

Thus (3.29) follows and the sufficiency of the condition is proved.

Now we prove the necessity. Let $\Xi_{l,m}(\kappa) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$. Then $\forall p \geq 0$, there exist $C \geq 0, q \geq 0$ such that

$$\|\Xi_{l,m}(\kappa)\varphi\|_p \leq C\|\varphi\|_{p+q}, \quad \varphi \in (E)_{\mathbb{C}}.$$

In particular, let $\varphi = I_m(f), \psi = I_l(g)$ with $f \in E_{\mathbb{C}}^{\widehat{\otimes} m}$, and $g \in E_{\mathbb{C}}^{\widehat{\otimes} l}$. Then by (3.15), $\eta_{\varphi,\psi}^{(l,m)} = l!m!g \otimes f$. Thus

$$
\begin{aligned}
|\langle \kappa, g \otimes f \rangle| &= \frac{1}{l!m!} |\langle\langle \Xi_{l,m}(\kappa)\varphi, \psi \rangle\rangle| \leq \frac{C}{l!m!} \|\varphi\|_{p+q} \|\psi\|_{-p} \\
&= \frac{C}{\sqrt{l!m!}} |g|_{-p} |f|_{p+q}.
\end{aligned}
$$

Now by Lemma 3.8, $\kappa \in E_{\mathbb{C}}^{\otimes l} \otimes E_{\mathbb{C}}^{*\widehat{\otimes} m}$. ∎

Corollary 3.10 *If $\kappa \in E_{\mathbb{C}}^{*\widehat{\otimes} l} \otimes E_{\mathbb{C}}^{\widehat{\otimes} m}$, then $\Xi_{l,m}(\kappa)$ can be extended to a continuous linear operator on $(E)_{\mathbb{C}}^*$.*

Proof. We have $t_{m,l}(\kappa) \in E_{\mathbb{C}}^{\widehat{\otimes} m} \otimes E_{\mathbb{C}}^{*\widehat{\otimes} l}$. Thus by Theorem 3.9,

$$\Xi_{l,m}(\kappa)^* = \Xi_{m,l}(t_{m,l}(\kappa)) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}}),$$

from which the desired conclusion follows. ∎

We now give some examples of integral kernel operators.

Example 1. If $y \in E_{\mathbb{C}}^*$, then $D_y = \Xi_{0,1}(y)$, $D_y^* = \Xi_{1,0}(y)$. More generaly, if $y_1, \cdots, y_m \in E_{\mathbb{C}}^*$, then $D_{y_1} \cdots D_{y_m} = \Xi_{0,m}(\otimes_{j=1}^m y_j)$, $D_{y_1}^* \cdots D_{y_m}^* = \Xi_{m,0}(\otimes_{j=1}^m y_j)$.

Example 2. Let $\kappa = \tau_1 \in E \otimes E^*$ be defined as

$$\langle \tau_1, g \otimes f \rangle = \langle g, f \rangle, \quad g \in E, f \in E^*$$

(note the symmetrization of τ_1 in $E^* \otimes E^*$ is the τ defined by (IV.1.2)). Then the operator K corresponding to τ_1 through (3.27) is the identity, and by Theorem 3.9, $\Xi_{1,1}(\tau_1) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$.

If $f, g \in E_{\mathbb{C}}$, then

$$
\begin{aligned}
\langle\langle \Xi_{1,1}(\tau_1)\mathcal{E}_f, \mathcal{E}_g \rangle\rangle &= \langle \tau_1, \eta_{\mathcal{E}_f, \mathcal{E}_g}^{(1,1)} \rangle \\
&= \langle \tau_1, g \otimes f \rangle e^{\langle f, g \rangle} = \langle g, f \rangle e^{\langle f, g \rangle}.
\end{aligned}
$$

On the other hand, let N be the number operator. Then

$$
\begin{aligned}
\langle\langle N\mathcal{E}_f, \mathcal{E}_g \rangle\rangle &= \langle\langle \sum_{n=1}^{\infty} \frac{1}{(n-1)!} I_n(f^{\otimes n}), \sum_{n=1}^{\infty} \frac{1}{n!} I_n(g^{\otimes n}) \rangle\rangle \\
&= \sum_{n=1}^{\infty} \frac{\langle f, g \rangle^n}{(n-1)!} = \langle f, g \rangle e^{\langle f, g \rangle}.
\end{aligned}
$$

Consequently, we have

$$N = \Xi_{1,1}(\tau_1),$$

and N can be formally expressed as

$$N = \int_{T^2} \tau_1(s,t) \partial_s^* \partial_t \, ds dt = \int_T \partial_t^* \partial_t \, dt. \tag{3.30}$$

Example 3. Consider $\Xi_{0,2}(\tau)$. By Theorem 3.9 we have $\Xi_{0,2}(\tau) \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$. $\Xi_{0,2}(\tau)$ is called the *Gross Laplacian*, and denoted by $\triangle_G$. Its formal expression is

$$\triangle_G = \int_{T^2} \tau(s,t) \partial_s \partial_t \, ds dt = \int_T \partial_t^2 \, dt. \tag{3.31}$$

The dual of $\triangle_G$ is $\triangle_G^* = \Xi_{2,0}(\tau)$ with the formal expression

$$\triangle_G^* = \int_{T^2} \tau(s,t) \partial_s^* \partial_t^* \, ds dt = \int_T \partial_t^{*2} \, dt.$$

In fact, $\forall \varphi \in (E)_{\mathbb{C}}$, $\forall F \in (E)_{\mathbb{C}}^*$, we have (verified by S-transform)

$$\triangle_G \varphi = \int_T \partial_t^2 \varphi \, dt, \tag{3.32}$$

$$\triangle_G^* F = \int_T \partial_t^{*2} F \, dt = I_2(\tau) \diamond F. \tag{3.33}$$

The integral in (3.32) is in Bochner sense. Furthermore, if $\varphi \sim \{f_n\}$, then $\triangle_G \varphi \sim \{h_n\}$, where

$$h_n = (n+2)(n+1)\tau \otimes_2 f_{n+2}, \quad n \geq 0.$$

3.3 Integral kernel representation for generalized operators

From §2 we know that any $A \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}}^*)$ has the following chaos decomposition:

$$A\varphi = \sum_{l,m=0}^{\infty} I_{l,m}(a_{l,m})\varphi, \quad \varphi \in (E)_{\mathbb{C}}, \tag{3.34}$$

where $I_{l,m}(a_{l,m})$ is defined by (2.12), i.e.,

$$\widehat{I_{l,m}(a_{l,m})}(f, g) = \langle a_{l,m}, g^{\otimes l} \otimes f^{\otimes m} \rangle. \tag{3.35}$$

The sequence $\{a_{l,m}\}$ is determined by A as follows:

$$\langle\langle A I_m(f^{\otimes m}), I_l(g^{\otimes l}) \rangle\rangle = l!m! \langle a_{l,m}, g^{\otimes l} \otimes f^{\otimes m} \rangle, \quad f, g \in E. \tag{3.36}$$

Similar to the integral kernel operators, we give a formal expression for $I_{l,m}(a_{l,m})$ in terms of ∂_t and ∂_t^*. To this end, let $\varphi, \psi \in (E)_{\mathbb{C}}$, $\varphi \sim \{f_n\}$, $\psi \sim \{g_n\}$. Then

$$
\begin{aligned}
&\langle\langle I_{l,m}(a_{l,m})\varphi, \psi \rangle\rangle \\
&= \langle\langle I_{l,m}(a_{l,m}) I_m(f_m), I_l(g_l) \rangle\rangle \\
&= l!m! \langle a_{l,m}, g_l \otimes f_m \rangle \\
&= l!m! \int_{T^{l+m}} a_{l,m}(s,t) g_l(s) f_m(t) \, ds_1 \cdots ds_l dt_1 \cdots dt_m;
\end{aligned} \tag{3.37}
$$

where $s = (s_1, \cdots, s_l), t = (t_1, \cdots, t_m)$. On the other hand, by (1.18),

$$\langle\langle \partial_{t_1} \cdots \partial_{t_m} \varphi, 1 \rangle\rangle = m! f_m(t_1, \cdots, t_m),$$

thus

$$l!m! g_l(s) f_m(t) = \langle\langle \langle\langle \partial_{t_1} \cdots \partial_{t_m} \varphi, 1 \rangle\rangle, \partial_{s_1} \cdots \partial_{s_l} \psi \rangle\rangle.$$

If we denote by P_0 the projection onto the 0-th chaos (i.e., $\forall F \in (E)^*_{\mathbf{C}}, P_0 F = \langle\langle F, 1 \rangle\rangle$), then

$$g_l(s_1, \cdots, s_l) f_m(t_1, \cdots, t_m) = \langle\langle \partial^*_{s_1} \cdots \partial^*_{s_l} P_0 \partial_{t_1} \cdots \partial_{t_m} \varphi, \psi \rangle\rangle. \tag{3.38}$$

Thus from (3.37) and (3.38) we get

$$\begin{aligned} &I_{l,m}(a_{l,m}) \\ &= \int_{T^{l+m}} a_{l,m}(s,t) \partial^*_{s_1} \cdots \partial^*_{s_l} P_0 \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m. \end{aligned} \tag{3.39}$$

We now give another decomposition of a generalized operartor in terms of integral kernel operators (called the *integral kernel representation*).

Theorem 3.11 *For any $A \in \mathcal{L}((E)_{\mathbf{C}}, (E)^*_{\mathbf{C}})$, there exist $\kappa_{l,m} \in E^{*\hat{\otimes} l}_{\mathbf{C}} \otimes E^{*\hat{\otimes} m}_{\mathbf{C}}, l, m \geq 0$ such that*

$$A\varphi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\varphi, \ \varphi \in (E)_{\mathbf{C}}, \tag{3.40}$$

where the series converges strongly in $(E)^*_{\mathbf{C}}$. If $A \in \mathcal{L}((E)_{\mathbf{C}}, (E)_{\mathbf{C}})$, then $\kappa_{l,m} \in E^{\hat{\otimes} l}_{\mathbf{C}} \otimes E^{*\hat{\otimes} m}_{\mathbf{C}}$, and the series in (3.40) converges in $(E)_{\mathbf{C}}$. The S-transform of A is

$$\widetilde{A}(f, g) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, g^{\otimes l} \otimes f^{\otimes m} \rangle, \ f, g \in E_{\mathbf{C}}. \tag{3.41}$$

Moreover, we can take

$$\kappa_{l,m} = \sum_{n=0}^{l \wedge m} \frac{(-1)^n}{n!} a_{l-n, m-n} \otimes \tau_n, \tag{3.42}$$

where $\{a_{l,m}\}$ is given by (3.36), $\tau_n \in E^{\hat{\otimes} n} \otimes E^{*\hat{\otimes} n}$ is determined by

$$\langle \tau_n, g^{\otimes n} \otimes f^{\otimes n} \rangle = \langle f, g \rangle^n, \ g \in E, f \in E^*. \tag{3.43}$$

$\{\kappa_{l,m}\}$ is called the integral kernel sequence of A.

Proof. Clearly, $\widetilde{A}$ is a $U_{0,0}$-functional. Thus there exist $B \in \mathcal{L}((E)_{\mathbf{C}}, (E)^*_{\mathbf{C}})$ and $\{\kappa_{l,m} \in E^{*\hat{\otimes} l}_{\mathbf{C}} \otimes E^{*\hat{\otimes} m}_{\mathbf{C}}, l, m \in \mathbf{N}_0\}$ such that $\widetilde{B}(f, g) = \widetilde{A}(f, g), \forall f, g \in E_{\mathbf{C}}$, and (3.41) holds. By (2.9), there exists $p_1 \geq 0$ such that, when p is sufficiently large,

$$|\kappa_{l,m}|^2_{-p} \leq C^2 (l!m!)^{-1} (2e^2 K)^{l+m} \|I_{p_1,p}\|^{2(l+m)}_{\mathrm{HS}}. \tag{3.44}$$

Thus from (3.23) and (3.20) we get

$$\|\Xi_{l,m}(\kappa_{l,m})\varphi\|_{-p} \leq C(1-\rho^{2p})^{-\frac{l+m}{2}} (2e^2 K)^{\frac{l+m}{2}} \|I_{p_1,p}\|^{l+m}_{\mathrm{HS}} |\varphi|_p.$$

Since

$$\begin{aligned} \|I_{p_1,p_1+k+1}\|^2_{\mathrm{HS}} &= \|A^{-(k+1)}\|^2_{\mathrm{HS}} \\ &= \sum_{j=0}^{\infty} \lambda_j^{-2(k+1)} \leq \rho^{2k} \|A^{-1}\|^2_{\mathrm{HS}}, \end{aligned}$$

if we take $p > p_1$ sufficiently large such that $\|I_{p_1,p}\|^2_{\mathrm{HS}} < (2e^2 K)^{-1}(1-\rho^{2p})$, then

$$\sum_{l,m=0}^{\infty} \|\Xi_{l,m}(\kappa_{l,m})\varphi\|_{-p} < \infty.$$

This means that the series in (3.40) converges strongly in $(E)^*_{\mathbf{C}}$, and by (3.26) and (3.41), we conclude that (3.40) holds. If $A \in \mathcal{L}((E)_{\mathbf{C}}, (E)_{\mathbf{C}})$, then by (3.41), $\kappa_{l,m} \in E^{\hat{\otimes} l}_{\mathbf{C}} \otimes E^{*\hat{\otimes} m}_{\mathbf{C}}$, and it can be similarly proved that the series in (3.40) converges in $(E)_{\mathbf{C}}$.

Finally, (3.42) follows readily from the following equality:

$$\begin{aligned} &\sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, g^{\otimes l} \otimes f^{\otimes m} \rangle \\ &= \sum_{k,j=0}^{\infty} \langle a_{k,j}, g^{\otimes k} \otimes f^{\otimes j} \rangle \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle f, g \rangle^n. \end{aligned} \qquad \blacksquare$$

Remark 1. Since $\kappa_{l,m}$ is not required to be (l, m)-symmetric, the sequence $\{\kappa_{l,m}\}$ is not unique. However, the (l, m)-symmetrization of $\kappa_{l,m}$ is unique.

Remark 2. Let A' and A'' be two generalized operators with integral kernel sequences $\{\kappa'_{l,m}\}$ and $\{\kappa''_{l,m}\}$, respectively. Then the integral kernel sequence of $A' \diamond A''$ is $\{\kappa_{l,m}\}$:

$$\kappa_{l,m} = \sum_{i+k=l, j+n=m} \kappa'_{i,j} \otimes \kappa''_{k,n}. \tag{3.45}$$

We should mention that all results in this section can be adapted for the case $\beta = 1$ provided we replace S-transforms by local S-transforms.

We now give some examples of integral kernel representations for generalized operators. The results can be proved by means of Taylor's expansions for symbols of operators.

Example 1. Let P_0 be the projection to the 0-th chaos. Then

$$P_0 = \sum_n (-1)^n (n!)^{-1} \Xi_{n,n}(\tau_n).$$

Example 2. For $y \in E^*$, the shift operator τ_y can be expressed as

$$\tau_y = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi_{0,n}(y^{\otimes n}).$$

As a consequence, we obtain the Taylor expansion for $\tau_y \varphi$ (see (1.23)):

$$\tau_y \varphi = \sum_{n=0}^{\infty} \frac{1}{n!} D_y^n \varphi, \quad \varphi \in (E)_{\mathbb{C}},$$

where the series converges in $(E)_{\mathbb{C}}$.

Example 3. Let $\varphi \in (E)^*_{\mathbb{C}}$, $\varphi \sim \{f_n\}$. As a multiplication operator, its integral kernel representation is given by

$$\varphi = \sum_{l,m=0}^{\infty} \binom{l+m}{m} \Xi_{l,m}(f_{l+m}).$$

Moreover, $\varphi \in \mathcal{L}((E)_{\mathbb{C}}, (E)_{\mathbb{C}})$ if and only if $\varphi \in (E)_{\mathbb{C}}$.

Example 4. For $\lambda \in \mathbb{C}$, the scaling operator σ_λ can be expressed as

$$\sigma_\lambda = \sum_{l,m=0}^{\infty} \frac{(\lambda-1)^l(\lambda^2-1)^m}{l!m!2^m} \Xi_{l,l+2m}(\eta \otimes \tau^{\otimes m}),$$

where τ is defined by (3.43).

Example 5. For $\lambda \in \mathbb{C}$,

$$\Gamma(\lambda) = \sum_{m=0}^{\infty} \frac{(\lambda-1)^m}{m!} \Xi_{m,m}(\tau_m).$$

§4. Applications to quantum physics

In this section we present some applications of the theory of generalized operators to quantum physics. We shall define quantum stochastic integrals by means of generalized operators, which generalizes the one in Hudson-Parthasarathy's sense. We shall give an interpretation of the Klein-Gordon field in terms of Wick product of generalized operators. Moreover, we shall study infinite dimensional classical Dirichlet forms in the framework of white noise analysis. As for the applications of the theory of generalized operators to infinite dimensional harmonic analysis and quantum probability, we refer the reader to Obata[4,5,6,8].

4.1 Quantum stochastic integration

We adopt the notations of Section 3.2. Let $(T, \mathcal{B}(T), \nu)$ be a measure space as specified there. Let $\{K_t, t \in T\}$ be an $\mathcal{L}((E)_{\mathbb{C}}, (E)^*_{\mathbb{C}})$-valued measurable process,

M a Borel subset of T. We may define integrals of the following form:

$$\int_M K_t W_{l,m}(dt) \equiv \int_M K_t \diamond \partial_t^{*l} \partial_t^m dt, \tag{4.1}$$

provided the r.h.s. exists as a Bochner integral in $\mathcal{L}((E)_{\mathbb{C}}, (E)^*_{\mathbb{C}})$. Here the l.h.s. is a formal notation. $W_{l,m}(dt) = \partial_t^{*l} \partial_t^m dt$ is called the *quantum white noise measure*. The S-transform of the integral defined by (4.1) is

$$G(\xi, \eta) = \int_M \widehat{K}_t(\xi, \eta) \eta(t)^l \xi(t)^m dt, \quad \xi, \eta \in E. \tag{4.2}$$

In particular, if $T = \mathbb{R}_+$, ν is the Lebesgue measure and $\{K_t\}$ is an adapted operator valued process on $L^2(E^*, \mu)$, the above integral is essentially the quantum stochastic integral in the sense of Hudson and Parthasarathy. Put

$$A_t^* = \int_0^t \partial_s^* ds, \quad A_t = \int_0^t \partial_s ds, \quad N_t = \int_0^t \partial_s^* \partial_s ds. \tag{4.3}$$

A_t^*, A_t and N_t are called the *quantum creation, annihilation* and *number process*, respectively. The quantum Brownian motion and quantum Poisson processes corresponding to the classical Brownian motion and Poisson processes are respectively

$$Q_t = A_t^* + A_t, \quad P_t^\lambda = N_t + \sqrt{\lambda} Q_t + \lambda t, \tag{4.4}$$

where $\lambda > 0$. In fact, both Q_t and P_t^λ are self-adjoint operators on $L^2(E^*, \mu)$, $\{Q_t 1\}$ is the standard Brownian motion and $\{P_t^\lambda 1\}$ is the Poisson process with intensity λ (see Meyer[3]).

The integral defined by (4.1) generalizes the one in Hudson-Parthasarathy's sense in two aspects: the first is that the integrand process $\{K_t\}$ takes values not exclusively in operators on $L^2(E^*, \mu)$ but in generalized operators; the second is that the integrand process is not required to be adapted one. Moreover, the integration domain is extended from $\mathbb{R}_+$ to a general measure space. These generalizations are useful for the study of random fields and quantum fields.

In what follows we shall use the Wick ordering of operator's product to derive the Itô formula for quantum stochastic integrals. As mentioned above, $Q_t = A_t^* + A_t$ is a quantum Brownian motion. By using the commutation relation $[A_t, A_t^*] = t$, we can easily obtain that $\forall n \in \mathbb{N}_0$,

$$Q_t^n = \sum_{2j+k+l=n} \frac{n!}{2^j j! k! l!} t^j A_t^{*k} A_t^l. \tag{4.5}$$

Let $f(x) = x^n$. Regarding $f(Q_t) = Q_t^n$ as a function of the variables t, A_t^* and

A_t, and according to the chain rule for differentiation, we obtain

$$\begin{aligned} df(Q_t) = &\sum_{2j+k+l=n} \frac{n!}{2^j j!\,k!\,l!}\big[t^j k A_t^{*k-1} A_t^l dA_t^* \\ &+ t^j l A_t^{*k} A_t^{l-1} dA_t + j t^{j-1} A_t^{*k} A_t^l dt\big] \\ = n &\sum_{2j+k+l=n-1} \frac{(n-1)!}{2^j j!\,k!\,l!} t^j A_t^{*k} A_t^l d(A_t^* + A_t) \\ &+ \frac{n(n-1)}{2} \sum_{2j+k+l=n-2} \frac{(n-2)!}{2^j j!\,k!\,l!} t^j A_t^{*k} A_t^l dt. \end{aligned}$$

Hence

$$df(Q_t) = f'(Q_t)dQ_t + \frac{1}{2} f''(Q_t)dt. \tag{4.6}$$

Thus we have obtained the *quantum Itô formula* for the case of polynomial functions.

In the above derivation, we have first expressed Q_t^n as a function of products of A_t^* and A_t in Wick ordering, then viewed A_t^* and A_t as ordinary variables, and differentiated Q_t^n according to ordinary chain rule. Since A_t and A_t^* are not commutative, in the derivation of (4.6), we have used the commutation relation $[A_t, A_t^*] = t$. This leads to the second order term in the Itô formula.

Once we have quantum stochastic integrals, we can solve quantum stochastic differential equations. By means of S-transforms, the latter can be converted to functional integral equations. The following are two examples.

Example 1. Consider the quantum stochastic differential equation

$$dX_t = X_t dA_t,$$

this is equivalent to the integral equation

$$X_t = X_0 + \int_0^t X_s \diamond \partial_s ds.$$

Taking S-transform on both sides gives

$$\widetilde{X}_t(\xi,\eta) = \widetilde{X}_0(\xi,\eta) + \int_0^t \widetilde{X}_s(\xi,\eta)\xi(s)ds.$$

Thus

$$\widetilde{X}_t(\xi,\eta) = \widetilde{X}_0(\xi,\eta)\exp\Big\{\int_0^t \xi(s)ds\Big\},$$

and consequently,

$$X_t = X_0 \diamond e^{A_t},$$

where X_0 may be a generalized operator.

Example 2. Consider the quantum stochastic integral equation

$$X_t = X_0 + \int_0^t X_s \diamond \partial_s^{*l}\partial_s^m ds.$$

Taking S-transform gives

$$\widetilde{X}_t(\xi,\eta) = \widetilde{X}_0(\xi,\eta) + \int_0^t \widetilde{X}_s(\xi,\eta)\eta(s)^l\xi(s)^m ds.$$

Hence

$$\widetilde{X}_t(\xi,\eta) = \widetilde{X}_0(\xi,\eta)\exp\Big\{\int_0^t \eta(s)^l\xi(s)^m ds\Big\}.$$

Let $\exp^\diamond K$ denote the "Wick exponential" of K: $\sum_{n=0}^\infty (n!)^{-1}K^{\diamond n}$. Then

$$X_t = X_0 \diamond \exp^\diamond(A_t^{*l}A_t^m).$$

It can be easily proved that $\exp^\diamond(A_t^{*l}A_t^m)$ is still a generalized operator. Thus the above equation has a unique solution in $\mathcal{L}((E),(E)^*)$.

4.2 Klein-Gordon field

Let $\square = \nabla_t^2 - \nabla_x^2 = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2}$ be the wave operator on Minkowski space $\boldsymbol{R}\times\boldsymbol{R}^3$. The physical free field $\{\phi(t,x) : (t,x)\in\boldsymbol{R}\times\boldsymbol{R}^3\}$, as operator valued distribution in certain sense, satisfies the Klein-Gordon equation ($m>0$ is a constant)

$$(\square + m^2)\phi = 0 \tag{4.7}$$

and the equal-time commutation relations

$$\begin{aligned} &[\phi(t,x),\phi(t,y)] = 0, \quad [\dot\phi(t,x),\dot\phi(t,y)] = 0, \\ &[\phi(t,x),\dot\phi(t,y)] = i\delta(x-y), \end{aligned}$$

where $\dot\phi(t,x) = \frac{\partial}{\partial t}\phi(t,x)$. According to the the formal derivation in physics,

$$\phi(t,x) = \int_{\boldsymbol{R}^3}\big[f_k^*(t,x)\partial_k^* + f_k(t,x)\partial_k\big]dk,$$

where ∂_k^* and ∂_k are the pointwise creation and annihilation operators on the Fock space $\Gamma(L^2(\boldsymbol{R}^3))$, and

$$\begin{aligned} f_k^*(t,x) &= \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{i\omega_k t - ikx}, \\ f_k(t,x) &= \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-i\omega_k t + ikx} \end{aligned}$$

are a pair of conjugate solutions of the classical field equation

$$(\square + m^2)f = 0, \tag{4.8}$$

where $\omega_k = \sqrt{k^2+m^2}$. The energy and momentum operators of the Klein-Gordon fields $\varphi(t,x)$ are

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3}[(\nabla_t\phi)^2 + (\nabla_x\phi)^2 + m^2\phi^2]dx, \tag{4.9}$$

$$P(t) = -\int_{\mathbb{R}^3}\nabla_t\phi\cdot\nabla_x\phi dx, \tag{4.10}$$

respectively.

From the mathematical point of view, the above derivation needs clarifying. In particular:

(i) ∂_k^* is not a well-defined operator on Fock space (its domain is merely $\{0\}$), and can only be interpreted as a generalized operator or an operator-valued distribution.

(ii) the products of operator-valued distributions such as $(\nabla_t\phi)^2$, $(\nabla_x\phi)^2$ and ϕ^2 are not ordinary products. They should be interpreted as Wick products.

In the sequel we shall legalize the above operations in the framework of generalized operators. Let

$$H = L^2(\mathbb{R}^3),\ E = S(\mathbb{R}^3),\ E^* = S^*(\mathbb{R}^3).$$

Consider the classical framework of white noise analysis. Denote $\mathcal{L}((E)_{\mathbb{C}},(E)^*_{\mathbb{C}})$ simply by $\mathcal{L}$. Regard the Klein-Gordon equation (4.7) as an abstract wave equation with respect to the function $\phi:\mathbb{R}\times\mathbb{R}^3\to\mathcal{L}$. Then it is easily verified that the $\mathcal{L}$-valued function

$$\phi(t,x) = \int_{\mathbb{R}^3}[f_k^*(t,x)\partial_k^* + f_k(t,x)\partial_k]dk$$

satisfies equation (4.7). The S-transform of $\phi(t,x)$ is

$$\widetilde{\phi(t,x)}(\xi,\eta) = \int_{\mathbb{R}^3}[f_k^*(t,x)\eta(k) + f_k(t,x)\xi(k)]dk,\ \xi,\eta\in S(\mathbb{R}^3),$$

which satisfies the classical wave equation (4.8).

By using Wick product, the renormalized energy and momentum operators are

$$H(t) = \frac{1}{2}\int_{\mathbb{R}^3}[(\nabla_t\phi)^{\diamond 2} + (\nabla_x\phi)^{\diamond 2} + m^2\phi^{\diamond 2}]dx,$$

$$P(t) = -\int_{\mathbb{R}^3}\nabla_t\phi\diamond\nabla_x\phi dx,$$

respectively. Their S-transforms are

$$\widetilde{H(t)}(\xi,\eta) = \frac{1}{2}\int_{\mathbb{R}^3}[(\nabla_t\tilde{\phi})^2 + (\nabla_x\tilde{\phi})^2 + m\tilde{\phi}^2]dx,$$

$$\widetilde{P(t)}(\xi,\eta) = -\int_{\mathbb{R}^3}\nabla_t\tilde{\phi}\cdot\nabla_x\tilde{\phi}dx,$$

respectively. By the above formulas, we can directly verify that $\forall\xi,\eta\in S(\mathbb{R}^3)$, $\frac{d}{dt}\widetilde{H(t)} = 0$, $\frac{d}{dt}\widetilde{P(t)} = 0$, hence $H(t) = H(0)$, $P(t) = P(0)$. These are just the laws of energy conservation and momentum conservation. By straightforward computations, we have (see Huang-Luo[2] and Luo[1]):

$$H(0) = \int_{\mathbb{R}^3}\omega_k\partial_k^*\partial_k dk,$$

$$P(0) = \int_{\mathbb{R}^3}k\partial_k^*\partial_k dk.$$

Thus starting from generalized operators and Wick calculus, we conclude that $H(t)$ and each component of $P(t)$ belong to $\mathcal{L}((E)_{\mathbb{C}},(E)_{\mathbb{C}})$ and $\mathcal{L}((E)^*_{\mathbb{C}},(E)^*_{\mathbb{C}})$. They are essentially self-adjoint operators on Fock space $\Gamma(H)$. Moreover, we can prove that the following Heisenberg equation holds:

$$\nabla_t\phi(t,x) = i[H(t),\phi(t,x)], \tag{4.11}$$

where the commutator in the r.h.s. is a generalized operator.

4.3 Infinite dimensional classical Dirichlet forms

Dirichlet form is a generalization of the classical Dirichlet integral. In 1959, Beurling and Deny introduced, for the first time, the notion of Dirichlet space and initiated the L^2-framework for potential theory. In the 70's, Fukushima, Silverstein and others developed systematically the theory of Dirichlet forms. This theory is a bridge connecting potential theory and Markov processes. Since the theory of Dirichlet forms has important applications in non-relativistic quantum mechanics and Euclidean quantum fields, it has been greatly advanced in recent years. The most important breakthrough is the framework of quasi-regular Dirichlet forms established by Ma and Albeverio. This framework is particularly suited to infinite dimensional analysis. The interested reader may consult Ma and Röckner[1].

In the following we give an application of white noise analysis in the theory of Dirichlet forms. The material is taken from Hida-Kuo-Potthoff-Streit[1].

Let X be a separable metric space, m a σ-finite measure on $\mathcal{B}(X)$. Then $H = L^2(X,m)$ is a separable Hilbert space. Let $(\mathcal{E},\mathcal{D})$ be a symmetric positive densely defined bilinear form on $L^2(X,m)$. Put $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v), u,v\in\mathcal{D}$. If $\mathcal{D}$ is complete under the $\mathcal{E}_1$-norm, then $(\mathcal{E},\mathcal{D})$ is called closable. If the following are satisfied:

$$u\in\mathcal{D}\Rightarrow u^+\wedge 1\in\mathcal{D},\ \mathcal{E}(u^+\wedge 1, u^+\wedge 1)\le\mathcal{E}(u,u),$$

then $(\mathcal{E},\mathcal{D})$ is said to have the contractivity property (or Markov property). A symmetric closed bilinear form $(\mathcal{E},\mathcal{D})$ on H with the Markov property is called a Dirichlet form.

In non-relativistic quantum mechanics, the dynamics of a system with d degrees of freedom is determined by a positive self-adjoint operator H on $L^2(\mathbb{R}^d, dx)$ (called the **Hamiltonian** or **energy operator**): $H = -\frac{1}{2}\Delta + V$, where Δ is the Laplace operator, V is the potential. Assume that H is essentially self-adjoint and bounded below on $C_0^\infty(\mathbb{R}^d)$. Then the smallest eigenvalue of H is simple, the corresponding eigenvector φ can be taken to be strictly positive and satisfying $\int_{\mathbb{R}^d} \varphi^2(x)dx = 1$. φ is called the ground state (or vacuum). For simplicity, we may assume that the smallest eigenvalue of H is 0. Put $\nu(dx) = \varphi^2(x)dx$, $Wf(x) = f(x)/\varphi(x)$. Then W is a unitary map from $L^2(\mathbb{R}^d, dx)$ to $L^2(\mathbb{R}^d, d\nu)$. Let $H_\nu = WHW^{-1}$. Then H_ν is a positive self-adjoint operator on $L^2(\mathbb{R}^d, \nu)$. H_ν is called the ground state representation of H. If φ is sufficiently smooth, then by integration by parts formula, $H_\nu = -\frac{1}{2}\Delta + b\nabla$, where $b = -\nabla \log \varphi$. However, when V is a singular potential, H_ν does not possesses the above expression. In this case, put

$$\mathcal{E}_\nu(u, v) = \frac{1}{2}\int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, d\nu, \quad u, v \in C_0^\infty(\mathbb{R}^d). \tag{4.12}$$

Then the smallest closed extension of $(\mathcal{E}_\nu, C_0^\infty(\mathbb{R}^d))$ is a Dirichlet form. It is called a *classical Dirichlet form*, which associates with a diffusion process. Based on this ground state representation, we may study Hamiltonians with singular potentials. In quantum field theory, the counterpart is an infinite dimensional Dirichlet form:

$$\mathcal{E}(\varphi, \psi) = \int_{E^*} D\varphi \cdot D\psi \, d\nu, \quad \varphi, \psi \in (E), \tag{4.13}$$

where $E = \mathcal{S}(\mathbb{R}^d)$, $E^* = \mathcal{S}^*(\mathbb{R}^d)$, ν is a finite Borel measure on $\mathcal{S}^*(\mathbb{R}^d)$,

$$D\varphi \cdot D\psi = \sum_{j=1}^\infty (D_{e_j}\varphi(D_{e_j}\psi). \tag{4.14}$$

Here and in what follows, we adopt the notations in Section 1.3 of this chapter and Section 1.3 of Chapter 4. In particular, A is the harmonic oscillator on $L^2(\mathbb{R})$, $\{e_j, j \geq 1\}$ is an orthonormal base of $L^2(\mathbb{R}^d)$ consisting of the eigenvectors of $A^{\otimes d}$. An important question is: under what conditions on the measure ν can the bilinear form defined by (4.13) be extended to a Dirichlet form? We shall discuss this question.

First, assume that ν is the measure associated with a Hida distribution Φ. We shall prove $D\varphi \cdot D\psi \in (E)$, thus (4.13) is well-defined and

$$\mathcal{E}(\varphi, \psi) = \langle\langle D\varphi \cdot D\psi, \Phi\rangle\rangle, \quad \varphi, \psi \in (E). \tag{4.15}$$

Henceforth denote by $\mathcal{E}_\Phi$ the bilinear form on (E) defined by (4.15).

In the following lemma, $\|\cdot\|_p$ does not denote the L^p-norm but denotes the Hilbert norm on $(E)_p$ (see Section 1.3 of Chapter 4).

Lemma 4.1 *If $\varphi, \psi \in (E)$, then $D\varphi \cdot D\psi \in (E)$. Moreover, there exists a constant $C > 0$ such that $\forall p \in \mathbb{N}_0$,*

$$\| |D\varphi|^2 - |D\psi|^2 \|_p \leq C\|\varphi - \psi\|_{p+2}(\|\varphi\|_{p+2} + \|\psi\|_{p+2}), \tag{4.16}$$

where $|D\varphi|^2 = D\varphi \cdot D\varphi$. In particular, the map $\varphi \mapsto |D\varphi|^2$ is continuous from (E) to (E).

Proof. Let $F \in (E)$ with $F \sim \{f_n\}$. By (1.25) in Chapter 4, we have

$$\sum_{n=0}^\infty n!\|A^{-1}\|^{-2n}|f_n|_p^2 \leq \sum_{n=0}^\infty n!|f_n|_{p+1}^2.$$

Noting $\|A^{-1}\| < 1$, by (3.11) in Chapter 4, there exists $K > 0$ such that

$$\|(D_{e_j}\varphi)^2\|_p \leq K\|D_{e_j}\varphi\|_{p+1}^2.$$

Now by (1.19), there exists a constant $K' > 0$ such that

$$\begin{aligned}\|(D_{e_j}\varphi)^2\|_p &\leq K'|e_j|_{-(p+1)}^2\|\varphi\|_{p+2}^2 \\ &\leq K'|e_j|_{-1}^2\|\varphi\|_{p+2}^2.\end{aligned}$$

Hence we have

$$\| |D\varphi|^2 \|_p \leq K'\|A^{-1}\|_{\mathrm{HS}}^2\|\varphi\|_{p+2}^2.$$

This means $|D\varphi|^2 \in (E)$.

In order to prove inequality (4.16), we rewrite the l.h.s. of (4.16) as

$$\| |D\varphi|^2 - |D\psi|^2 \|_p = \|\sum_{k=1}^\infty D_{e_k}(\varphi - \psi)D_{e_k}(\varphi + \psi)\|_p.$$

Similar, we get (4.16). ∎

Since ν is a Radon measure on $(\mathcal{S}^*(\mathbb{R}^d), \mathcal{B})$, the set of polynomial functionals $\mathcal{P}$ is dense in $L^2(\nu)$. But $\mathcal{P} \subset (E)$, the bilinear form on $L^2(\nu)$ defined by (5.16) is thus densely defined. Moreover, it is positive and symmetric. In the following we shall prove that $\mathcal{E}_\Phi$ is closable under some conditions on Φ.

Let $l^2(\nu) = L^2(\mathbb{R}^d) \otimes L^2(\nu)$. Then the gradient operator D is a densely defined linear operator from $L^2(\nu)$ to $l^2(\nu)$. Let D' denote the adjoint of D. If we can show that D is closable, then $D'D$ will be essentially self-adjoint (see Theorem 1.7 in Chapter 1) and $\mathcal{E}_\Phi$ is thus closable. Consequently, by Theorem 1.5 in Chapter 1, in order to prove that $\mathcal{E}_\Phi$ is closable, it suffices to prove that the adjoint D' of D from $L^2(\nu)$ to $l^2(\nu)$ is densely defined.

Let $\mathcal{C}$ be the algebra generated by $\{\sin W_\xi, \cos W_\xi, \xi \in E\}$, where $W_\xi(x) = \langle x, \xi\rangle, x \in \mathcal{S}^*(\mathbb{R}^d)$. It can be easily proved that the algebra on complex field generated by $\{e^{iW_\xi}, \xi \in E\}$ is dense in $(E)_{\mathbb{C}}$. Thus $\mathcal{C}$ is dense in (E). From this we conclude that the closability of $(\mathcal{E}, \mathcal{C})$ is equivalent to that of $(\mathcal{E}, (E))$, and

they possess the same closure. We now study the closability of $(\mathcal{E}, \mathcal{C})$. For this purpose, it suffices to study the closability of the operator D from $L^2(\nu)$ to $l^2(\nu)$ defined on $\mathcal{C}$.

In the statement of the following lemma, we adopt the Sobolev norm $\|\cdot\|^{\sim}_{k,p}$ introduced in Section 3.4 of Chapter 2, i.e., $\|F\|^{\sim}_{k,p} = \|(I+N)^{k/2}\|_{L^p(\mu)}$, where $k \in \boldsymbol{N}, p > 1$.

Lemma 4.2 *Let Φ be a measurable function on $\mathcal{S}^*(\boldsymbol{R}^d)$, with $\Phi > 0$, μ-a.e., such that for some $p > 1$, $\Phi^{1/2} \in \boldsymbol{D}_1^{2p}$. Put $d\nu = \Phi d\mu$. Then the operator D from $L^2(\nu)$ to $l^2(\nu)$ with domain $\mathcal{C}$ is densely defined and closable.*

Proof. First, by (3.41) in Chapter 2, there exists a constant $C > 0$ such that

$$\|\Phi\|^{\sim}_{1,p} = \|\Phi^{1/2}\Phi^{1/2}\|^{\sim}_{1,p} \leq C\|\Phi^{1/2}\|^{\sim 2}_{1,2p}.$$

Thus by assumption, $\Phi \in \boldsymbol{D}_1^p$. Hence, Φ belongs to the (L^p)- domain of D and $D\Phi = 2\Phi^{1/2}D\Phi^{1/2}$. Denote by $l_0^2(\nu)$ the subspace of $l^2(\nu)$ consisting of F of the following form:

$$F = \sum_k e_k \otimes F_k,$$

where $F_k \in \mathcal{C}$, and there are only finitely many non-zero F_k. Clearly, $l_0^2(\nu)$ is dense in $l^2(\nu)$. Since $\forall \varphi \in \mathcal{C}$,

$$\begin{aligned} \int (\Phi^{-1}(D_{e_k}\Phi)\varphi)^2 d\nu &= \int \Phi^{-1}(D_{e_k}\Phi)^2\varphi^2 d\mu \\ &\leq \|\varphi\|_\infty^2 \int (\Phi^{-1/2}D_{e_k}\Phi)^2 d\mu \\ &= 4\|\varphi\|_\infty^2 \int (D_{e_k}\Phi^{1/2})^2 d\mu \\ &\leq 4\|\varphi\|_\infty^2 \|\Phi^{1/2}\|^{\sim 2}_{1,2}, \end{aligned}$$

Therefore the linear operator W_Φ from $l^2(\nu)$ to $L^2(\nu)$ defined by

$$W_\Phi F = \sum_k \Phi^{-1}(D_{e_k}\Phi)F_k \tag{4.17}$$

is densely defined.

Let $\varphi \in \mathcal{C}, F \in l_0^2(\nu)$. Then

$$\begin{aligned} &(\varphi, (D^* - W_\Phi)F)_{L^2(\nu)} \\ &= \int (D(\varphi\Phi), F)_{L^2(\boldsymbol{R}^d)} d\mu - \int \varphi(D\Phi, F)_{L^2(\boldsymbol{R}^d)} d\mu \\ &= \int (D\varphi, F)_{L^2(\boldsymbol{R}^d)} \Phi d\mu = (D\varphi, F)_{L^2(\nu)}. \end{aligned}$$

This implies

$$D'F = (D^* - W_\Phi)F, \quad F \in l_0^2(\nu).$$

In particular, D' is densely defined. ∎

The following theorem is one of the main results of this section.

Theorem 4.3 *Under the conditions of Lemma 4.2, $(\mathcal{E}, \mathcal{C})$ is closable, and its closure $(\mathcal{E}, \overline{\mathcal{C}})$ is a Dirichlet form on $L^2(\nu)$. Here $\overline{\mathcal{C}}$ is the completion of $\mathcal{C}$ with respect to the $\mathcal{E}_1$-norm.*

Proof. By Lemma 4.2, $(\mathcal{E}, \overline{\mathcal{C}})$ is a positive, symmetric, closed bilinear form on $L^2(\nu)$. In order to prove that $(\mathcal{E}, \overline{\mathcal{C}})$ is a Dirichlet form, it suffices to verify that it possesses the Markov property (i.e., contractivity). To this end, let $\varphi \in \mathcal{C}, g \in C_b^1(\boldsymbol{R})$. Assume that $\sup_x |\varphi(x)| < R$. By Weierstrass' theorem from mathematical analysis, there exists a sequence of polynomials $\{g_n, n \in \boldsymbol{N}\}$ such that g_n' converges uniformly on $[-R, R]$ to some g'. Put

$$g_n(t) = \int_{-R}^t g_n'(s)ds - g(-R).$$

Then $\{g_n, n \in \boldsymbol{N}\}$ is a sequence of polynomials which converges uniformly on $[-R, R]$ to g. Clearly, $g_n \circ \varphi$ converges to $g \circ \varphi$ in $L^2(\nu)$. The following estimate shows that for $m, n \to \infty$, $a_{n,m} \equiv \mathcal{E}(g_n \circ \varphi - g_m \circ \varphi, g_n \circ \varphi - g_m \circ \varphi) \to 0$:

$$\begin{aligned} a_{n,m} &= \int |D(g_n \circ \varphi - g_m \circ \varphi)|^2 d\nu \\ &= \int |g_n' \circ \varphi - g_m' \circ \varphi|\,|D\varphi|^2 d\nu \\ &\leq \sup_{|t| \leq R} |g_n'(t) - g_m'(t)| \mathcal{E}(\varphi, \varphi). \end{aligned}$$

Thus, the closability of $(\mathcal{E}, \mathcal{C})$ implies $g \circ \varphi \in \mathcal{C}$.

We now show that $\varphi \in \overline{\mathcal{C}} \Rightarrow g \circ \varphi \in \overline{\mathcal{C}}$. Take any sequence $\{\varphi_n, n \in \boldsymbol{N}\}$ from $\mathcal{C}$ such that $\mathcal{E}_1(\varphi - \varphi_n, \varphi - \varphi_n) \to 0$ as $n \to \infty$. In particular, $\varphi_n \to \varphi$ in $L^2(\nu)$ and consequently, $g \circ \varphi_n \to g \circ \varphi$. Moreover, we have

$$\begin{aligned} &\mathcal{E}(g \circ \varphi_n - g \circ \varphi_m, g \circ \varphi_n - g \circ \varphi_m) \\ &= \int |g' \circ \varphi_n D\varphi_n - g' \circ \varphi_m D\varphi_m|^2 d\nu \\ &\leq 2\int |g' \circ \varphi_n - g' \circ \varphi_m|^2 |D\varphi_n|^2 d\nu \\ &\quad + 2\int |g' \circ \varphi_m|^2 |D(\varphi_n - \varphi_m)|^2 d\nu \\ &\leq 2\int |g' \circ \varphi_n - g' \circ \varphi_m|^2 |D\varphi_n|^2 d\nu \\ &\quad + 2\sup_{t \in \boldsymbol{R}} |g'(t)|^2 \int |D(\varphi_n - \varphi_m)|^2 d\nu. \end{aligned}$$

Clearly, when $n, m \to \infty$, the two terms in the r.h.s. of the last inequality tend to zero, thus $g \circ \varphi_n$ tends to $g \circ \varphi$ in the $\mathcal{E}_1$-norm. Hence $g \circ \varphi \in \overline{\mathcal{C}}$.

Finally, we prove the contractivity of $(\mathcal{E}, \bar{\mathcal{C}})$. According to a well-known result in the theory of Dirichlet forms (see Ma Röckner[1]), this is equivalent to verify that $(\mathcal{E}, \overline{\mathcal{C}})$ has the following property: $\forall \epsilon > 0$, there exists $C_\epsilon : \mathbb{R} \to [-\epsilon, 1+\epsilon]$ such that

(a) $C_\epsilon(t) = t, t \in [0,1]; 0 \le C_\epsilon(t) - C_\epsilon(s) \le t - s, t, s \in \mathbb{R}, s \le t$;

(b) $\forall \varphi \in \overline{\mathcal{C}}$, $C_\epsilon \circ \varphi \in \overline{\mathcal{C}}$, and $\mathcal{E}(C_\epsilon \circ \varphi, C_\epsilon \circ \varphi) \le \mathcal{E}(\varphi, \varphi)$.

Such a C_ϵ satisfying (a) can be easily constructed. The first condition of (b) has been proved to be satisfied, the second condition of (b) is actually satisfied for any C_ϵ satisfying (a) (note $|C_\epsilon'| < 1$).

$$\begin{aligned}\mathcal{E}(C_\epsilon \circ \varphi, C_\epsilon \circ \varphi) &= \int |DC_\epsilon \circ \varphi|^2 \, d\nu \\ &= \int (C_\epsilon' \circ \varphi)^2 |D\varphi|^2 \, d\nu \\ &\le \int |D\varphi|^2 \, d\nu = \mathcal{E}(\varphi, \varphi).\end{aligned}$$

The proof of the theorem is complete. ∎

The following theorem gives another sufficient condition for the closability of $(\mathcal{E}, \mathcal{C})$.

Theorem 4.4 *Let $\Phi \in (E)^*_+$. If $D\Phi = B \cdot \Phi$, where $B \in E^* \otimes (E), B \cdot \Phi$ is an element of $E^* \otimes (E)^*$ defined as follows:*

$$\langle\langle B \cdot \Phi, \eta \otimes \varphi \rangle\rangle = \langle\langle \Phi, \langle B, \eta \rangle \varphi \rangle\rangle, \quad \eta \in E, \varphi \in (E),$$

then $(\mathcal{E}_\Phi, \mathcal{C})$ is closable and $(\mathcal{E}_\Phi, \mathcal{C})$ is a Dirichlet form on $L^2(\nu)$.

Proof. By assumption, $\forall \eta \in E, \varphi \in (E)$, we have

$$\begin{aligned}\langle\langle D\Phi, \eta \otimes \varphi \rangle\rangle &= \langle\langle B \cdot \Phi, \eta \otimes \varphi \rangle\rangle \\ &= \langle\langle \Phi, \langle B, \eta \rangle \varphi \rangle\rangle,\end{aligned} \tag{4.18}$$

where $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$, $\langle B, \eta \rangle \in (E)$. By (4.18) we easily conclude that $\forall F \in l_0^2(\nu)$, $\varphi \in (E)$,

$$\langle\langle D\Phi, \varphi \cdot F \rangle\rangle = \langle\langle \Phi, (B, F)\varphi \rangle\rangle, \tag{4.19}$$

where

$$(B, F) = \sum_k \langle B, e_k \rangle \cdot F_k.$$

Hence, $\forall \varphi \in \mathcal{C}$, $F \in l_0^2(\nu)$,

$$\begin{aligned}(\varphi, D^* F)_{L^2(\nu)} &= \langle\langle \Phi, \varphi D^* F \rangle\rangle \\ &= \langle\langle D(\Phi\varphi), F \rangle\rangle \\ &= \langle\langle \Phi, (D\varphi, F)_{L^2(\mathbb{R}^d)} \rangle\rangle + \langle\langle D\Phi, \varphi \cdot F \rangle\rangle \\ &= (D\varphi, F)_{L^2(\nu)} + \langle\langle \Phi, (B, F)\varphi \rangle\rangle \\ &= (D\varphi, F)_{L^2(\nu)} + (\varphi, (B, F))_{L^2(\nu)}.\end{aligned}$$

This implies

$$D'F = D^* F - (B, F), \quad F \in l_0^2(\nu).$$

Thus D' is densely defined and $(\mathcal{E}, \mathcal{C})$ is closable. Moreover, $(\mathcal{E}, \overline{\mathcal{C}})$ is a Dirichlet form on $L^2(\nu)$ ∎

As an application of Theorem 4.4, we consider the Gaussian measure ν on $(\mathcal{S}^*(\mathbb{R}^d), \mathcal{B})$ with characteristic functional

$$\hat{\nu}(\xi) = \int e^{i\langle x, \xi \rangle} \nu(dx) = \exp\{-\tfrac{1}{2}(\xi, K\xi)\},$$

where K is a positive invertible self-adjoint operator on $L^2(\mathbb{R}^d)$, and K is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. Clearly, ν corresponds to a Hida distribution, denoted by Φ (i.e., $\Phi = \frac{d\nu}{d\mu}$). By means of S-transform and using (1.22), we can prove that $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$D_f \Phi = \langle \cdot, (K^{-1} - I) f \rangle \Phi.$$

Put

$$B = \sum_k e_k \otimes \langle \cdot, (K^{-1} - I) e_k \rangle.$$

Then $B \in \mathcal{S}'(\mathbb{R}^d) \otimes (E)$, and $D\Phi = B \cdot \Phi$. Thus by Theorem 4.4, there is a Dirichlet form on $L^2(\nu)$ associated with ν through (4.13).

Appendix A

Hermite polynomials and Hermite functions

Real Hermite polynomials are defined to be

$$H_n(u) \equiv (-1)^n e^{u^2/2} \frac{d^n}{du^n} e^{-u^2/2}, \quad u \in \mathbb{R}, n \in \mathbb{N}_0, \tag{A.1}$$

which are coefficients in expansion of power series for $\exp\{tu - t^2/2\}$ as function of t:

$$\exp\{tu - t^2/2\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(u), \quad t, u \in \mathbb{R}. \tag{A.2}$$

By this expansion formula we have:

Theorem A.1 *Hermite polynomials have the following expression:*

$$H_n(u) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k u^{n-2k}}{2^k k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{A.3}$$

Conversely,

$$u^n = n! \sum_{k=0}^{[n/2]} \frac{H_{n-2k}(u)}{2^k k!(n-2k)!}, \quad n \in \mathbb{N}_0. \tag{A.4}$$

$\{H_n, n \in \mathbb{N}\}$ *satisfy the following differential equations*

$$H_n'(u) = nH_{n-1}(u), \qquad n \geq 1, \tag{A.5}$$

$$H_n''(u) - uH_n'(u) + nH_n(u) = 0, \qquad n > 0 \tag{A.6}$$

and recursion formula:

$$H_0(u) \equiv 1, \qquad H_1(u) = u,$$

$$H_{n+1}(u) = uH_n(u) - nH_{n-1}(u), \qquad n \geq 1, \tag{A.7}$$

as well as multiplication formula:

$$H_m(u)H_n(u) = \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}(u). \tag{A.8}$$

Moreover, for any $\lambda \in \mathbb{R}$ it holds that

$$H_n(\lambda u) = n! \sum_{k=0}^{[n/2]} \frac{(\lambda^2 - 1)^k \lambda^{n-2k}}{2^k k!(n-2k)!} H_{n-2k}(u). \tag{A.9}$$

Proof. Replacing the power series of e^{tu} and $e^{-t^2/2}$ with respect to t into eq. (A.2) and comparing the coefficients of t^n on both sides, we obtain eqs. (A.3) and (A.4). Differentiating eq. (A.2) with respect to u and comparing the coefficients of power series we get (A.5) and (A.6). Again from eq. (A.2) we know

$$\begin{aligned}
\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} H_m(u)H_n(u) &= \exp\left\{(s+t)u - \frac{(s+t)^2}{2} + st\right\} \\
&= \sum_{j=0}^{\infty} \frac{(s+t)^j}{j!} H_j(u) \sum_{k=0}^{\infty} \frac{s^k t^k}{k!} \\
&= \sum_{j,k=0}^{\infty} \frac{H_j(u)}{j!k!} \sum_{l=0}^{j} \binom{j}{l} s^{l+k} t^{j-l+k}.
\end{aligned}$$

Letting $l + k = m$, $j - l + k = n$ in the last expression, we have

$$\sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} \sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} H_{m+n-2k}.$$

The multiplication formula (A.8) is obtained by comparing the coefficients of $s^m t^n$. In particular, the recursion formula (A.7) is obtained by letting $m = 1$ in eq. (A.8). Finally, it follows from eq. (A.2) that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\lambda u) &= \exp\left\{\lambda t u - \frac{t^2}{2}\right\} \\
&= \exp\left\{\lambda t u - \frac{\lambda^2 t^2}{2} + \frac{(\lambda^2 - 1)t^2}{2}\right\} \\
&= \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} H_j(u) \sum_{k=0}^{\infty} \frac{(\lambda^2 - 1)^k t^{2k}}{2^k k!}.
\end{aligned}$$

Letting $j + 2k = n$ in the last expression, we obtain

$$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/2]} \frac{(\lambda^2 - 1)^k \lambda^{n-2k}}{2^k k!(n-2k)!} H_{n-2k}(u).$$

by comparing the coefficients of t^n, we then have eq. (A.9). ∎

Considering the Gaussian measure on $\mathbb{R}$:

$$\gamma(du) \equiv (2\pi)^{-1/2} \exp\{-u^2/2\} du$$

and the Hilbert space $L^2(\mathbb{R},\gamma)$, we have

Theorem A.2 *Hermite polynomials constitute an orthogonal system in $L^2(\mathbb{R},\gamma)$:*

$$\int_{\mathbb{R}} H_m(u)H_n(u)\gamma(du) = n!\delta_{mn}, \qquad m,n\in\mathbb{N}_0. \tag{A.10}$$

Denote $i=\sqrt{-1}$. Then

$$H_n(u) = \int_{\mathbb{R}} (u+iv)^n\gamma(dv), \qquad n\in\mathbb{N}_0, \tag{A.11}$$

moreover,

$$H_n(u+v) = \sum_{k=0}^{n}\binom{n}{k}u^k H_{n-k}(v), \qquad n\in\mathbb{N}_0. \tag{A.12}$$

When $t^2<1$, we have

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(u)H_n(v) = \frac{1}{\sqrt{1-t^2}}\exp\left\{-\frac{t^2u^2-2tuv+t^2v^2}{2(1-t^2)}\right\}. \tag{A.13}$$

Proof. It follows from eq. (A.2) that

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}\int_{\mathbb{R}} H_m(u)H_n(u)\gamma(du)\\
&=\int_{\mathbb{R}}\exp\left\{(s+t)u-\frac{s^2+t^2}{2}\right\}\gamma(du)\\
&=\exp\left\{-\frac{s^2+t^2}{2}+\frac{(s+t)^2}{2}\right\}=e^{st}\\
&=\sum_{n=0}^{\infty}\frac{(st)^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $s^m t^n$ we obtain eq. (A.10). Using contour integration we have

$$\int_{\mathbb{R}}\exp\{t(u\pm iv)\}\gamma(dv) = \exp\left\{tu-\frac{t^2}{2}\right\}.$$

By expansion in power series of t (using eq. (A.2) for right-hand side) and comparing the coefficients of t^n we prove eq. (A.11). From eq. (A.11) we know

$$\begin{aligned}
H_n(u+v) &= \int_{\mathbb{R}}(u+v+iy)^n\gamma(dy)\\
&=\sum_{k=0}^{n}\binom{n}{k}u^k\int_{\mathbb{R}}(v+iy)^{n-k}\gamma(dy),
\end{aligned}$$

which implies eq. (A.12). Again by eq. (A.11) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(u)H_n(v)\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}\exp\{t(u+ix)(v+iy)\}\gamma(dx)\gamma(dy).
\end{aligned}$$

A direct computation of the integral yields eq. (A.13). ∎

It follows from eq. (A.4) and multiplication formula (A.8) that Hermite polynomials constitute a linear base of polynomial ring. In view of eq. (A.10) and density of polynomials in $L^2(\mathbb{R},\gamma)$, we know that $\{(n!)^{-1/2}H_n\}$ is an orthonormal base of $L^2(\mathbb{R},\gamma)$. Now consider the Hilbert space $L^2(\mathbb{R})=L^2(\mathbb{R},du)$, where du is Lebesgue measure. For $f\in L^2(\mathbb{R})$, define

$$Jf(u) \equiv \pi^{1/4}e^{u^2/4}f(u/\sqrt{2}). \tag{A.14}$$

Then

$$\|Jf\|^2_{L^2(\mathbb{R},\gamma)} = \|f\|^2_{L^2(\mathbb{R})}.$$

Moreover,

$$J^{-1}f(u) = \pi^{-1/4}e^{-u^2/2}f(\sqrt{2}u). \tag{A.15}$$

Hence $J: L^2(\mathbb{R})\longrightarrow L^2(\mathbb{R},\gamma)$ is an isomorphism for Hilbert spaces. Let

$$\begin{aligned}
h_n(u) &\equiv (n!)^{-1/2}J^{-1}H_n(u)\\
&=(n!)^{-1/2}\pi^{-1/4}e^{-u^2/2}H_n(\sqrt{2}u).
\end{aligned} \tag{A.16}$$

Then $\{h_n, n\in\mathbb{N}_0\}$ constitute an orthonormal base of $L^2(\mathbb{R})$. They are called *Hermite functions*. By definition and properties of Hermite polynomials we have

$$h_n'(u)+uh_n(u) = \sqrt{2n}\,h_{n-1}(u), \qquad n\geq 1. \tag{A.17}$$

In addition, the following estimates are very useful, for the proof see Hille-Phillips[1] or G.Szegö[1].

Theorem A.3 *For any fixed $u\in\mathbb{R}$, we have*

$$h_n(u) = O(n^{-1/4}), \tag{A.18}$$

$$\int_0^u h_n(v)dv = O(n^{-3/4}). \tag{A.19}$$

Moreover,

$$\|h_n\|_{L^\infty} \equiv \sup_{u\in\mathbb{R}}|h_n(u)| = O(n^{-1/12}), \tag{A.20}$$

$$\|h_n\|_{L^1} \equiv \int_{\mathbb{R}}|h_n(u)|du = O(n^{1/4}). \tag{A.21}$$

Since $H_n(u) = (n!)^{1/2}\pi^{1/4}e^{u^2/4}h_n(u/\sqrt{2})$, it follows from (A.20) that

$$|\, H_n(u) \,| \leq c(n!)^{1/2}e^{u^2/4}. \tag{A.22}$$

More precisely, we may take $c = 1.2$ in the above inequality and (A.22) is then called Cramér's estimate (cf. Erdélyi[1], p.208).

Appendix B

Locally convex spaces and their dual spaces

We briefly introduce some basic notions of topological linear spaces which are frequently used in the book. For details see Bourbaki[1], Schaefer[1] or Treves[1].

1. Semi-norms, norms and H-norms

Let X be a linear space on field $\mathbb{K}$ (real field $\mathbb{R}$ or complex field $\mathbb{C}$) and p a non-negative real-valued function on X, if

(N.1) $\quad p(x+y) \leq p(x) + p(y), \qquad x, y \in X;$

(N.2) $\quad p(\lambda x) = |\lambda| p(x), \qquad x \in X, \lambda \in \mathbb{K},$

then p is called a *semi-norm on* X. If, furthermore,

(N.3) $\quad x \neq 0 \Rightarrow p(x) > 0,$

then it is called a *norm*. If, moreover, the following "parallelogram identity" holds:

(N.4) $\quad p(x+y)^2 + p(x-y)^2 = 2p(x)^2 + 2p(y)^2, \; x, y \in X,$

then it is called a *Hilbertian norm* (*H-norm* for short). The function p with properties (N.1), (N.2) and (N.4) is called an *H semi-norm*.

Let p be a semi-norm on X, define

$$N_p \equiv p^{-1}(0) = \{x \in X : p(x) = 0\}.$$

By properties (N.1) and (N.2), N_p is a linear subspace of X (if p is a norm, then $N_p = \{0\}$). Let

$$\widehat{X}_p \equiv X/N_p$$

be the quotient space, that is, the linear space of all equivalence classes $\widehat{x}$ for equivalence relation $x \sim y$ defined by $p(x-y) = 0$. The quotient map is denoted by

$$Q_p : X \longrightarrow \widehat{X}_p, \tag{B.1}$$

namely, $\widehat{x} = Q_p x$ is the equivalence class containing x. Define a function on $\widehat{X}_p$ by

$$\widehat{p}(Q_p x) = p(x).$$

It is easy to see that $\tilde{p}$ is a norm on $\widetilde{X}_p$ and $(\widetilde{X}_p, \tilde{p})$ is a normed space. By completion we obtain a Banach space $(\overline{X}_p, \bar{p})$. If this Banach space is separable, we call semi-norm p a *separable semi-norm*.

If p is an H semi-norm on X, then the Banach space thus obtained is in fact a Hilbert space.

Let p and q be two semi-norms on X. If $\exists c > 0$ such that

$$p(x) \leq cq(x), \qquad \forall x \in X,$$

then p is said to be *bounded by* q and denoted by $p \prec q$. In this case $N_q \subset N_p$, q-equivalence implies p-equivalence, hence

$$I_{pq} \equiv Q_p Q_q^{-1} \,:\, \widetilde{X}_q \longrightarrow \widetilde{X}_p \tag{B.2}$$

extends to a continuous linear operator from $\overline{X}_q$ to $\overline{X}_p$.

Let p and q be two H semi-norms on X. If there exists an orthonormal base $\{e_n\}$ of $\overline{X}_q$ such that

$$\sum_{n=1}^{\infty} p(I_{pq}e_n)^2 < \infty, \tag{B.3}$$

then p is said to be *HS bounded by* q and denoted by $p \prec_{\mathrm{HS}} q$. It is easy to prove that the sum in (B.3) is independent of the choices of orthonormal base $\{e_n\}$ and is equal to square of Hilbert-Schmidt norm of the operator I_{pq}.

2. Locally convex topological linear spaces, bounded sets

If a linear space X equipped with topology $\mathcal{T}$ such that addition and scalar product operations are continuous, then $(X, \mathcal{T})$ is called a *topological linear space* or *topological vector space*.

By virtue of continuity of addition operation, the neighborhood system of any point in a topological linear space can be obtained by translation of $\mathcal{N}(0)$, the neighborhood system of point 0, hence its topology is entirely determined by $\mathcal{N}(0)$.

Let V be a subset of X. If $\forall x \in X$, $\exists \lambda_0$ such that $\lambda x \in V$ holds for any $\lambda \in \mathbb{K}$ with $|\lambda| \leq \lambda_0$, then V is called an *absorbing set*. It is easy to see from the continuity of scalar production that any neighborhood of 0 is an absorbing set.

In a Banach space $(X, \|\cdot\|)$, as a base of $\mathcal{N}(0)$, we choose open balls $\{x, \|x\| < \epsilon\}$ with different radii ($\epsilon > 0$). However, in infinite dimensional analysis, it is too restrictive to consider the topological linear spaces generated by a single norm. One needs to consider the topological linear spaces $(X, \mathcal{T})$ generated by some family $\Gamma = \{p_\alpha ; \alpha \in A\}$ of semi-norms, where A is an arbitrary set. The base of $\mathcal{N}(0)$ consists of the following sets:

$$\{x : p_{\alpha_j}(x) < \epsilon_j,\ j = 1, 2, \cdots, n\}, \tag{B.4}$$

where $n \in \mathbb{N}, \epsilon_j > 0, \alpha_j \in A\ (j = 1, 2, \cdots, n)$. In order to make $(X, \mathcal{T})$ a Hausdorff topological space, instead of (N.3), the totality Γ should satisfy that

$$x \neq 0 \Rightarrow \exists \alpha \in A,\ p_\alpha(x) > 0. \tag{B.5}$$

Any topological linear space generated by a family Γ of semi-norms satisfying (B.5) is called a *locally convex space* (LCS for short). This definition is equivalent to the geometric one, that is, any Hausdorff topological linear space having a convex neighborhood base is a locally convex space.

Let Γ_1 and Γ_2 be two families of semi-norms on X. If the topology generated by Γ_1 is weaker than that generated by Γ_2, then we say that Γ_1 is *weaker than* Γ_2 and denote $\Gamma_1 \prec \Gamma_2$; if $\Gamma_1 \prec \Gamma_2$ and $\Gamma_2 \prec \Gamma_1$, then we say that Γ_1 is *equivalent to* Γ_2 and denote $\Gamma_1 \sim \Gamma_2$. Families of semi-norms which are equivalent to each other generate the same topology.

A locally convex space is metrizable if and only if its topology can be generated by a countable family of semi-norms. A complete metrizable locally convex space is called a *Fréchet space*. All Banach spaces are Fréchet spaces.

In a normed space, the bounded sets can be defined by norm. However, in a general topological linear space X, a subset B is said to be *bounded set* if it can be absorbed by any neighborhood of 0, that is,

$$\forall U \in \mathcal{N}(0),\ \exists \lambda \in \mathbb{K} \text{ such that } B \subset \lambda U. \tag{B.6}$$

If X is a locally convex space whose topology is generated by a family Γ of semi-norms, then a subset B is bounded if and only if

$$\sup_{x \in B} p(x) < \infty, \qquad \forall p \in \Gamma. \tag{B.7}$$

A locally convex space is normable if and only if it has bounded neighborhoods of 0. A topological space is said to be *locally compact* if it has compact neighborhoods of any point. A locally convex space is locally compact if and only if it is finite dimensional.

3. Projective topologies and projective limits

Let X be a linear space, $\{X_\alpha, \mathcal{T}_\alpha ; \alpha \in A\}$ be a family of locally convex spaces, where topology $\mathcal{T}_\alpha$ is generated by a family Γ_α of semi-norms. Let

$$f_\alpha \,:\, X \longrightarrow X_\alpha, \quad \alpha \in A$$

be a family of linear maps satisfying $\bigcap_\alpha f_\alpha^{-1}(0) = \{0\}$. The *weakest* locally convex topology $\mathcal{T}$ in X such that each map f_α is continuous is called *projective topology* with respect to $(X_\alpha, \mathcal{T}_\alpha, f_\alpha ; \alpha \in A)$. It is generated by the family $\{p_\alpha \circ f_\alpha ; p_\alpha \in \Gamma_\alpha, \alpha \in A\}$ of semi-norms. A linear map T from any locally convex space Y to

X is $\mathcal{T}$-continuous if and only if $\forall\alpha\in A$, $f_\alpha\circ T$ is $\mathcal{T}_\alpha$-continuous. A subset B of X is bounded if and only if $\forall\alpha\in A$, $f_\alpha(B)$ is bounded in X_α.

Example 1 Let X be a linear subspace of locally convex space $(Y,\mathcal{T})$, $f: X\longrightarrow Y$ be the natural imbedding. Then the projective topology in X with respect to $(Y,\mathcal{T},f)$ is the *induced topology* of $\mathcal{T}$ in subspace X, denoted by $\mathcal{T}|_X$.

Example 2. Let $X=\prod_{\alpha\in A}X_\alpha$ be product space, f_α be the coordinate projection of X onto X_α. Then the projective topology in X with respect to $(X_\alpha,\mathcal{T}_\alpha,f_\alpha;\alpha\in A)$ is their *product topology*, denoted by $\prod_{\alpha\in A}\mathcal{T}_\alpha$.

Example 3. Let $\{X_n,\mathcal{T}_n;n\in\mathbb{N}\}$ be a sequence of locally convex spaces such that

$$X_1\supset X_2\supset\cdots\supset X_n\supset\cdots.$$

Suppose that when $m\ge n$, $\mathcal{T}_n|_{X_m}\prec\mathcal{T}_m$ and that

$$X=\bigcap_n X_n,$$

f_n being natural imbedding of X into X_n. Then the projective topology $\mathcal{T}$ in X with respect to $(X_n,\mathcal{T}_n,f_n;n\in\mathbb{N})$ is called the *projective limit topology* of this projective sequence of locally convex spaces. The space X equipped with this topology is called the *topological projective limit* of sequence $(X_n,\mathcal{T}_n)$ and denoted by $X=\varprojlim X_n$. (The definition can be extended to the case of any directed partially ordered subscript set). A linear functional f on X is $\mathcal{T}$-continuous if and only if $\exists n\in\mathbb{N}$, such that f extends to a $\mathcal{T}_n$-continuous functional on X_n. If, moreover, for any $m\ge n$, $\mathcal{T}_n|_{X_m}=\mathcal{T}_m$, then the projective limit is said to be *strict*.

4. Inductive topologies and inductive limits

Let X be a linear space, $\{X_\alpha,\mathcal{T}_\alpha;\alpha\in A\}$ be a family of locally convex spaces. Suppose that

$$g_\alpha : X_\alpha\longrightarrow X,\quad \alpha\in A$$

are linear maps satisfying $X=\mathrm{span}\{\bigcup_\alpha g_\alpha(X_\alpha)\}$. The *strongest* locally convex topology $\mathcal{T}$ in X such that each map g_α is continuous is called the *inductive topology* with respect to $(X_\alpha,\mathcal{T}_\alpha,g_\alpha;\alpha\in A)$. A linear map T from X to any locally convex space Y is $\mathcal{T}$-continuous if and only if $\forall\alpha\in A, T\circ g_\alpha$ is $\mathcal{T}_\alpha$-continuous.

Example 1. Let M be a closed linear subspace of locally convex space $(Y,\mathcal{T})$, $X=Y/M$ be the quotient space, $g : Y\longrightarrow X$ be the quotient map. Then the inductive topology in X with respect to $(Y,\mathcal{T},g)$ is the *quotient topology*.

Example 2. Let $\{X_\alpha,\mathcal{T}_\alpha;\alpha\in A\}$ be a family of locally convex spaces, $X=\bigoplus_{\alpha\in A}X_\alpha$ be their algebraic direct sum (namely the linear subspace of product space $\prod_{\alpha\in A}X_\alpha$ consists of all elements containing only finite non-zero coordinates), g_α be the natural imbedding from X_α into X. Then the inductive

topology in X with respect to $(X_\alpha,\mathcal{T}_\alpha,g_\alpha;\alpha\in A)$ is the *direct sum topology*, denoted by $\bigoplus_{\alpha\in A}\mathcal{T}_\alpha$.

Example 3. Let $\{X_n,\mathcal{T}_n;n\in\mathbb{N}\}$ be a sequence of locally convex spaces satisfying that

$$X_1\subset X_2\subset\cdots\subset X_n\subset\cdots$$

and $\mathcal{T}_m|_{X_n}\prec\mathcal{T}_n$ whenever $m\ge n$. Suppose that

$$X=\bigcup_n X_n,$$

g_n is the natural imbedding from X_n into X. Then the inductive topology $\mathcal{T}$ in X with respect to $(X_n,\mathcal{T}_n,g_n;n\in\mathbb{N})$ is called the *inductive limit topology* of this inductive sequence of locally convex spaces. The space X equipped with this topology is called the *topological inductive limit* of sequence $(X_n,\mathcal{T}_n)$ and denoted by $X=\varinjlim X_n$. (The definition can also be extended to the case of directed subscript sets). If, moreover, for any $m\ge n$, $\mathcal{T}_m|_{X_n}=\mathcal{T}_n$, then the inductive limit is said to be *strict*.

A subset B of X is bounded in the inductive limit topology if and only if $\exists n_0\in\mathbb{N}$ so that $B\subset X_{n_0}$ and B is bounded in X_{n_0}. A sequence $\{x_n\}$ converges to x if and only if $\exists n_0\in\mathbb{N}$ so that $\{x_n\}\subset X_{n_0}$ and $x_n\to x$ in X_{n_0}.

Note that the inductive limit topology needs not be a Hausdorff topology. But if the limit is strict, then it is a Hausdorff topology.

5. Dual spaces and weak topologies

Let X,Y be two linear spaces. If there is a bilinear functional $\langle x,y\rangle$ on $X\times Y$ satisfying the separation axioms:

$$\begin{aligned}&\langle x,y\rangle=0\ ,\ \forall y\in Y\Rightarrow x=0,\\ &\langle x,y\rangle=0\ ,\ \forall x\in X\Rightarrow y=0,\end{aligned}\tag{B.8}$$

then we say that X and Y constitute a *dual system*, or briefly, $\langle X,Y\rangle$ is a *duality* (of linear spaces). For example, all linear functionals on a linear space X constitute a linear space X', the *algebraic dual* space of X. For any $x\in X$ and $f\in X'$, define

$$\langle x,f\rangle\equiv f(x)$$

as a bilinear functional. Then $\langle X,X'\rangle$ is a duality.

If $(X,\mathcal{T})$ is a locally convex space, then all $\mathcal{T}$-continuous linear functionals on X constitute a linear space X^*, the *topological dual* space of $(X,\mathcal{T})$. Obviously, X^* is a linear subspace of X' and $\langle X,X^*\rangle$ is a duality, too.

Let $\langle X,Y\rangle$ be a duality. The topology in X generated by semi-norms

$$p_y(x)\equiv|\langle x,y\rangle|\ ,\quad y\in Y\tag{B.9}$$

is called the *weak topology* with respect to the duality and denoted by $\sigma(X,Y)$. It is the weakest locally convex topology such that all linear functionals $f_y(x) = \langle x,y\rangle, y \in Y$ are continuous. By symmetry, we can define the weak topology $\sigma(Y,X)$ in Y.

6. Compatibility and Mackey topology

Let $\langle X,Y\rangle$ be a duality. A topology $\mathcal{T}$ in X is said to be *compatible* with this duality if the linear space X^* of all $\mathcal{T}$-continuous linear functionals on X coincides with Y (any element y in Y can be looked as a linear functional $\langle\cdot,y\rangle$ on X, so Y can be looked as a linear subspace of the algebraic dual space X' of X). Obviously, the weak topology $\sigma(X,Y)$ is the weakest of compatible topologies.

In order to introduce stronger topologies, we consider the following families of semi-norms:

$$p_S(x) \equiv \sup_{y\in S} |\langle x,y\rangle|, \quad S \in \mathfrak{S}. \tag{B.10}$$

where $\mathfrak{S}$ is some nonempty class of subsets of Y covering the space Y. It follows from condition (B.7) that, if and only if p_S is finite, that is, S is a $\sigma(Y,X)$-bounded set, p_S is a semi-norm. The topology in X generated by semi-norms (B.10) is the topology of uniform convergence on every set S in $\mathfrak{S}$, and is called $\mathfrak{S}$*-topology*. In particular, if $\mathfrak{S}$ consists of all finite sets in Y, then we obtain the topology of pointwise convergence, namely the weak topology $\sigma(X,Y)$.

In order to obtain the strongest compatible topology, we note that p_S is continuous (in some topology) if and only if $\forall\epsilon > 0$, $\exists U \in \mathcal{N}(0)$ so that $p_S(x) < \epsilon, \forall x \in U$, namely, S is an *equicontinuous set* of functionals on X. Any compatible topology in X is a topology of uniform convergence on some equicontinuous sets in Y. By the **Alaoglu Bourbaki** theorem (for example, cf. Taylor & Lay[1], p.166), any equicontinuous set is relatively $\sigma(Y,X)$-compact. If we take $\mathfrak{S}$ as all absolutely convex [2] weakly compact subsets of Y, then this $\mathfrak{S}$-topology is called *Mackey topology*, denoted by $\tau(X,Y)$. We have the following important theorem:

Mackey-Arens Theorem (cf. Schaefer[1], p.131) Let $\langle X,Y\rangle$ be a duality. Then a locally convex topology $\mathcal{T}$ in X is compatible with this duality if and only if

$$\sigma(X,Y) \prec \mathcal{T} \prec \tau(X,Y). \tag{B.11}$$

All compatible topologies have the same family of bounded sets and same family of closed convex sets. According to this theorem, the closures of a convex set (especially a linear subspace) with respect to all compatible topologies coincide.

[2]A set V in a linear space X is said to be absolutely convex if $\lambda x+\mu y\in V$ whenever $x,y\in V$, $\lambda,\mu\in\mathbb{K}$ with $|\lambda|+|\mu|\leq 1$.

Let $(X,\mathcal{T})$ be a locally convex space. In view of the duality $\langle X, X^*\rangle$, it is obvious that

$$\sigma(X,X^*) \prec \mathcal{T} \prec \tau(X,X^*).$$

Therefore, any weakly bounded set is bounded.

7. Strong topologies and reflexivity

Let $(X,\mathcal{T})$ be a locally convex space, X^* be its topological dual space, $\mathfrak{S}$ be the class of all $\sigma(X^*,X)$-bounded sets in X^*. This $\mathfrak{S}$-topology, being the topology of uniform convergence on every $\sigma(X^*,X)$-bounded sets, is called the *strong topology* and denoted by $\beta(X,X^*)$. In order to distinguish between the original space $(X,\mathcal{T})$ and the locally convex space $(X,\beta(X,X^*))$ equipped with strong topology, we denote the latter by X_β (similarly, we have X_σ, X_τ etc.). If $(X,\mathcal{T})$ coincides with X_β, then it is called a *barreled* [3] space. If $(X,\mathcal{T})$ coincides with X_τ, then it is called a *Mackey space*. All Fréchet spaces are barreled. All barreled spaces are Mackey spaces. Denote by $(X_\beta)^*$ the linear space consisting of all strongly continuous linear functionals on X (similarly, we have $(X_\sigma)^*, (X_\tau)^*$, etc.), it follows from Mackey-Arens Theorem that, as linear spaces,

$$(X_\sigma)^* = X^* = (X_\tau)^* \subset (X_\beta)^*. \tag{B.12}$$

By symmetry, we have weak* topology $\sigma(X^*,X)$, Mackey topology $\tau(X^*,X)$ and strong topology $\beta(X^*,X)$ in X^*. To specify different topologies in X^*, we denote them by X^*_σ, X^*_τ and X^*_β (note that $X^*_\beta \neq (X_\beta)^*$), respectively. X^*_β is called the *strong dual space* of X.

All strongly continuous functionals on X^* constitute a linear space $X^{**} \equiv (X^*_\beta)^*$ which is called the *bidual* space of X. In general, we have $X \subset X^{**}$. If $X = X^{**}$ and $\mathcal{T} = \beta(X,X^*)$, that is, $(X,\mathcal{T})$ coincides with its strong bidual space $X^{**}_\beta = (X^*_\beta)^*_\beta$, then X is called a *reflexive space*.

A locally convex space X is reflexive if and only if X is barreled and in which all bounded sets are relatively weakly compact. In particular, A Banach space is reflexive if and only if its unit balls are weakly compact.

8. Dual maps

Let X,Y be locally convex spaces, X^*,Y^* be their topological dual spaces respectively. Denote by $\mathcal{L}(X,Y)$ the linear space consisting of all continuous linear maps from X to Y. If $T \in \mathcal{L}(X,Y)$, then for any $f \in Y^*$ we have $f\circ T \in X^*$. The linear map: $f \longmapsto f\circ T$ is denoted by

$$T^* : Y^* \longrightarrow X^*.$$

[3]Any closed absolutely convex absorbing set is called a barrel. In a barreled space, all barrels are neighborhoods of 0.

and is called the *dual map* of T. In view of dualities $\langle X, X^*\rangle$ and $\langle Y, Y^*\rangle$, the relation between T and T^* is as follows:

$$\langle Tx, f\rangle = \langle x, T^* f\rangle, \quad x \in X, f \in Y^*. \tag{B.13}$$

It follows that T and T^* are σ-continuous, that is,

$$T \in \mathcal{L}(X_\sigma, Y_\sigma)\ , \quad T^* \in \mathcal{L}(Y^*_\sigma, X^*_\sigma). \tag{B.14}$$

The dual map T^* is an injection if and only if $\mathcal{R}(T)$, the range of T, is dense in Y (or equivalently, weakly dense in Y, since the closures of a linear subspace in all compatible topologies are the same).

If X, Y are Mackey spaces, then continuity is equivalent to σ-continuity. In particular, if X, Y are Banach spaces and X is continuously densely imbedded into Y, then the strong dual space Y^* is also continuously densely imbedded into X^*.

Consider a sequence of Hilbert spaces

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset X_{n+1} \supset \cdots,$$

if $\forall n \in \mathbb{N}, X_{n+1}$ is continuously densely imbedded into X_n and $X = \varprojlim X_n$ is their topological projective limit, then by duality,

$$X_1^* \subset X_2^* \subset \cdots \subset X_n^* \subset X_{n+1}^* \subset \cdots,$$

where $\forall n \in \mathbb{N}, X_n^*$ is continuously densely imbedded into X_{n+1}^*, and $X^* = \varinjlim X_n^*$ is their topological inductive limit.

Since a countably Hilbertian space X (cf. Chapter I, §3) is reflexive and the inductive limit topology is the strongest locally convex topology such that each imbedding is continuous, in its dual space X^*, the inductive limit topology, strong topology and Mackey topology are equivalent.

9. Uniformly convex spaces and Banach-Saks' theorem

Let X be a normed space. If $\forall \epsilon \in (0,2), \exists\, \delta > 0$ such that $\forall x, y \in X$,

$$\|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \longrightarrow \|\tfrac{1}{2}(x+y)\| \le 1 - \delta, \tag{B.15}$$

then X is called a *uniformly convex space*.

Example 1. (Hilbert space) By the parallelogram identity, for $\|x\| \le 1, \|y\| \le 1$ and $\|x-y\| \ge \epsilon$ we have $\|x+y\|^2 \le 4 - \epsilon^2$. Letting $\delta = 1 - (1 - \epsilon^2/4)^{1/2}$ we obtain the uniform convexity for Hilbert spaces.

Example 2. (L^p space) Let $1 < p < \infty$, By Clarkson inequality:

$$\|\tfrac{1}{2}(f+g)\|_p^p + \|\tfrac{1}{2}(f-g)\|_p^p \le \tfrac{1}{2}(\|f\|_p^p + \|g\|_p^p) \quad (2 \le p < \infty), \tag{B.16}$$

$$\|\tfrac{1}{2}(f+g)\|_p^q + \|\tfrac{1}{2}(f-g)\|_p^q \le (\tfrac{1}{2}(\|f\|_p^p + \|g\|_p^p))^{q-1} \ (1 < p \le 2) \tag{B.17}$$

(where q is the conjugate exponent of p), for $p \ge 2$, we have

$$\|\tfrac{1}{2}(f+g)\|_p \le (1 - (\epsilon/2)^p)^{1/p},$$

and take $\delta = \epsilon^p/2^p p + o(\epsilon^p)$; for $p \le 2$, we have

$$\|\tfrac{1}{2}(f+g)\|_p \le (1 - (\epsilon/2)^q)^{1/q},$$

and take $\delta = (p-1)\epsilon^2/8 + o(\epsilon^2)$. Then we obtain the uniform convexity of L^p spaces.

Every uniformly convex Banach space is reflexive. As we know, all bounded sets in a reflexive Banach space are relatively *weakly* compact. However, any uniformly convex Banach space has the so-called *Banach-Saks property*:

Banach-Saks-Kakutani Theorem *If $\{x_n\}$ is a bounded sequence in a uniformly convex space X, then it has a subsequence $\{x_{n_j}\}$ so that the average sequence $S_k \equiv \frac{1}{k}\sum_{j=1}^{k} x_{n_j}$ strongly converges in X.*

For details see Diestel[1]. Furthermore, it can be proved that: if A is a closed convex subset in a uniformly convex Banach space X, then $\forall x \in X$ there exists a unique element $P_A(x) \in A$ such that

$$\|P_A(x) - x\| = \inf_{y \in A} \|y - x\|.$$

Comments

Chapter I

§1. The main contents of Sections 1.1–1.4 are taken from Kato[1], Chapter 5 and 6. For Section 1.5 we refer to Kuo[1].

§2. The definition of tensor product for Hilbert spaces is taken from Reed-Simon[1]. The notion of Fock space was initiazed by Fock[1]. A mathematically rigorous theory for second quantization first appeared in Cook[1], more detailed discussion can be found in Segal[2] and Simon[1].

§3. The notion of countably normed space was introduced by Gel'fand, see Gel'fand-Shilov[1]. The nuclear space was defined by Grothendieck[1] in finding a general class of locally convex spaces in which the Schwartz kernels theorem holds. For its general theory see Schaefer[1],Treves[1] and Itô[2], for a systematic account of theory of countably Hilbertian nuclear spaces see Gel'fand-Vilenkin[1]. The projective tensor product of general locally convex spaces is the π-tensor product defined by Treves[1]. There is another kind of tensor product: ϵ-tensor product. However, in case of nuclear spaces, these two kinds of tensor products coincide. In case of Hilbert spaces, they are different from that defined in §2.

§4. The contents of Section 4.1 are taken from Kuo[1] and Skorohod[2]. Sections 4.2–4.3 are based on Kuo[1], the simple proof of Fernique theorem is taken from Da Prato-Zubczyk[1]. For generalized forms of Minlos theorem and Gross theorem see Yan[2].

Chapter II

The fundamental work on stochastic calculus of variation is Malliavin[1]. Since then many authors have devoted to its rigorous mathematical theory. Among them, Stroock[1,2] and Kusuoka-Stroock[1,3] systematically developed a theory of infinite dimensional symmetric diffusion semi-groups; by a more direct method of Girsanov transformation Bismut[1] obtained the integration by parts formula on Wiener spaces; Shigekawa[1], S. Watanabe[1], Ikeda-Watanabe[1], Meyer[2] and others developed a method of Sobolev spaces for Wiener functionals and established a unified theory for infinite dimensional Sobolev spaces. For systematic accounts of the theory see S. Watanabe[1], Ikeda-Watanabe[3], Norris[1], Ocone[2], Huang[4], Malliavin[5], Ustunel[4] or Nualart[1].

§1. The notion of abstract Wiener space was introduced by Gross[1]. Since its differential structure is entirely determined by the Cameron-Martin subspace H, Itô[3,4] has devoted to establish a stochastic calculus of variation which is only based on H and independent of any other additional structures. This study goes back to the framework of Gaussian probability space due to Segal[1]. In Malliavin[5], Nualart[1] and the present book, the basic theory of stochastic calculus of variation were developed under this general framework.

The notions of irreducible Gaussian probability space, numerical model and intrinsic properties are taken from Malliavin[5]. The chaos decomposition for square integrable functionals was first obtained by Wiener[2]. Main results of this paragraph belong to Itô[1], some parts of proofs are taken from Nualart[1].

§2. The hyper-contractivity of Ornstein-Uhlenbeck semigroups was first proved by Nelson[1]. Here we adopt a simple proof due to Neveu[1]. Cameron-Martin[1] first discovered the quasi-invariance of Wiener measure (Theorem 2.5). Its presentation and proof for general case of Gaussian probability spaces are from Malliavin[5]. A systematic exposition of theory of Sobolev spaces for Wiener functionals can be found in S. Watanabe[1] and Sugita[1,2].

§3. The L^p-multiplier theorem (Theorem 3.8) was obtained by Meyer[2]. The simple proof is taken from Shigekawa[4]. In the special case of $p=2$, Meyer's inequalities (Theorem 3.15) were discovered by M. Krée- P. Krée[1]. In general case they were proved by Meyer[2] using Littlewood-Payley inequalities. The simplest proof based on boundedness in L^p of Hilbert transformation is due to Pisier[1]. Generalized functionals were introduced by S. Watanabe[1]. A synthesis for formulas in Malliavin calculus can be found in Nualart-Zakai[4]. Proposition 3.23 is taken from Sugita[3].

§4. The main results about existence and smoothness of densities of Wiener functionals were first obtained by Malliavin[1] using an integration by parts formula. His method has been developed by Stroock[1], Bismut[1] and Shigekawa[1]. Bouleau-Hirsch[1,2] weakened conditions for existence of density by using Dirichlet forms (Theorem 4.7). S. Watanabe[1] defined the composition of distributions with Wiener functionals and obtained an important result about smoothness of the density (Theorem 4.9). Example of Donsker δ- function is taken from Ikeda-Watanabe[3]. Another approach based on white noise analysis see Kuo[3] or Examples 2.24 and 2.25 in Chapter IV. Example about density of maximum of a continuous process is from Nualart-Vives[1]. Some probabilistic proofs of results which have not been proved so far by analytic method can be found in Kusuoka-Stroock[4].

Chapter III

§1. The anticipating stochastic integral with respect to Brownian motion was introduced by Skorohod[1]. Gaveau-Trauber[1] proved that Skorohod integration is equivalent to the divergence operator δ. Proofs based on the chaos decompo-

sition in this paragraph are due to Nualart-Pardoux[1], Nualart-Zakai[1,2] and Yan[1]. Theorem 1.7 was first obtained by Clark[1] (see also Haussmann[1]) under the assumption that F is Fréchet differentiable. Ocone[1] extended it to the case of $F\in\mathbb{D}_1^2$ and Karatzas-Ocone-Li[1] to the case of $F\in\mathbb{D}_1^1$. Here the simplest proof is taken from Yan[1] (see also Nualart-Zakai[2]). A unified treatment for Clark's formula see Wu[3]. The detailed proof of Theorem 1.9 can be found in Ikeda-Watanabe[1] or Stroock[3], here we give a simple proof using Picard iteration and Lemma 1.4. The probabilistic proof of Hörmander's theorem was first given by Malliavin[1] (see Ikeda-Watanabe[1] or Huang[4]), here the much simpler proof is due to Norris[1], where the key Lemma 1.11 is from Stroock[3]. For a different proof see Bismut[1]. Some improvement of Hörmander's conditions can be found in Kusuoka-Stroock[3].

§2. Most part of this paragraph are taken from Malliavin[5] and Sugita[3], but some improvements have been made in the proofs. Malliavin[2] introduced the notions of (k,p)-capacities and slim sets and initiated the research field of quasi-sure analysis. The question about invariance of capacities was put forward by Itô and solved by Albeverio, Fukushima et al.[1]. Here the proof of Theorem 2.15 is taken from Malliavin[5]. Sugita[3] proved that Meyer-Watanabe's positive generalized functionals are measures in the framework of abstract Wiener spaces. Similar results in the framework of white noise spaces were obtained by Kondratiev-Samoylenko[1] and Yokoi[1] (see also Chapter IV, Theorem 4.9). For a discussion of their relation see Huang[5]. As for the quasi-sure sample properties of stochastic processes we refer to Fukushima[1], Takeda[1], Yoshida[1], Denis[1], Ren[1,2,5] and references cited in Ren[4].

§3. The material concerning approximation of Skorohod integral as well as Stratonovich integral by Riemannian sums is mainly taken from Nualart-Pardoux[1]. There have been many other approaches to anticipating stochastic integration and Itô formula, for example, see Hitsuda[1], Sevljakov[1], Ogawa[1], Sekiguchi-Shiota[1], Kuo-Russek[1], Asch-Potthoff[1] and Ustunel[2].

The material concerning anticipating Girsanov transformation and stochastic differential equations is taken from Kusuoka[1] and Buckdahn[2,3,4]. For related results see Ramer[1], Buckdahn[1], Enchev[1], Enchev-Stroock[1], Ustunel-Zakai[3,5] and Y. N. Zhang[1]. For other types of equations and approaches we refer to Ocone-Pardoux[1], Buckdahn-Nualart[1] and references cited in Pardoux[1].

Malliavin calculus has a wide range of applications. For example, for a probabilistic proof of Atiyah-Singer's index theorem see Bismut[5] or Watanabe[3], for applications to filtering problems see Bismut-Michel[1], for results concerning asymptotic properties of heat kernels see Watanabe[2] and Ikeda[1], for investigation of stochastic oscillatory integrals see Gaveau-Moulinier[1], for research on the relation between independence and orthogonality of gradients of random variables on Wiener spaces see Ustunel-Zakai[1,2]. Moreover, for stochastic calculus

of variation involving processes with jumps see Bismut[3], Bichteler-Gravereaux-Jacod[1] and Wu[1,2]. For other applications and further developments of theories we refer to Malliavin[4] and references therein.

Chapter IV

§1. The classical references of Wiener-Itô-Segal isomorphism are Wiener[2], Itô[1] and Segal[2]. The notion of Wick tensor product $:x^{\otimes n}:$ originated from the Wick ordering in quantum physics which was initiated by Wick[1]. It seems not appropriate to call Wick tensor products as Wick orderings as appeared in some references. The classical framework for white noise analysis constructed via second quantization is due to Kubo-Takenaka[1]. Meyer-Yan[2] and Kondratiev-Leukert-Potthoff-Streit-Westerkamp[1] further developed this construction to a general Gel'fand triplet. To establish a general framework for white noise analysis via an extended second quantization was first put forward by Kondratiev-Streit[1]. It can be applied to non-Gaussian analysis (see Kondratiev-Streit-Westerkamp-Yan[1]). Recently, several new frameworks for white noise analysis appeared to fit the need of different applications or theoretical interests (for example, see Meyer-Yan[3], Potthoff-Timpel[1], Huang-Song[1], Imkeller-Yan[2]).

§2. In the classical framework of white noise analysis, characterizations of spaces of distributions and its two important consequences were given by Potthoff-Streit[1], those of testing functional spaces were obtained by Kuo-Potthoff-Streit[1]. A refinement of above results can be found in Yan[7]. T. S. Zhang[1] obtained another kind of characterizations for functional spaces. Results in the framework of general Gel'fand triplet were given in Kondratiev-Leukert-Potthoff et al.[1]. Relative results in Sections 2.1 and 2.3 under general framework are due to Kondratiev-Streit[1], those in Section 2.2 are due to Kondratiev-Leukert-Streit[1]. Example 2.23 is taken from Kubo-Takenaka[1]. The Donsker δ-functional in Example 2.24 was investegated by Kuo[3]. Example 2.26 is taken from Kondratiev-Streit[1]. The local time of self-intersection of multi-dimensional Brownian motion was first investigated by means of white noise analysis by H. Watanabe[1], here Example 2.27 is taken from He-Yang-Yao-Wang[1].

§3. The product formula (3.10) for functionals is classical (see the comment in Meyer[3]). Lemma 3.1 and Theorem 3.2 are extensions of relative results in Potthoff-Yan[1] in the case of $\rho=0$. The notion of Wick product of functionals goes back to that of "s-product" in quantum field theory introduced by Wick[1] (see Simon[1]). In 1965, Hida-Ikeda[1] introduced the Wick product in probability theory. Meyer-Yan[1] defined the Wick product of distributions under the framework of white noise analysis by means of S-transform. For widespread applications of Wick product in stochastic analysis see Holden et al.[1]. White noise approach to Feynman integrals was initiated by Hida-Streit[1] and developed by Meyer-Yan[1], Hu-Meyer[1] and De Faria-Potthoff-Streit[1]. In Section 3.3, under the framework of Meyer-Yan[1], we introduce some results obtained

by Khandekar-Streit[1].

§4. The moment characterizations of distribution spaces were first given by Kondratiev-Streit[1], here we have simplified the proof by means of the renormalization operator of Yan[6]. In classical framework of white noise analysis, measure representation of positive distributions was given by Yokoi[1], corresponding result under general setting (Theorem 4.9) was obtained by Kondratiev-Streit[1].

Chapter V

§1. A systematic account of analytical calculus for distributions can be found in Potthoff-Yan[1]. For further developments see Kuo-Potthoff-Yan[1] and Yan[10]. Extensions of their results under general framework constitute contents of the paragraph.

§2. The notion of symbols of operators in Fock spaces originated from Berezin[2] and Krée-Rączka[1]. In the framework of white noise analysis, Obata[3] obtained a characterization for symbols of generalized operators which is a natural extension of the characterization for distributions given by Potthoff-Streit[1]. However, his proof involves estimation of integral kernel operators and is not fit to the general framework. A statement of this result under general frameworks was presented in Obata[8], here the simple proof is taken from Luo-Yan[1]. The chaos decomposition for operators was initiated by Berezin[2]. Under the framework of white noise analysis, Huang[6] obtained a chaos decomposition for generalized operators. The notion of Wick product of generalized operators originated from that of Wick ordering for products of creation and annihilation operators in quantum field theory. Its mathematical definition was given by Huang-Luo[1]. Results in section 2.2 are taken from Luo-Yan[1].

§3. The notion of integral kernel operators also came from Berezin[2]. Under the framework of white noise analysis, Kubo-Takenaka[1] represented some kind of operators via Hida's differential operator and its dual operator. A systematic investigation of integral kernel operators can be found in Hida-Obata-Saitô[1]. The integral kernel representations for generalized operators (Theorem 3.11 and formula (3.39)) are due to Huang[6] and Obata[4] where it is called Fock expansions. Here the proof is simplified by means of chaos decomposition of operators. Main reference of this paragraph is Obata[4].

§4. Applications of white noise analysis to quantum probability were first put forward by Huang[1], where the notion of quantum white noise measures was introduced. The contents of Sections 4.1 and 4.2 are taken from Huang-Luo[1] and Luo[1], that of Section 4.3 are from Hida-Kuo-Potthoff-Streit[1]. For further applications of white noise analysis to infinite dimensional Dirichlet forms see Albeverio-Hida et al.[1,2], Hida-Potthoff-Streit[1] and Razafimanantena[1]. For applications of theory of generalized operators to infinite dimensional harmonic analysis and quantum probability see Obata[4,5,6,8].

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Subject Index

Index of Symbols

$\doteq$ (defined as); $\Longrightarrow$ (imply); $\Longleftrightarrow$ (if and only if);
$\prec$ (topologically weaker than); $\sim$ (equivalent to); ■ (end of proof);
$\cong$ (isomorphic to); $\hookrightarrow$ (continuously and densely embedded into);
$\uparrow$ (increasingly converges to); $\uparrow\uparrow$ (strictly increasingly converges to);
$\downarrow$ (decreasingly converges to); $\downarrow\downarrow$ (strictly decreasingly converges to);
$x \longmapsto f(x)$ (mapping f); 1_A (indicator of set A);
$f|_A$ (restriction of f to A); $\overline{\mathcal{F}}^{\mu}$ (completion of σ-algebra $\mathcal{F}$ w.r.t. μ);
$a\wedge b \equiv \min(a,b)$; $a\vee b \equiv \max(a,b)$; $x^{+} \equiv x\vee 0$; $x^{-} \equiv -(x\wedge 0)$;
$|\cdot|_X$, $\|\cdot\|_X$ (norm in space X); $(\cdot,\cdot)_H$ (inner product in space H);
X' (algebraic dual of space X, B5);
X^{*} (topological dual of space X, B5);
$\langle\cdot,\cdot\rangle$, $\langle\langle\cdot,\cdot\rangle\rangle$ (canonical bilinear form, B5);
$\partial_j \equiv \partial/\partial x_j$ $(x=(x_1,\cdots,x_m)\in \mathbb{R}^m)$;
$\partial_\alpha \equiv \partial_1^{\alpha_1}\cdots\partial_m^{\alpha_m}$ $\alpha=(\alpha_1,\cdots,\alpha_m)\in \mathbb{N}_0^m$);
$x^\alpha \equiv x_1^{\alpha_1}\cdots x_m^{\alpha_m}$; $|\alpha| \equiv \sum_j \alpha_j$; $\alpha! \equiv \prod_j (\alpha_j!)$;
$\otimes$ (tensor product, I2.1, I2.3, I3.3); $\hat{\otimes}$ (symmetric tensor product, I2.1);
$\bar{\otimes}$ (projective tensor product, I2.3); $\oplus$ (direct sum I1.1, I2.2);
$\otimes_r$ (contraction of tensor product, III.3, IV3.1, V3.1);
$x^{\diamond n}$; (Wick tensor product, IV1.1); $\diamond$ (Wick product, IV3.2, V2.2);
$\hat{A}$ (symbol of operator A, V2.1);
A^T (transpose of matrix A); $A^*(E_C^1)$ (IV3.3);
$\bar{A}$ (closure of operator A, I1.1-1.3; S-transform of operator A, V2.2);
a.a. (almost all); a.e. (almost everywhere); a.s. (almost surely);
$\mathcal{B}(T)$ (Borel σ-algebra of topological space T);
$\mathcal{B}(X,Y)$ (I3.3); $\mathcal{B}(X,Y)$ (B7); X_β (B7);
$\mathbb{C}$ (complex numbers); $X_C = X + iX$ (complexification of space X, IV2);
C^k, C^∞ (k-times, infinitely continuously differentiable functions);
C_b^k, C_b^∞ (functions in C^k, C^∞ with bounded derivatives);
C_0^k, C_0^∞ (functions in C^k, C^∞ with compact support);
$C_{k,p}$ (III2.1); $C_b^1(H)$ (III3.3);
$\mathcal{D}(A)$ (I1.1); D, δ (II2.1, II2.2); D_h, δ_h (II2.2);
D_t (II2.3); D_t^1, D_t, ∇ (III3.1);
δ_{ij} (Kronecker symbol); $d\Gamma(A)$ (I2.3);

$\mathbb{D}_p^k(E)$, $\mathbb{D}_p^{\infty}(E)$, $\mathbb{D}^{\infty}(E)$ (II2.3);
$\widetilde{\mathbb{D}}_p^k(E)$, $\mathbb{D}^{-\infty}(E)$ (II3.4); $\widetilde{\mathbb{D}}_p^k(H)$ (III3.1);
det (determinant); $\det_2$ (Carleman-Fredholm determinant);
$\mathbb{E}(\cdot)$ (mathematical expectation); $\mathbb{E}(\cdot|\mathcal{G})$ (conditional expectation);
$\mathcal{E}(h), \mathcal{E}_h$ (exponential vector (functional), I2.3, II1.3, IV2.1);
$\hat{\varepsilon}_h$ (I2.1); e_A (III2.3); ess (III2.2);
$(E), (E)^*, (E)^\beta$ (IV1.2); $(E)_+^{-\beta}$, $(E)_+^*$ (IV4.3);
$\mathcal{F}$ (Fourier transform); $\mathcal{F}(H)$ (I2.2);
$\mathcal{G}(A)$ (I1.1); $\Gamma(H)$ (I2.2); $\Gamma(A)$ (I2.3); $\gamma_{k,p}^{C}$ (III2.1);
H_n, h_n (A); H_n (II1.2); $(H_{p,q,\beta})$ (IV1.2) H^2 (II4.2);
$Hol_0(E_C)$ (IV2.2); $H_G(U), H(U)$ (IV2.1);
$\|\cdot\|_{HS}$ (I1.5, I2.1); $\prec_{HS}$ (B1);
I_n (II1.3) I_{pr} (B1); $(I-\mathcal{L})^{r/2}$ (II3.2); Im (imaginary part);
$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (scalar field); $\mathcal{K}(H,K)$ (I1.4);
$\|\cdot\|_{k,p}$ (II2.3); (k,p)-q.s., (k,p)-q.e. (III2.1)
l^2 (space of square-summable sequences); $L(X,Y)$, $\mathcal{L}(X,Y)$ (I1.1);
$\mathcal{L}_{(1)}(H,K)$, $\mathcal{L}_{(2)}(H,K)$ (I1.5);
$L^p(\Omega,\mathcal{F},\mu;E)$, $L^p(\mu;E)$, L^p (II1.1);
$\widehat{L}^2$ (I2.1); $L^{\infty-}$, L^{1+} (II2.2);
$\mathcal{L}$ (II1.2, II2.1, II3.1); (L^2) (IV1.2);
Λ, Λ_n (I2.1, II1.2); $\Lambda^{(n)}$ (IV2.4); $_{loc}\mathbb{D}_p^k(E)$ (III3.2);
$\overline{\lim}$ (limit superior); $\underline{\lim}$ (limit inferior); $\varprojlim$, $\varinjlim$ (B3);
$\hat{\mu}$ (Fourier transform of measure μ, I1.4), $\mathcal{M}^\beta(E^-)$ (IV4.3);
$\mathbb{N}$ (set of natural numbers); $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$;
$\mathbb{N}_0^{\mathbb{N}}$ (set of all sequences of non-negative integral numbers);
$\mathcal{N}(0)$ (B2); $\mathcal{N}(A)$ (I1.1);
ONB (orthonormal base); $(\Omega,\mathcal{F},\mu;H)$ (II1.1); $\mathcal{P}(E)$ (II1.1);
$\mathbb{Q}$ (field of rational numbers); $\mathbb{Q}_+$ (non-negative rational numbers);
q.s., q.e. (III2.1); $Q = (I-\mathcal{L})^{-1/2}$ (II3.3); Q_ε (B1);
$\mathcal{R}(A)$ (I1.1); $\rho(A)$ (I1.4); Re (real part);
$\mathbb{R}$ (real numbers); $\mathbb{R}_+$ (non-negative real numbers);
$\mathbb{R}^d$ (d-dimensional real space); $\mathbb{R}^\infty$ (space of real sequences);
$(\mathbb{R}^\infty, \mathcal{B}^\infty, \gamma^\infty; l^2)$ (numerical model, II1.2);
$\mathcal{S}(\mathbb{R}^m)$, $\mathcal{S}'(\mathbb{R}^m)$ (I3.2); $S_\Psi(E)$ (II1.1);
S_n S_r (permutation group, I2.1, V3.2); $s_{r,n}(\kappa)$ (V3.2);
$\sigma(\mathcal{G})$ (σ-algebra generated by $\mathcal{G}$);
$\sigma(f_\alpha, \alpha\in\Gamma)$ (σ-algebra generated by $\{f_\alpha \in \Gamma\}$);
$\sigma(A), \sigma_p(A)$ (I1.4); $\sigma(X,Y)$ (B5); $\Sigma = (\sigma_{ij})$ (II4.1);
span (linear span); spec (spectral set); supp (support);
$\mathcal{S}$ (IV1.1); τ_n (V3.3); $\tau(X,Y)$ (B6);
Tr (I1.5); T_t (II2.1, II3.1); $t_{n,r}(\kappa)$ (V3.2);
$\mathcal{W}(A)$ (I1.1); $W^{k,p}(\mathbb{R}^m)$ (Sobolev space);